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ON UNIQUENESS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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It is known, see for example [1], that there exists a continuous function $f : R^2 \to R$ such that for every point $(a, b) \in R^2$ and for every $\varepsilon > 0$ the initial-value problem

$$(1) \qquad y' = f(x, y)\,, \quad y(a) = b\,,$$

has more than one solution in the interval $\langle a, a + \varepsilon)$ as well as in the interval $(a - \varepsilon, a\rangle$.

Let $D \subset R^2$, let $f : D \to R$ be a function continuous in D. We shall say that the differential equation

$$(2) \qquad y' = f(x, y)$$

has the property of uniqueness in the forward (backward) direction in D, if for every $(a, b) \in D$ and for every $\varepsilon > 0$ the initial-value problem (1) has at most one solution in the interval $\langle a, a + \varepsilon)$ (resp. in the interval $(a - \varepsilon, a\rangle)$.

It is well-known, see for example [1], that if the function f of the variable y is for each x nonincreasing (nondecreasing) in D, then Equation (2) has the property of uniqueness in the forward (backward) direction. Therefore, the following question is natural: to what degree the property of uniqueness in the forward direction can be violated in case of a differential equation with the property of backward uniqueness?

Theorem 1. *Let $D \subset R^2$, let $f : D \to R$ be a continuous function in D. Let Equation (2) have the property of uniqueness in the backward direction. For any $a \in R$ let A be the set of all $b \in R$ such that $(a, b) \in D$ and, for some $\varepsilon > 0$, the initial-value problem (1) has more than one solution in the interval $\langle a, a + \varepsilon)$. Then A is at most countable.*

Proof. Let $\varepsilon > 0$. Denote by $A(\varepsilon)$ the set of all $b \in R$ such that $(a, b) \in D$ and the initial-value problem (1) has more than one solution in the interval $\langle a, a + \varepsilon)$. Therefore, if $b \in A(\varepsilon)$, then there are two solutions y_1, y_2 of (1) defined in an interval which contains $\langle a, a + \varepsilon)$ and such that $y_1(a + \varepsilon) < y_2(a + \varepsilon)$. Denote $I(b) = \langle y_1(a + \varepsilon), y_2(a + \varepsilon)\rangle$. To every $b \in A(\varepsilon)$ we have thus assigned a nondegenerate interval $I(b)$. (This interval depends on the functions y_1, y_2, and consequently, it is not uniquely determined. The only important point is that it is nondegenerate.) Obviously $I(b_1) \cap I(b_2) = \emptyset$ if $b_1 \neq b_2$, for we suppose that the equation has the property of backward uniqueness. As any set of nondegenerate disjoint intervals is at most countable, the set $A(\varepsilon)$ is also at most countable.

Let $\{r_n\}$ be a sequence containing just all the positive rational numbers. Now, it suffices to take into account that $A = \bigcup_{n=1}^{+\infty} A(r_n)$ and that the countable union of at most countable sets is, again, at most countable. The theorem is proved.

The previous theorem shows the worst possibility of nonuniqueness in the forward direction in case Equation (2) has the property of backward uniqueness. In the following theorem we shall see that this "worst" case may, as a matter of fact, occur. The symbols $C\langle a, b\rangle$ and $C^1\langle a, b\rangle$ will have the usual meaning of the sets of all continuous and continuously differentiable functions on the interval $\langle a, b\rangle$, respectively.

Theorem 2. *Let $a < b$, let $f_0, f_1 \in C^1\langle a, b\rangle$ be any two functions such that $f_0(a) < f_1(a)$ and $f_1 - f_0$ is increasing on the interval $\langle a, b\rangle$. Let*

$$G = \{(x, y) : a \leqq x \leqq b,\ f_0(x) \leqq y \leqq f_1(x)\}\ .$$

Then there is a continuous function $\varphi : G \to R$ such that:

1. *For each $x \in \langle a, b\rangle$, φ is a nondecreasing function of the variable y on the interval $\langle f_0(x), f_1(x)\rangle$;*

2. *For each $c \in \langle a, b)$, there is a countable set $H(c)$, dense in the interval $\langle f_0(c), f_1(c)\rangle$ and such that for each $d \in H(c)$, $\varepsilon \in (0, b - c\rangle$, the initial-value problem*

$$y' = \varphi(x, y)\,, \quad y(c) = d \tag{3}$$

has more than one solution in the interval $\langle c, c + \varepsilon\rangle$.

Note. The first assertion of the theorem implies that the differential equation $y' = \varphi(x, y)$ has the property of backward uniqueness in D.

Before proving the theorem we shall state several lemmas and introduce some notations.

For $f : \langle a, b\rangle \to R$, $g : \langle a, b\rangle \to R$ we shall write $f \prec g$ if $f(x) \leqq g(x)$ for each $x \in \langle a, b\rangle$. In the space $C^1\langle a, b\rangle$ we introduce the norm, for example, by the rule

$$\|f\| = \max_{a \leqq x \leqq b} (|f(x)| + |f'(x)|)$$

for $f \in C^1\langle a, b\rangle$.

Lemma 1. Let $f, g \in C^1\langle a, b\rangle$, $f \prec g$. *Define* $h : \langle a, b\rangle \to R$ *by* $h(x) = \frac{1}{2}(f(x) + g(x))$, $x \in \langle a, b\rangle$. *Then* $h \in C^1\langle a, b\rangle$, $f \prec h \prec g$, $\|h - f\| = \|g - h\| = \frac{1}{2}\|g - f\|$. *In addition, if* $g - f$ *is increasing on an interval* $I \subset \langle a, b\rangle$, *then both the functions* $h - f$, $g - h$ *are increasing on* I *as well.*

Proof. Obvious.

Lemma 2. *Let* $f \in C^1\langle u, v\rangle$ *be increasing on* $\langle u, v\rangle$, $u < v$. *Then there is a function* $g \in C^1\langle u, v\rangle$ *such that*:

1. g *is increasing on* $\langle u, v\rangle$, $g(u) = g'(u) = 0$;
2. $f - g$ *is also increasing on* $\langle u, v\rangle$.

Proof. Choose a number $q \in (0, 1)$ and define a function $h : \langle u, v\rangle \to R$ by

$$h(x) = \min(q f'(x), x - u), \quad x \in \langle u, v\rangle .$$

Clearly $h \in C\langle u, v\rangle$, $h(u) = 0$, $0 \leqq h(x) \leqq q f'(x)$ for $x \in \langle u, v\rangle$. As f is increasing on $\langle u, v\rangle$, there is no nondegenerate interval I such that h is identically equal to zero on I. The function g defined by

$$g(x) = \int_u^x h(t)\, dt , \quad x \in \langle u, v\rangle$$

is, therefore, increasing on $\langle u, v\rangle$. In addition, $g \in C^1\langle u, v\rangle$, $g(u) = g'(u) = 0$. For any two real numbers x, y, $u \leqq x < y \leqq v$, we have $g(y) - g(x) = \int_x^y h(t)\, dt \leqq q \int_x^y f'(t)\, dt = q(f(y) - f(x)) < f(y) - f(x)$ so that $f(y) - g(y) > f(x) - g(x)$. The function $f - g$ is, therefore, increasing on $\langle u, v\rangle$ and the lemma is proved.

Lemma 3. *Let* $c \in \langle a, b\rangle$, $f, g \in C^1\langle a, b\rangle$, $f \prec g$ *and let* $g - f$ *be increasing on* $\langle c, b\rangle$. *Then there is a function* $h \in C^1\langle a, b\rangle$, $f \prec h \prec g$, *such that*: 1. $h(x) = f(x)$ *for each* $x \in \langle a, c\rangle$; 2. *Both the functions* $h - f$, $g - h$ *are increasing on* $\langle c, b\rangle$.

Proof. The function p, $p(x) = g(x) - f(x)$, $x \in \langle c, b\rangle$ is, by the hypothesis of the lemma, increasing on $\langle c, b\rangle$ and $p \in C^1\langle c, b\rangle$. According to Lemma 2, there is a function $q \in C^1\langle c, b\rangle$, increasing on $\langle c, b\rangle$, $q(c) = q'(c) = 0$ and such that $p - q$ is also increasing on $\langle c, b\rangle$. Define $h : \langle a, b\rangle \to R$ by $h(x) = f(x)$ for $x \in \langle a, c\rangle$, $h(x) = f(x) + q(x)$ for $x \in \langle c, b\rangle$. Because of $q(c) = q'(c) = 0$ we have $h \in C^1\langle a, b\rangle$. Since $h(x) - f(x) = q(x)$, $x \in \langle c, b\rangle$, $h - f$ is, consequently, increasing on $\langle c, b\rangle$. Since $g(x) - h(x) = g(x) - f(x) - q(x) = p(x) - q(x)$, $x \in \langle c, b\rangle$, then the fact that $p - q$ is increasing on $\langle c, b\rangle$ implies that the function $g - h$ is increasing on $\langle c, b\rangle$, and the lemma is proved.

Note. In the sense of the above given construction, the function h is said to be obtained by splitting f at point c in the direction to g. In addition to f, we no whave another function h, which coincides with f on the interval $\langle a, c\rangle$ and $f \prec h \prec g$.

Let f_0, f_1 be two functions fulfilling the assumptions of Theorem 2. Denote

$$F = \{f : f \in C^1\langle a, b\rangle,\ f_0(x) \leqq f(x) \leqq f_1(x),\ x \in \langle a, b\rangle\}\ .$$

Let A be a subset of F. We shall say that A has the property V, if the following three conditions hold:

1. A is finite.

2. For any two functions $f, g \in A$, either $f \prec g$ or $f \succ g$. (A is, therefore, linearly ordered with respect to the relation $\prec$.)

3. If $f, g \in A, f \prec g$, then $g - f$ is nondecreasing on $\langle a, b\rangle$. If, in addition, $f(c) < g(c)$ for some $c \in \langle a, b\rangle$, then $g - f$ is increasing on $\langle c, b\rangle$.

Suppose $A \subset F$ has the property V. Let $f, g \in A, f \neq g, f \prec g$. We shall say that f, g are adjacent in A if, for each $h \in A$ such that $f \prec h \prec g$, either $h = f$ or $h = g$.

Lemma 4. *For each $n \in N$ there is a set $S_n \subset F$ with the property V, such that the following five conditions hold:*

1. $f_0 \in S_n$, $f_1 \in S_n$;

2. *if $f, g \in S_n$ are any two functions adjacent in S_n, then*

$$\|g - f\| \leqq 2^{-n}C \tag{4}$$

where $C = 2\|f_1 - f_0\|$;

3. *if $f \in S_n$, $f \neq f_1$, then $f(x) < f_1(x)$ for each $x \in \langle a, b\rangle$;*

4. $S_n \subset S_{n+1}$;

5. *for each $f \in S_n$, $f \neq f_1$, and for each interval $I \subset \langle a, b\rangle$ of length $1/n$, there is a number $c \in I$ and a function $g \in S_{n+1}$ such that $g(x) = f(x)$ for $x \in \langle a, c\rangle$, $g(x) > f(x)$ for $x \in (c, b\rangle$.*

Note. If the set S_{n+1} fulfils the conditions 4 and 5 we shall say that S_{n+1} has the property V_n with respect to S_n.

Proof. We shall construct the sequence $\{S_n\}$ inductively.

1. Assume S_1 contains just the two functions f_0, f_1. Clearly S_1 fulfils the first three conditions of the lemma.

2. Suppose $n \geqq 1$, let S_n be already defined fulfilling the first three conditions of the lemma. First, we shall construct an auxiliary set $R_n \subset F$ with the property V which should have the property V_n with respect to S_n. In the beginning, choose $r \in N$ and real numbers $x_0 < x_1 < \ldots < x_r$ such that $x_0 = a$, $x_r = b$, $x_i - x_{i-1} \leqq 1/n$ for $i = 1, 2, \ldots, r$. Obviously the set of all points $(x_i, f(x_i))$ where $f \in S_n$, $f \neq f_1$ and $i = 0, 1, \ldots, r - 1$, is finite. We denote these points $A_1, A_2, \ldots, A_s$ (the order is of no importance). Denote $T_1 = S_n$. Considering, at first, the point $A_1 = (u, v)$, there is a function $f \in T_1$ such that $f(u) = v$ and, if $h \in T_1$ is any other function which fulfils $h(u) = v$, then $h \prec f$. Furthermore, there is a function $g \in T_1$, $g \succ f$, which is

adjacent to f in T_1. Such a function must exist because of $f(u) < f_1(u)$. (As a matter of fact, $f(x) < f_1(x)$ for all $x \in \langle a, b \rangle$.) Now, we split f at u in the direction to g, in the sense of Lemma 3, thus obtaining a function $h \in C^1\langle a, b \rangle$, $f \prec h \prec g$, which coincides with f just on $\langle a, u \rangle$. Joining h to T_1 we obtain a set T_2. Lemma 3 implies that, again, T_2 has the property V. Now, considering A_2 we obtain a new function by splitting an appropriate function at the x-coordinate of A_2 (by the same method as in the previous case, considering, of course, the set T_2 instead of T_1). Joining this new function to T_2 we get a set T_3 which also has the property V. In this way, we successively consider all the points A_i, $i = 1, 2, \ldots, s$, thus obtaining the set T_{s+1} which, again, has the property V. At last we denote $R_n = T_{s+1}$. It is easy to see that R_n has the property V_n with respect to S_n. In addition, R_n obviously fulfils the first three conditions of our lemma (of course, after replacing in them S_n by R_n).

3. Now, for each two functions $f, g \in R_n$ which are adjacent in R_n and $f \prec g$ we construct a function h by the rule $h(x) = \frac{1}{2}(f(x) + g(x))$, $x \in \langle a, b \rangle$. By the induction hypothesis, Inequality (4) holds for f, g. Hence, Lemma 1 implies $h \in C^1\langle a, b \rangle$, $f \prec h \prec g$ and $\|h - f\| = \|g - h\| \leqq C\,2^{-n-1}$. Joining the set of all functions obtained in this way to the set R_n we get the set S_{n+1}. Obviously S_{n+1} has the property V. Moreover, S_{n+1} fulfils the first three conditions of the lemma. Since $S_{n+1} \supset R_n$ and R_n has the property V_n with respect to S_n, then S_{n+1} also has the property V_n with respect to S_n. The lemma is proved.

In the following, for a given function $f : D \to R$ the symbol $\langle f \rangle$ should denote the graph of f, that is, the set of all points $(x, f(x))$ where $x \in D$.

Let $S_1, S_2, \ldots$ be sets which fulfil the hypotheses of Lemma 4. Denote

$$\langle S_n \rangle = \bigcup_{f \in S_n} \langle f \rangle\,, \quad S = \bigcup_{i=1}^{+\infty} S_i\,, \quad \langle S \rangle = \bigcup_{i=1}^{+\infty} \langle S_i \rangle\,.$$

Now, we define a function $\varphi : \langle S \rangle \to R$ by the following rule: if $(x, y) \in S$, then there is $n \in N$ and a function $f \in S_n$ such that $y = f(x)$. So we define

$$\varphi(x, y) = f'(x)\,.$$

Because of the properties of the set S_n and in view of the inclusion $S_n \subset S_{n+1}$, $n \in N$, the value $\varphi(x, y)$ depends neither on n nor on f. The function φ is, therefore, uniquely determined.

Consider now the properties of φ. For each $x \in \langle a, b \rangle$ let

$$I_x = \{(x, y) : f_0(x) \leqq y \leqq f_1(x)\}\,,$$
$$P_x = \langle S \rangle \cap I_x.$$

Lemma 4 (the second property, Relation (4)) implies that P_x is dense in I_x so that $\langle S \rangle$ is dense in G. Furthermore, P_x is countable, since S_n is finite for each $n \in N$.

Lemma 5. *For each $x \in \langle a, b \rangle$, φ is a nondecreasing function of the variable y on $\langle S \rangle$, that is, on P_x.*

Proof. Let $(x, c) \in P_x$, $(x, d) \in P_x$, $c < d$. Then there are $n \in N$ and two functions $f, g \in S_n$ such that $f(x) = c$, $g(x) = d$. Since S_n has the property V, we have $f \prec g$ so that $g - f$ is nondecreasing on $\langle a, b \rangle$. Hence $g'(x) \geqq f'(x)$, that is, $\varphi(x, d) \geqq \varphi(x, c)$, which proves the lemma.

Lemma 6. *For each $x \in \langle a, b \rangle$, φ is uniformly continuous on P_x.*

Proof. Let $\varepsilon > 0$. Then there is $n \in N$ such that $C\, 2^{-n} < \frac{1}{2}\varepsilon$, where $C = 2\|f_1 - f_0\|$. The set S_n contains at most a finite number of functions, hence $\langle S_n \rangle$ intersects I_x in a finite number of points (belonging, therefore, to P_x) (x, y_i), $i = 0, 1, \ldots, r$, $y_0 < y_1 < \ldots < y_r$ where $y_0 = f_0(x)$, $y_r = f_1(x)$. Because of Inequality (4) we have $y_i - y_{i-1} < \frac{1}{2}\varepsilon$, $\varphi(x, y_i) - \varphi(x, y_{i-1}) < \frac{1}{2}\varepsilon$ for $i = 1, 2, \ldots, r$. Let $\delta = \min_{1 \leqq i \leqq r} (y_i - y_{i-1})$. Since φ is nondecreasing in the variable y, then for (x, u), $(x, v) \in P_x$, $0 < v - u < \delta$, the inequality $\varphi(x, v) - \varphi(x, u) < \varepsilon$ must hold. Since $\varepsilon > 0$ was arbitrary, the lemma is proved.

Since the function φ is uniformly continuous on P_x and P_x is dense in I_x, the function φ can be uniquely continuously extended onto the whole I_x. This extension we denote, again, by φ. Since x is an arbitrary point from $\langle a, b \rangle$, the function φ is, therefore, extended onto the whole set G.

Lemma 7. *The function φ is continuous on G.*

Proof. On the one hand, φ is continuous and nondecreasing (in the variable y) on I_x for each $x \in \langle a, b \rangle$. On the other hand, φ is continuous on each curve $\langle f \rangle$ where $f \in S$. We shall only show that φ is continuous at each interior point $(c, d) \in G$. The continuity at the boundary points of G can be proved by an easy modification. So let $a < c < b$, $f_0(c) < d < f_1(c)$, let $\varepsilon > 0$. The continuity of φ on I_c implies the existence of a number $\Delta > 0$ such that, if $y \in (d - \Delta, d + \Delta)$, then

$$\varphi(c, d) - \varepsilon < \varphi(c, y) < \varphi(c, d) + \varepsilon . \tag{5}$$

Since P_c is dense in I_c, there are functions $f, g \in S$ such that $d - \Delta < f(c) < d < g(c) < d + \Delta$. As f, g are continuous and, moreover, φ is continuous on $\langle f \rangle$, $\langle g \rangle$, there is a number $\delta > 0$ such that, if $x \in (c - \delta, c + \delta)$, then $f(x) < g(x)$ and

$$\varphi(x, f(x)) > \varphi(c, f(c)) - \varepsilon ; \quad \varphi(x, g(x)) < \varphi(c, g(c)) + \varepsilon .$$

Now, using Relation (5) we obtain

$$\varphi(x, f(x) > \varphi(c, d) - 2\varepsilon ; \quad \varphi(x, g(x)) < \varphi(c, d) + 2\varepsilon .$$

Since φ is nondecreasing in the variable y, the inequality

$$\varphi(c, d) - 2\varepsilon < \varphi(x, y) < \varphi(c, d) + 2\varepsilon$$

must be true in $U = \{(x, y) : c - \delta < x < c + \delta, f(x) < y < g(x)\}$. Because $\varepsilon > 0$ was arbitrary, the last relation implies that φ is continuous at the point (c, d). The lemma is proved.

Proof of Theorem 2. We shall show that the just constructed function φ fulfils all the conditions of our theorem. First, by Lemma 7, φ is continuous on G. Furthermore, the first condition of the theorem follows from Lemma 5. It remains to prove the second condition. Let $c \in \langle a, b)$, $H(c) = \{y : (c, y) \in P_c, y < f_1(c)\}$. The set P_c is dense in I_c, so that $H(c)$ is dense in $\langle f_0(c), f_1(c)\rangle$. Moreover, $H(c)$ is at most countable. Let $d \in H(c)$, $\varepsilon \in (0, b - c\rangle$ be arbitrary. Because of $d \in H(c)$ we have $(c, d) \in P_c$ and, therefore, there are $n \in N$ and a function $f \in S_n$ such that $d = f(c)$. From $d < f_1(c)$ it follows that $f \neq f_1$. The function f is, of course, a solution of the initial-value problem (3). We shall show that there exists another solution of (3) which differs from f at least at one point of the interval $\langle c, c + \varepsilon)$. As $S_n \subset S_{n+1}$, $n \in N$, we may assume n to be so large that $n^{-1} < \varepsilon$. Now, by the fifth property of Lemma 4, there is a number $u \in \langle c, c + \varepsilon)$ and a function $g \in S_{n+1}$ such that $g(x) = f(x)$ for $x \in \langle a, u\rangle$, $g(x) > f(x)$ for $x \in (u, b\rangle$. The function g is, therefore, another solution of the initial-value problem (3) which, moreover, differs from f in the interval $(u, b\rangle$. We see that the set $H(c)$ fulfils all the requirements. The theorem is proved.

By an easy modification of the proof given above the following theorem can be proved.

Theorem 3. *There is a continuous function $f : R^2 \to R$ such that*

1. *the differential equation $y' = f(x, y)$ has the property of backward uniqueness;*
2. *for any $a \in R$ there is a countable set $H(a)$ dense in R and such that for each $\varepsilon > 0$, $b \in H(a)$, the initial-value problem $y' = f(x, y)$, $y(a) = b$ has more than one solution on the interval $\langle a, a + \varepsilon)$.*

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