Časopis pro pěstování matematiky

K. Howalla; Abdallah N. Dabboucy; R. Tout

An extremal problem for some classes of oriented graphs

Časopis pro pěstování matematiky, Vol. 108 (1983), No. 1, 53--69

Persistent URL: `http://dml.cz/dmlcz/118158`

AN EXTREMAL PROBLEM FOR SOME CLASSES OF ORIENTED GRAPHS

K. Howalla, A. N. Dabboucy, R. Tout, Beirut

(Received November 6, 1981)

INTRODUCTION AND NOTATION

Let $\mathscr{G}_k$ be the set of oriented graphs (directed graphs with no 2-cycle) satisfying property (P_k): For all couples of points x and y, there exist at most k distinct directed paths from x to y. In previous paper the authors gave the values of $f_k(p) = \max \{q \mid G(p, q) \in \mathscr{G}_k\}$ for $k = 1, 2$ where $G(p, q)$ denotes a graph with p points and q arcs. We shall give here the value of $f_3(p)$ and characterize those graphs in $\mathscr{G}_k$ which have the maximum number $f_k(p)$ of arcs for $k = 1, 2, 3$.

Under an oriented graph $G(X, U)$ we shall understand a directed graph without loops and 2-cycles, with the set of points X and set of arcs U. If $|X| = p$, $|U| = q$, we also write $G(p, q)$. In such a graph, $d_G(x)$, for $x \in X$, denotes the sum of the out-degree and in-degree of x and $\delta(G) = \min \{d_G(x) \mid x \in X\}$.

Arcs will be denoted by (u, v) etc., non-directed paths by $[u, v, w \ldots]$ etc.

We shall also denote, for a real t, by $]\,t\,[$ the integer satisfying $t \leqq\,]\,t\,[< t + 1$, by $[t]$ the integer satisfying $t - 1 < [t] \leqq t$.

In section 1, we shall say that a graph $G(p, q)$ satisfies the relation (R) if

$$q \leqq 2(p - 2) + [\tfrac{1}{4}(p - 2)^2]\,.$$

1. EVALUATION OF $f_3(p)$

Theorem 1.1. $f_3(p) = 2(p - 2) + [\frac{1}{4}(p - 2)^2]$ *for* $p \geqq 6$.

Proof. We shall first establish two lemmas.

Lemma 1. *For every graph* $G(p, q) \in \mathscr{G}_3$ *we have* $\delta(G) \leqq\,]\frac{1}{2}(p + 1)[$.

Proof. If not then there exists a graph $G_0 \in \mathscr{G}_3$ such that $\delta(G_0) \geqq\,]\frac{1}{2}(p + 1)[+ 1$. Therefore, for any two adjacent points x and y of G_0,

$$d_{G_0}(x) + d_{G_0}(y) \geqq 2]\tfrac{1}{2}(p + 1)[+ 2 \geqq p + 3\,.$$

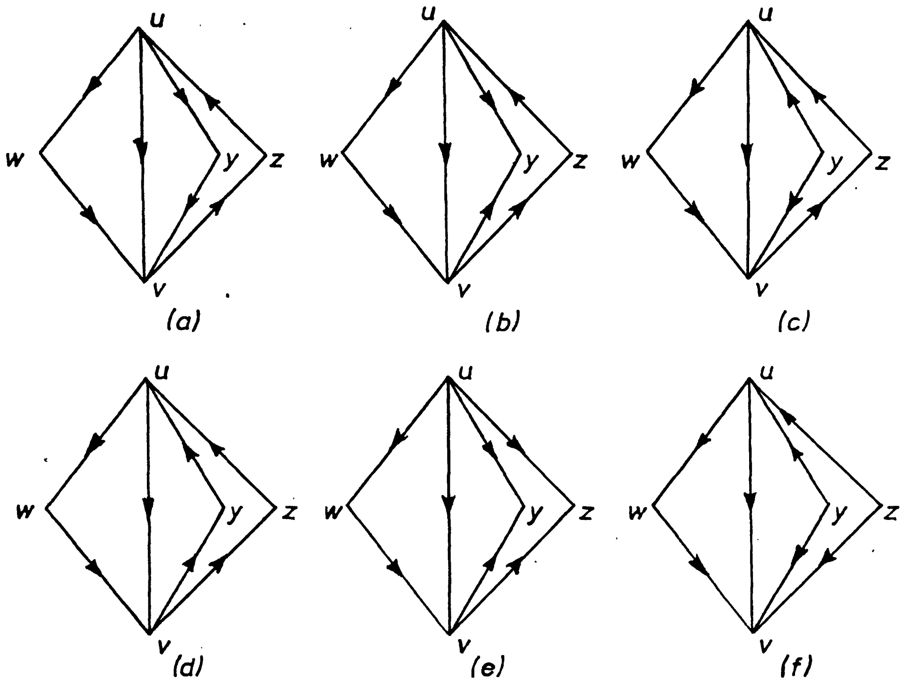


Fig. 1.

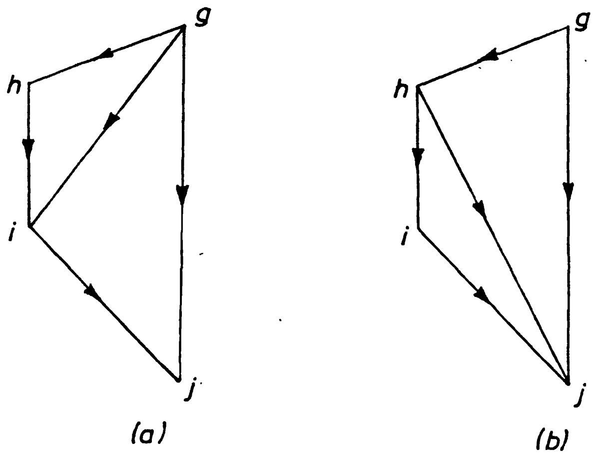


Fig. 2.

This implies the existence of at least three points of $X \setminus \{x, y\}$ adjacent simultaneously to x and y. We shall use this fact to show that all triangles in G_0 are 3-cycles and obtain a contradiction. Suppose this is not true and let u, v, w be points of

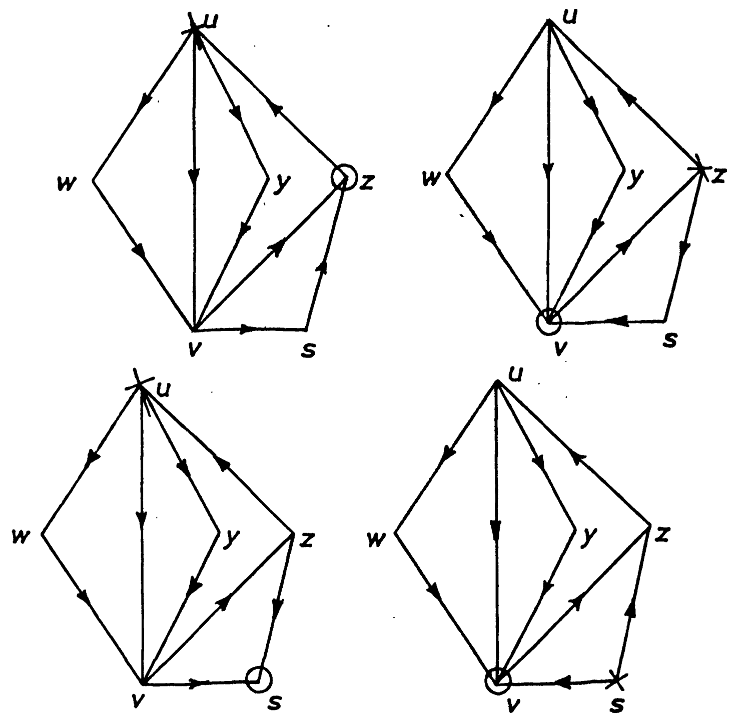


Fig. 3.

a triangle formed by the arc (u, v) and the (yet non-oriented) path $[u, w, v]$. By the above observation, there exist two points y and z, different from w, adjacent to or from u and v. All the six possible orientations of the edges $[u, y]$, $[u, z]$, $[v, y]$ and $[v, z]$ are given in Fig. 1. We shall show that all these graphs are "forbidden" subgraphs in G_0. It suffices to prove that the graphs in Fig. 1-a, 1-d and the graphs in Fig. 2-a and 2-b (subgraphs of 1-b, 1-c, 1-e and 1-f) are forbidden subgıaphs in G_0.

Let us consider the graph in Fig. 1-a. It is easy to see that z can be adjacent, to or from, neither w nor y by property (P_3). Thus there is a point s different from u, y and w which is adjacent to or from both v and z. All the four possible orientations of the edges $[v, s]$ and $[z, s]$ create four or more distinct paths from a point in G_0 to another point (see Fig. 3 where X marks the starting point of four or more distinct paths to the point marked O).

For the subgraph shown in Fig. 1-d, we see that y and z cannot be adjacent by (P_3) and since $d_{G_0}(y) + d_{G_0}(z) \geqq p + 3$, there are at least five points in $X \setminus \{y, z\}$ which are simultaneously adjacent to or from both y and z. Thus there is a point s different from u, v, w which is adjacent to or fıom both y and z. However, all the four possible orientations of the edges $[y, s]$ and $[z, s]$ lead to four or moıe distinct paths from a point to another (see Fig. 4 where again X marks the starting point of four or more paths to the point marked O).

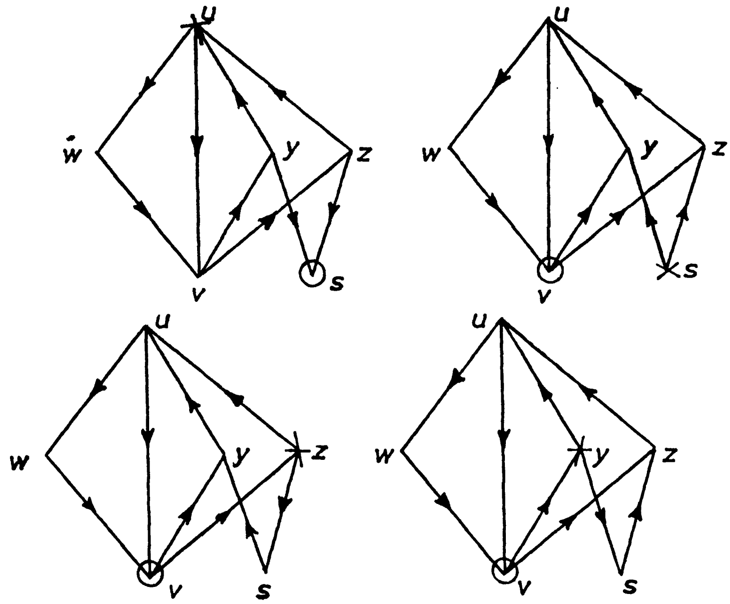
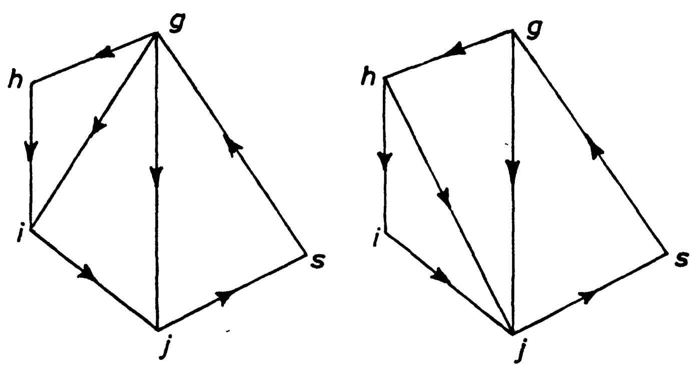


Fig. 4.



(a) Fig. 5 (b)

Now for each of the subgraphs given in Fig. 2-a and Fig. 2-b there is at least one point *s* different from *h* and *i*, adjacent to or from *g* and *j*. The only possible orientations of the edges [*g*, *s*] and [*j*, *s*] are from *j* to *s* and from *s* to *g* (see Fig. 5). It is easily seen that *s* can be adjacent neither to or from *h* nor to or from *i*. Thus there exists at least one point *r* different from *g*, *h* and *i* which is simultaneously adjacent to or from *j* and *s*. All the foui possible orientations of [*j*, *r*] and [*s*, *r*] lead, however, to four or more distinct paths in G_0 (see Fig. 6 which corresponds to Fig. 5-a; a similar set of figures may be given for Fig. 5-b).

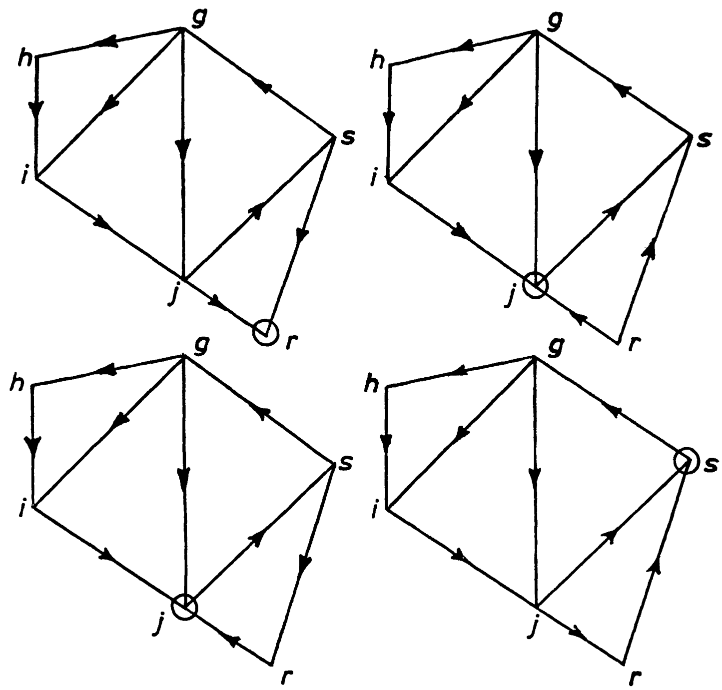


Fig. 6.

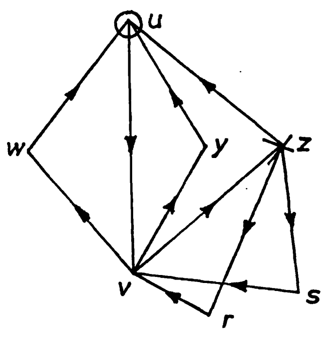


Fig. 7.

It follows that every triangle in G_0 is a 3-cycle. Hence to any arc (u, v) of $G_0 \in \mathscr{G}_3$ there correspond three distinct paths of length two from v to u in G_0, say $[v, w, u]$, $[v, y, u]$ and $[v, z, u]$ (see Fig. 7). However, since z cannot be adjacent to or from y or w by (P_3), there exist at least two points r and s different from u and adjacent simultaneously to or from v and z. The induced orientation on the new edges leads then to more than three paths from z to u, in contradiction to (P_3). The lemma is proved.

Lemma 2. *Let a graph* $G(p, q)$ *have a point* x *such that* $d_G(x) \leqq \,]\tfrac{1}{2}(p + 1)[$. *Then* $G(p, q)$ *satisfies the relation* (R) *if* $G' = G \setminus \{x\}$ *(obtained from* G *by omitting the point* x *and the arcs incident with* x*) satisfies the relation* (R).

Proof. Let the relation (R) for G' be fulfilled:

$$q - d_G(x) \leqq 2(p - 3) + [\tfrac{1}{4}(p - 3)^2] .$$

Then

$$q \leqq \,]\tfrac{1}{2}(p + 1)[+ 2(p - 3) + [\tfrac{1}{4}(p - 3)^2] .$$

However, by inspecting the four cases mod 4 it follows easily that

$$]\tfrac{1}{2}(p + 1)[+ 2(p - 3) + [\tfrac{1}{4}(p - 3)^2] = 2(p - 2) + [\tfrac{1}{4}(p - 2)^2]$$

so that $G(p, q)$ satisfies (R).

To finish the proof of theorem 1.1, we shall show by induction that all graphs in $\mathscr{G}_3$ satisfy the relation (R).

Let us show first that any graph $G(6, q) \in \mathscr{G}_3$ satisfies (R), i.e. $q \leqq 12$. Suppose there is a graph $G_0(6, q) \in \mathscr{G}_3$ such that $q \geqq 13$. By lemma 1, there exists a point x in G_0 such that $\delta(G_0) = d_{G_0}(x) \leqq 4$. Let $G_0'(5, q') = G_0(6, q) \setminus \{x\}$. Since $G_0' \in \mathscr{G}_3$, there exists in G_0' a point y such that $d_{G_0'}(y) = \delta(G_0') \leqq 3$. Let again $G_0''(4, q'') = G_0'(5, q') \setminus \{y\}$. One obtains then the following inequalities:

$$6 \geqq q'' = q' - d_{G_0'}(y) \geqq q' - 3 = (q - d_{G_0}(x)) - 3 \geqq$$
$$\geqq q - 4 - 3 \geqq 13 - 7 = 6 .$$

Thus $q'' = 6$, $d_{G_0'}(y) = \delta(G_0') = 3$, $d_{G_0}(x) = \delta(G_0) = 4$ and $q = 13$. Now since $d_{G_0}(y) = 3$ and $d_{G_0}(y) \geqq 4$, it follows that y is joined by an arc to (or from) x in G_0 and $d_{G_0}(y) = d_{G_0}(x) = 4$. We shall prove that no graph $G(6, 13)$ obtained from $G''(4, 6)$ by adding two points x and y joined by an arc and such that $d_G(x) = d_G(y) = 4$ can belong to $\mathscr{G}_3$. There are three graphs $G_i''(4, 6)$ in $\mathscr{G}_3$, $i = 1, 2, 3$ (see Fig. 8).

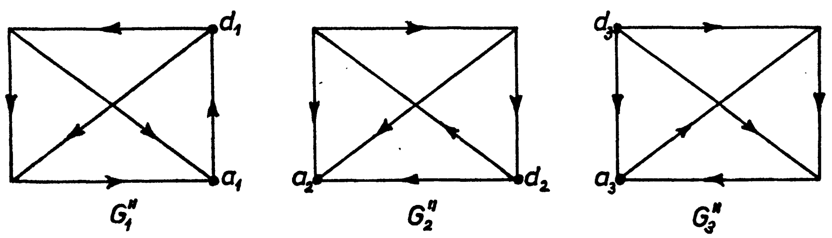


Fig. 8.

Each contains a point d_i from which there are three distinct paths to another point a_i in G_i'', $i = 1, 2, 3$. Since $\delta(G) = 4$, this implies that d_i is adjacent to (or from) x or y in $G(6, 13)$. It is, however, easy to see that the graph $G(6, 13)$ cannot belong to $\mathscr{G}_3$, a contradiction. Thus $G(6, q) \in \mathscr{G}_3$ always satisfies the inequality $q \leqq 12$, i.e. the relation (R).

Now we are able to finish the proof of theorem 1.1. We have shown that all $G(p, q)$ from $\mathscr{G}_3$ satisfy (R) for $p = 6$. Let $G(n, q) \in \mathscr{G}_3$ for $n > 6$ and assume this assertion is true for all graphs $G(p, q) \in \mathscr{G}_3$ for which $p \leqq n - 1$. By lemma 1, there is a point x in G such that $d_G(x) \leqq]\frac{1}{2}(p + 1)[$.

Since $G' = G \setminus \{x\}$ is in $\mathscr{G}_3$, it satisfies (R) by the induction hypothesis. Therefore, G satisfies (R) by lemma 2. Hence

$$f_3(p) \leqq 2(p - 2) + [\tfrac{1}{4}(p - 2)^2] .$$

However, the complete tripartite graph (A, B, C, U) with orientation from A to B, from A to C and from C to B where $|A| =]\frac{1}{2}(p - 2)[$, $|B| = [\frac{1}{2}(p - 2)]$ and $|C| = 2$ clearly belongs to $\mathscr{G}_3$ and

$$|U| = q = 2(p - 2) + [\tfrac{1}{4}(p - 2)^2] .$$

Therefore,

$$f_3(p) = 2(p - 2) + [\tfrac{1}{4}(p - 2)^2] .$$

2. CHARACTERIZATIONS OF EXTREMAL GRAPHS

In [1], we have found that foı $p \geqq 4$,

$$f_1(p) = [\tfrac{1}{4}p^2] ,$$
$$f_2(p) = [\tfrac{1}{2}(p - 1)] + [\tfrac{1}{4}p^2] ;$$

the result of the previous section was that

$$f_3(p) = 2(p - 2) + [\tfrac{1}{4}(p - 2)^2] \quad \text{for} \quad p \geqq 6$$

where

$$f_k(p) = \max \{q \mid G(p, q) \in \mathscr{G}_k\} , \quad k = 1, 2, 3 .$$

In this section, we shall give characterizations of all graphs $G(p, f_k(p))$ in $\mathscr{G}_k$, $k = 1, 2, 3$.

Theorem 2.1. *Every graph $G(p, f_1(p))$ in $\mathscr{G}_1$, $p \geqq 5$, is a complete bipartite graph (A, B, U) with arcs oriented from A to B where either $|A| =]\frac{1}{2}p[$ and $|B| = [\frac{1}{2}p]$, or $|A| = [\frac{1}{2}p]$ and $|B| =]\frac{1}{2}p[$. These two cases are distinct iff p is odd.*

Proof. We shall first establish three lemmas. For brevity, we call $\mathscr{G}_1^m$ the set of all graphs $G(p, f_1(p))$, $p \geqq 5$.

Lemma 3. *No graph $G \in \mathscr{G}_1^m$ with $p \geqq 5$ points contains any cycle.*

Proof. Let $G_0(p, f_1(p)) \in \mathscr{G}_1$ contain a cycle C_n of length $n \geqq 3$. Contracting C_n to a single point, we obtain a graph $G(p - n + 1, f_1(p) - n)$ which again belongs to $\mathscr{G}_1$. Hence

$$(1) \qquad [\tfrac{1}{4}p^2] - n \leqq [\tfrac{1}{4}(p - n + 1)^2] ,$$

a contradiction since (1) is not true for $p \geqq 5$ and $n \geqq 3$.

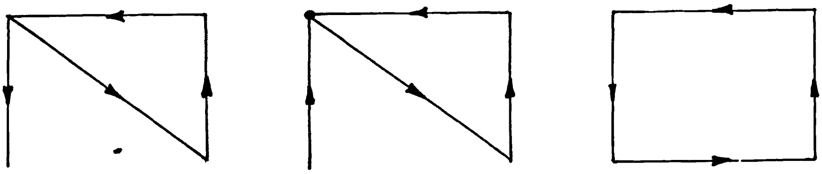


Fig. 9.

Remark. It is easy to see that for $p = 4$, the only graphs in $\mathscr{G}_1^m$ which contain cycles are those in Fig. 9.

Lemma 4. *If $G(p, f_1(p)) \in \mathscr{G}_1$, $p \geqq 5$ then $\delta(G) \geqq [\frac{1}{2}p]$.*

Proof. Since for any point x of G the graph $G' = G \setminus \{x\}$ belongs to $\mathscr{G}_1$, we have

$$[\tfrac{1}{4}p^2] - d_G(x) \leqq [\tfrac{1}{4}(p-1)^2].$$

Hence

$$d_G(x) \geqq [\tfrac{1}{4}p^2] - [\tfrac{1}{4}(p-1)^2] = [\tfrac{1}{2}p] \tag{2}$$

so that

$$\delta(G) \geqq [\tfrac{1}{2}p].$$

Lemma 5. *No graph in $\mathscr{G}_1^m$ with $p \geqq 5$ points contains a directed path of length greater than one.*

Proof. Let $G_0(p, f_1(p))$ belonging to $\mathscr{G}_1^m$ contain a directed path L_m of length $m \geqq 2$. Any point in G_0 which does not belong to L_m is joined by an arc with at most one point of L_m by (P_1) and lemma 3.

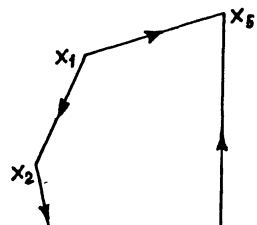


Fig. 10.

From this and (2), we obtain the double inequality

$$(m+1)[\tfrac{1}{2}p] \leqq \sum_{i=1}^{m+1} d(x_i) \leqq p - m - 1 + 2(m-1) + 2. \tag{3}$$

Hence

$$(m+1)[\tfrac{1}{2}p] \leqq p + m - 1. \tag{4}$$

A simple calculation shows that (4) is not true for $p > 5$ and $m \geqq 2$. For $p = 5$, (4) implies $m \leqq 2$. Since $m \geqq 2$, we have to consider only the case $p = 5$ and $m = 2$. The inequality (3) yields then the equality

$$3[\tfrac{5}{2}] = 6 = \sum_{i=1}^{3} d(x_i) = 6$$

which implies $d(x_i) = 2$, $i = 1, 2, 3$. The corresponding graph is that in Fig. 10 which is a contradiction since $q = 5 < f_1(5) = 6$.

Let us finish now the proof of theorem 2.1. Let $G \in \mathcal{G}_1^m$; By lemma 5, G contains no directed path of length greater than one. It follows easily that G is a bipartite graph (A, B, U) with arcs oriented, say, from A to B. If $|A| = t$ then $|B| = p - t$ so that

$$q = [\tfrac{1}{4}p^2] \leqq t(p - t).$$

This implies easily that $t =]\tfrac{1}{2}p[$ or $t = [\tfrac{1}{2}p]$. The proof is complete.

Theorem 2.2. *Every graph $G(p, f_2(p))$ in $\mathcal{G}_2$ with $p \geqq 6$ is a complete tripartite graph (A, B, C, U) with arcs oriented from A to B, from A to C and from C to B where either $|A| =]\tfrac{1}{2}(p - 1)[$, $|C| = 1$ and $|B| = [\tfrac{1}{2}(p - 1)]$, or $|A| = [\tfrac{1}{2}(p - 1)]$, $|C| = 1$ and $|B| =]\tfrac{1}{2}(p - 1)[$. These two cases are distinct iff p is even.*

Proof. We shall prove first two lemmas.

Lemma 6. *For any graph $G(p, f_2(p))$ in $\mathcal{G}_2$, $p \geqq 5$, we have $\delta(G) =]\tfrac{1}{2}p[$ and if y is a point in G for which $f_G(y) =]\tfrac{1}{2}p[$ then the graph $G' = G \setminus \{y\}$ is in $\mathcal{G}_2^m$ as well.*

Proof. If x is any point of a graph $G \in \mathcal{G}_2^m$, $G' = G \setminus \{x\}$ belongs to $\mathcal{G}_2$ so that

$$f_2(p) - d_G(x) \leqq f_2(p - 1).$$

This implies

$$d_G(x) \geqq p - 1 + [\tfrac{1}{4}(p - 1)^2] - (p - 2) - [\tfrac{1}{4}(p -)^2] =]\tfrac{1}{2}p[.$$

Hence

$$\delta(G) \geqq]\tfrac{1}{2}p[.$$

However, in [1] we have proved that

$$\delta(G) \leqq]\tfrac{1}{2}p[.$$

Thus

$$\delta(G) =]\tfrac{1}{2}p[.$$

On the other hand, let a point y of $G(p, f_2(p)) \in \mathcal{G}_2$ satisfy $d_G(y) =]\tfrac{1}{2}p[$; the above calculation shows that

$$f_2(p) - d_G(y) = f_2(p - 1),$$

i.e. that $G' = G \setminus \{y\} \in \mathcal{G}_2^m$.

Lemma 7. *$\mathscr{G}_2^m$ contains only two graphs of order $p = 6$. Both are complete tripartite graphs (A, B, C, U) with arcs oriented from A to B, from A to C and from C to B. For one of them, $|A| = 3$, $|B| = 2$, $|C| = 1$, for the other $|A| = 2$, $|B| = 3$, $|C| = 1$.*

Proof. We have $f_2(6) = 11$. Let thus $G(6, 11) \in \mathscr{G}_2^m$. By lemma 6, $\delta(G) = 3$ and if a point x of G satisfies $d_G(x) = 3$ then the graph $G'(5, 8) = G(6, 11) \setminus \{x\}$ is in $\mathscr{G}_2^m$ and $\delta(G') = 3$. The only possible distribution of degrees of vertices in $G'(5, 8)$ is

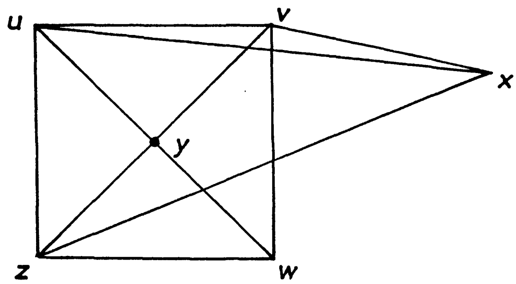


Fig. 11.

$(4, 3, 3, 3, 3)$ and that of $G(6, 11)$ is $(4, 4, 4, 4, 3, 3)$. It follows easily that then G with deleted orientation is the graph in Fig. 11. Denote by $G_0(6, 11)$ and $G_0'(5, 8) = G_0 \setminus \{x\}$ these non-oriented graphs. We shall investigate the possible orientations of the arcs of G_0' and deduce then those of G_0. Without difficulty one finds that the only possible orientations of G_0' are those two in Fig. 12 (any other orientation gives

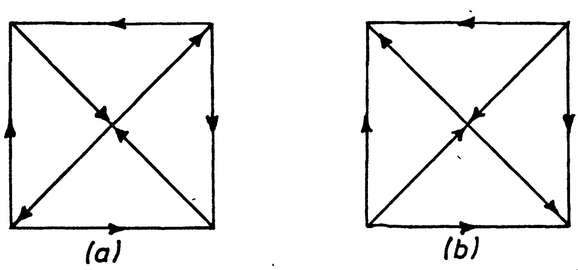


Fig. 12.

more than two directed paths from one point to another). While the graph (a) does not create any possible graph G_0, the graph (b) leads to the two described in the lemma.

To complete the proof of theorem 2.2, we shall use induction with respect to p. For $p = 6$, the theorem is true by lemma 7. Suppose that $p \geqq 6$ and that the theorem is true for all graphs in $\mathscr{G}_2^m$ of order p. Let $G(p + 1, f_2(p + 1)) \in \mathscr{G}_2^m$. By lemma 6, there is a point x of G with the minimum degree $]\frac{1}{2}(p + 1)[$. The graph $G' = G \setminus \{x\}$

has order p and since it belongs to $\mathscr{G}_2^m$ by lemma 6, it is a complete bipartite graph (A_1, B_1, C_1, U_1) oriented from A_1 to B_1, from A_1 to C_1 and from C_1 to B_1, with either

$$|A_1| =]\tfrac{1}{2}(p-1)[\,, \quad |B_1| = [\tfrac{1}{2}(p-1)]\,, \quad |C_1| = 1 \quad \text{(case 1)} \quad \text{or}$$

$$|A_1| = [\tfrac{1}{2}(p-1)]\,, \quad |B_1| =]\tfrac{1}{2}(p-1)[\quad \text{and} \quad |C_1| = 1 \quad \text{(case 2)}\,.$$

Let us notice first that there cannot be any arc to x from a point of A_1: since there must be at least one more arc to or from x from or to C_1 or B_1, one gets four cases

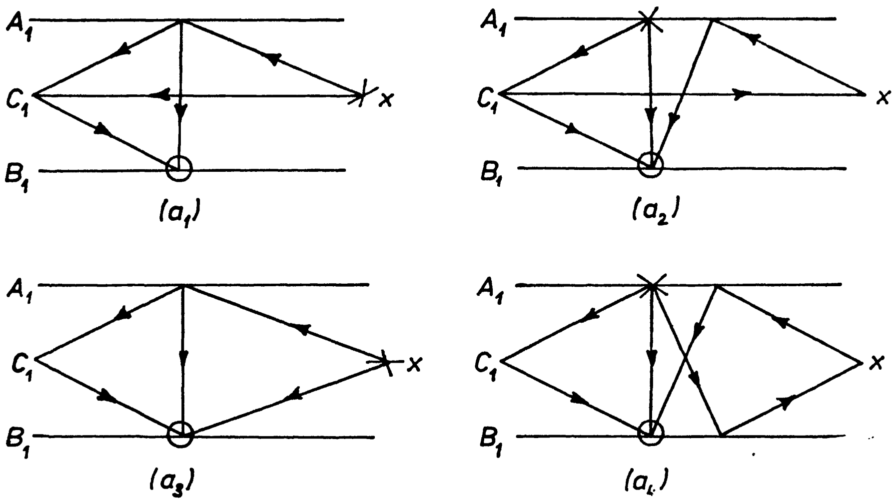


Fig. 13a.

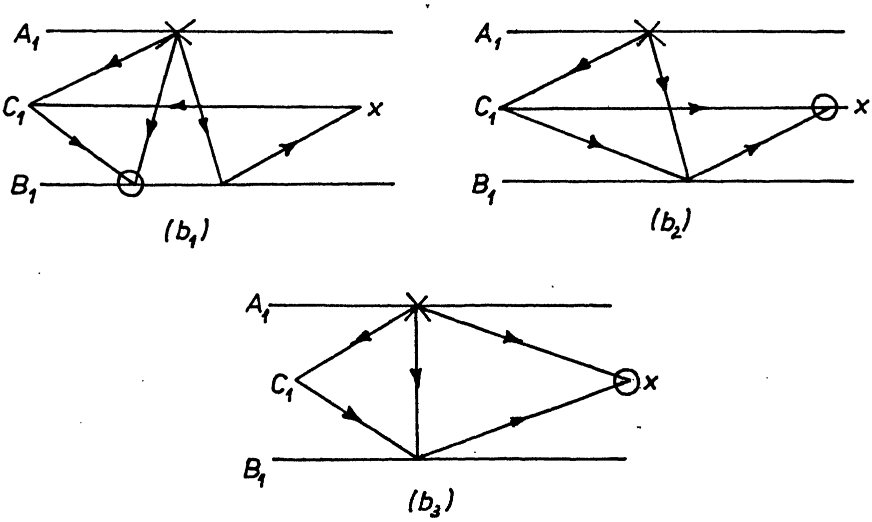


Fig. 13b.

in Fig. 13-a. Each of them leads to a contradiction. Similarly, there cannot be any arc to x from a point in B_1 by Fig. 13-b and 13-a. There is also no pair of arcs from A_1 to x and from x to B_1 (Fig. 14), no pair of arcs from A_1 to x and from x to C_1 (Fig. 15(a)) as well as no pair of arcs from C_1 to x and from x to B_1 (Fig. 15(b)).

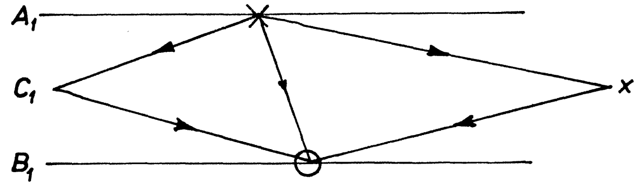


Fig. 14.

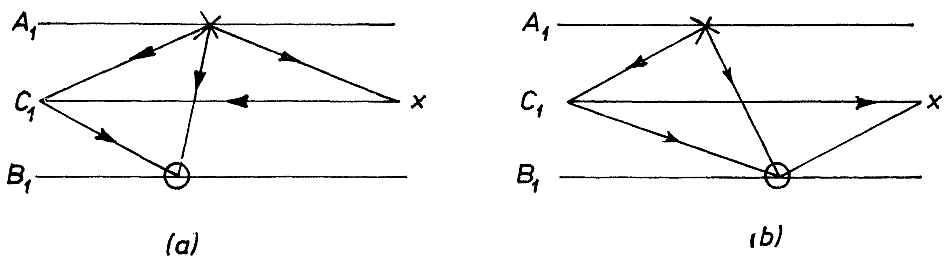


Fig. 15.

It follows that either there are arcs to x from all the points in $A_1 \cup C_1$ in case 2, or there are arcs from x to all points in $C_1 \cup B_1$ in case 1. In each case, one obtains that G is a tripartite graph of order $p + 1$ which satisfies the conditions in the theorem. The rest is obvious.

Theorem 2.3. *Every graph $G(p, f_3(p))$ in $\mathscr{G}_3^m$ with $p \geqq 7$ is a complete tripartite graph (A, B, C, U) with arcs oriented from A to B, from A to C and from C to B, where either $|A| =]\frac{1}{2}(p-2)[$, $|B| = [\frac{1}{2}(p-2)]$, $|C| = 2$, or $|A| = [\frac{1}{2}(p-2)]$, $|B| =]\frac{1}{2}(p-2)[$, $|C| = 2$. Both cases coincide iff p is even.*

Proof. We shall first prove a lemma.

Lemma 8. *For any graph $G(p, f_3(p))$ in $\mathscr{G}_3^m$, $p \geqq 7$, we have $\delta(G) =]\frac{1}{2}(p+1)[$. If y is a point of G such that $d_G(y) =]\frac{1}{2}(p+1)[$ then the graph $G' = G \setminus \{y\}$ belongs again to $\mathscr{G}_3^m$.*

Proof. If x is any point of a graph $G \in \mathscr{G}_3^m$ with $p \geqq 3$ points, then $G' = G \setminus \{x\}$ is in $\mathscr{G}_3$. Hence

$$f_3(p) - d_G(x) \leqq f_3(p-1)$$

which implies

$$d_G(x) \geqq f_3(p) - f_3(p - 1) =]\tfrac{1}{2}(p + 1)[\,.$$

Thus

$$\delta(G) \geqq]\tfrac{1}{2}(p + 1)[\,.$$

By lemma 1,

$$\delta(G) \leqq]\tfrac{1}{2}(p + 1)[$$

so that

$$\delta(G) =]\tfrac{1}{2}(p + 1)[\,.$$

If a point y in G satisfies $d_G(y) =]\tfrac{1}{2}(p + 1)[$, the above calculation shows that

$$f_3(p) - d_G(y) = f_3(p - 1)\,,$$

i.e. that $G' = G \setminus \{y\}$ belongs to $\mathscr{G}_3^m$.

Returning to the proof of theorem 2.3, we shall investigate graphs in $\mathscr{G}_3^m$ with six points. Let $G(6, 12)$ be such a graph. By lemma 1, $\delta(G) \leqq 4$. Let x be a point from $G(6, 12)$ for which $d_G(x) = \delta(G)$ and let $G'(5, q') = G(6, 12) \setminus \{x\}$. Since $G'(5, q') \in \mathscr{G}_3$, $\delta(G') \leqq 3$ by lemma 1. Let y be a point from $G'(5, q')$ of the smallest degree, i.e. $d_{G'}(y) = \delta(G')$. If $G''(4, q'') = G' \setminus \{y\}$ then

$$6 \geqq q'' = q' - \delta(G') = (q - \delta(G)) - \delta(G') = 12 - \delta(G) - \delta(G')$$

so that

$$\delta(G) + \delta(G') \geqq 6\,.$$

But

$$\delta(G) + \delta(G') \leqq 4 + 3 = 7\,.$$

We have thus two cases:

Case A. $\delta(G) = 4$ and $\delta(G') = 3$.

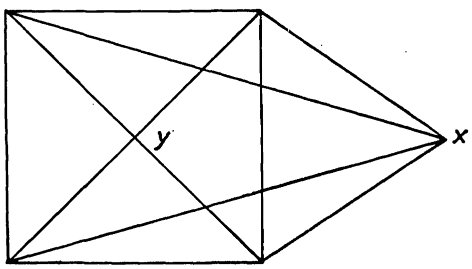


Fig. 16.

Then the only point y in G' which is not joined with x by an arc (of any orientation) has degree four. The distribution of the degrees in G' is thus $(4, 3, 3, 3, 3)$, that in G is $(4, 4, 4, 4, 4, 4)$. The graph G with omitted orientation is given in Fig. 16. It is not difficult to see that the subgraph G' of G cannot contain three directed paths from a point of degree three to another point of degree three. Therefore, the only graphs which can serve as G' are those in Fig. 17. Here, the graph in Fig. 17(c) cannot be completed into G in $\mathcal{G}_3^m$ while the remaining two can be completed in a single way, both resulting in the graph $G_1(6, 12)$, the complete tripartite graph (A, B, C, U) with arcs oriented from A to B, from A to C and from C to B, $|A| = |B| = |C| = 2$.

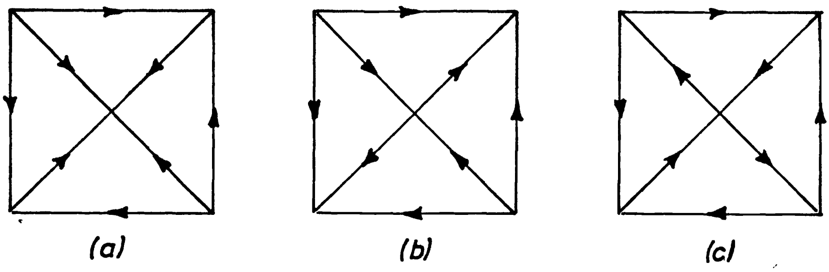


Fig. 17.

Case B. $\delta(G) = \delta(G') = 3$.

The distribution of the degrees in $G'(5,9)$ will then be $(4, 4, 4, 3, 3)$ and G' has as its subgraphs one of the graphs in Fig. 8. Let us denote in each of them (Fig. 18) one or more points as d_i (departure) and one or more points as a_i (arrival) in such a way that there are exactly three directed paths from each d_i to each a_j. Thus, for $i =$

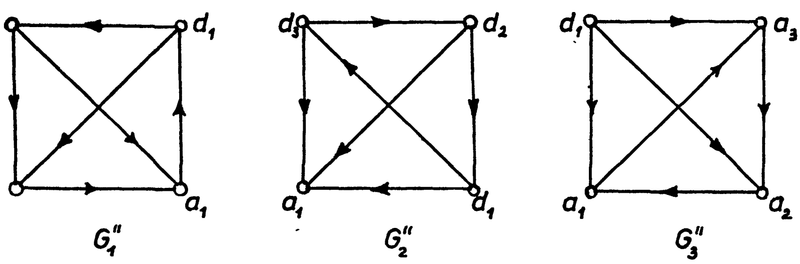


Fig. 18.

$= 1, 2, 3$, $G_i'' = G'(5, 9) \setminus \{y\}$ where y is a point joined with three points of G_i''. Observe that if y is adjacent to or from a point d_k then the only possibility is that y is adjacent from two points different from the a_j's. Similarly, if y is adjacent to or from a point a_k then the only possibility is that y is adjacent to two points different from the d_j's. It is easily seen that no graph $G'(5, 9)$ contains the graph G_1'' as subgraph and that only the graphs G_2', G_3' in Fig. 19 contain G_2'', G_3'', respectively. (Observe that G_2'', G_3'' as well as G_2', G_3' arise from each other by change of orientation of arcs.) An analogue reasoning allows to construct the graphs $G(6, 12)$ using the graphs

$G_i'(5, 9)$ and one obtains the two graphs in Fig. 20. (Observe that the point y is a point of type a_j in G_2' and a point of type d_j in G_3'.)

Let us turn now to graphs in $\mathscr{G}_3^m$ with seven points and prove the theorem holds in this case. Let $G_0(7, 16)$ be a graph in $\mathscr{G}_3^m$. By lemma 8, $\delta(G_0) = 4$ and if for a point x_0, $d_{G_0}(x_0) = 4$ then $G(6, 12) = G_0(7, 16) \setminus \{x_0\}$ is in $\mathscr{G}_3^m$. One can thus use the

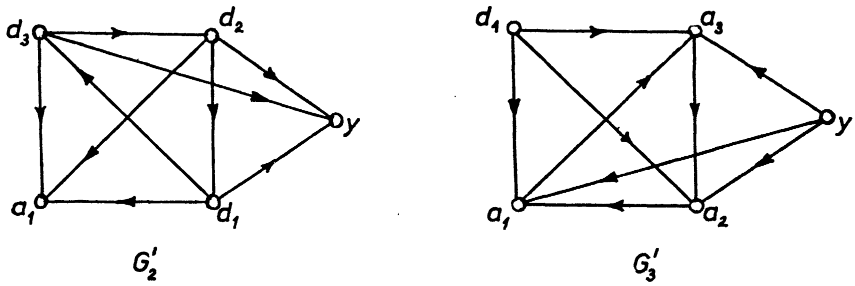


Fig. 19.

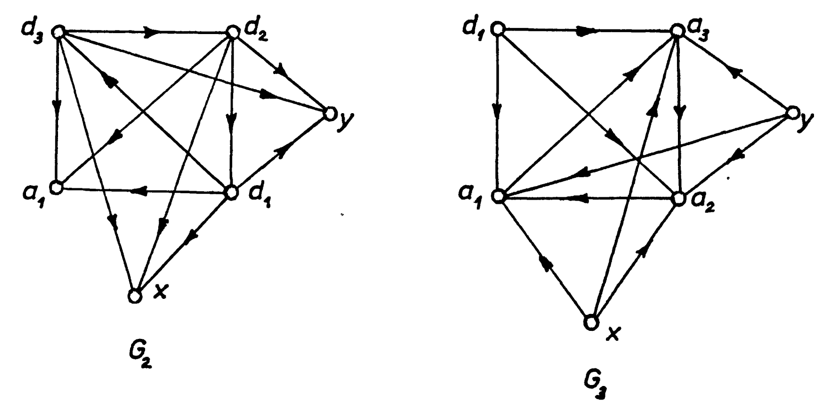


Fig. 20.

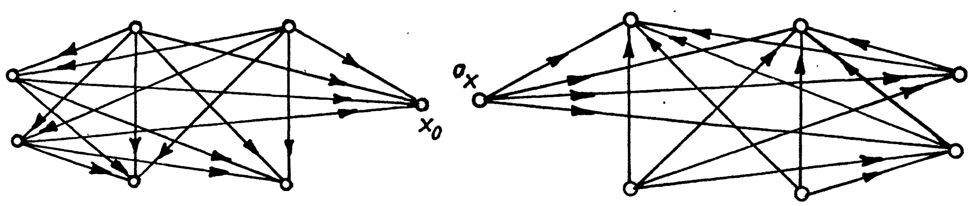


Fig. 21.

described technique again. We obtain the graphs in Fig. 21 from the graph $G_1(6, 12)$. These are both complete tripartite as asserted. (The graphs $G_2(6, 12)$ and $G_3(6, 12)$ do not yield any graph $G_0(7, 16)$ since $d_{G_0}(x_0) = 4$ on one side and on the other side x_0 cannot be adjacent to or from both a point d_i and a_j.)

We shall now complete the proof by induction with respect to p. We proved the theorem is true for $p = 7$. Thus assume a graph $G(p + 1, f_3(p + 1))$ belongs to $\mathscr{G}_3$ while the theorem is true for the graphs in $\mathscr{G}_3^m$ with $p \geqq 7$ points. By lemma 8, there is a point x in G which has the minimum degree $]\frac{1}{2}(p + 2)[$. Moreover, the graph $G' = G \setminus \{x\}$ also belongs to $\mathscr{G}_3^m$ by the same lemma. By the induction hypothesis, G' is a complete tripartite graph (A_1, B_1, C_1, U_1) oriented from A_1 to B_1, from A_1 to C_1 and from C_1 to B_1 and such that $|A_1| =]\frac{1}{2}(p - 2)[$, $|B_1| = [\frac{1}{2}(p - 2)]$, $|C_1| = 2$ (case 1), or, if p is odd, $|A_1| = [\frac{1}{2}(p - 2)]$, $|B_1| =]\frac{1}{2}(p - 2[$, $|C|_1 = 2$ (case 2).

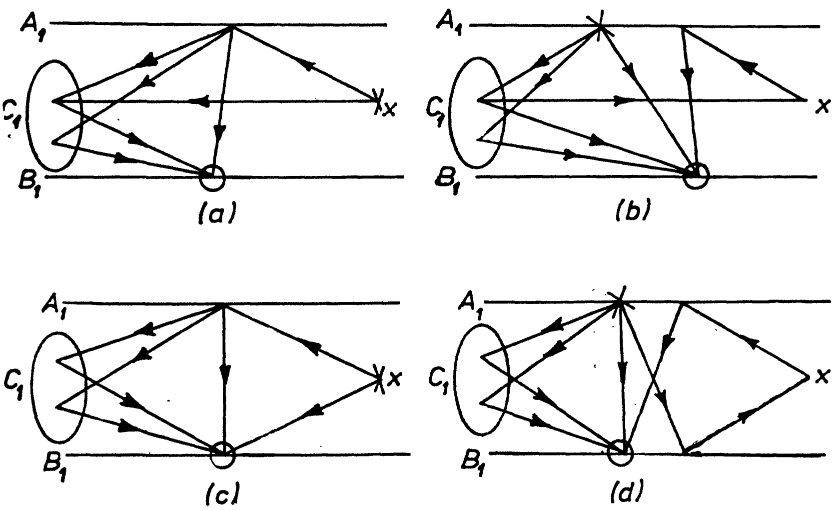


Fig. 22.

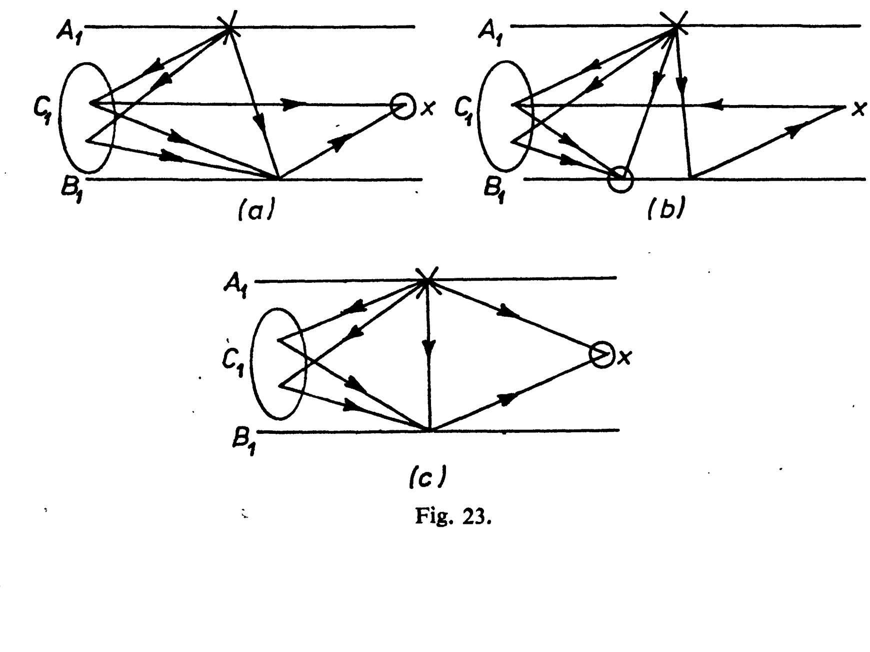


Fig. 23.

Notice that there cannot be an arc from x to a point in A_1 (Fig. 22) since there is always another arc incident with x with the other end-point in C_1 or B_1. Similarly, there cannot be an arc from a point in B_1 to x (Fig. 23 and Fig. 22d). There is also no pair of arcs from a point in $A_1 \cup C_1$ to x and from x to a point in B_1 (Fig. 24, 25) and similarly, no pair of arcs from A_1 to x and from x to $B_1 \cup C_1$ (Fig. 24, 26).

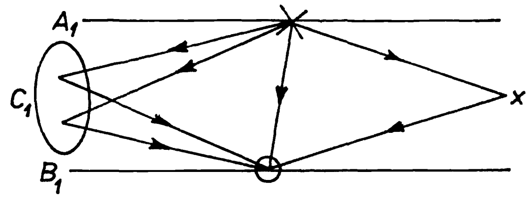


Fig. 24.

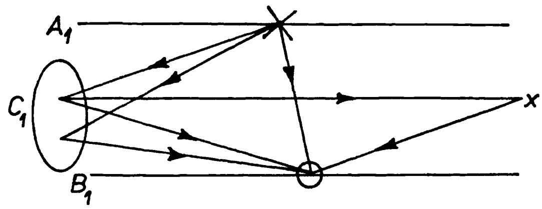


Fig. 25.

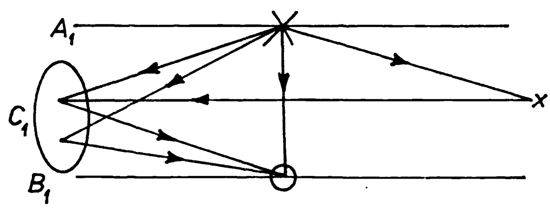


Fig. 26.

Therefore, x is either adjacent from all the points of $A_1 \cup C_1$ in case 1 if p is even and in case 2, or x is adjacent to all the points of $B_1 \cup C_1$ in case 1. In other cases, the degree of x would not be minimal. The graph G is then again a tripartite graph with the properties described in the theorem. The proof is complete.

References

[1] *K. Howalla, A. N. Dabboucy, R. Tout:* On the maximum number of arcs in some classes of graphs. Čas. pěst. mat. *107* (1982), 388–392.

Authors' address: Lebanese University, Faculty of Science, Hadeth, Beirut, Lebanon.