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Časopis pro pěstování matematiky, roč. 107 (1982), Praha

NOTE ON THE OSCILLATION OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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In this paper we are concerned with the oscillatory behavior of solutions of the nonlinear differential equation with deviating argument

$$y^{(n)}(t) + p(t) f(y(g(t))) = 0 , \quad n \geqq 2 , \tag{1}$$

and of the linear differential equation with delayed argument

$$y^{(n)}(t) + p(t)\, y(g(t)) = 0 , \quad n \geqq 3 . \tag{2}$$

A solution $y(t)$ of the equation (1) or (2) is called oscillatory if it has arbitrarily large zeros, and it is called nonoscillatory otherwise.

Lemma 1 (*Kiguradze*). *Let $y(t)$ be a solution of equation* (1) *or* (2) *satisfying the condition*

$$y(t) > 0 \quad \text{for} \quad t \in [t_0, \infty) ,$$

and let

$$y^{(n)}(t) \leqq 0 \quad \text{for} \quad t \in [t_0, \infty) .$$

Then there exist $t_1 \in [t_0, \infty)$ and an integer $l \in \{0, 1, \ldots, n\}$ such that $l + n$ is odd and

$$y^{(i)}(t) > 0 \quad \text{for} \quad t \in [t_1, \infty) \quad (i = 0, \ldots, l - 1) , \tag{3}$$

$$(-1)^{i+l}\, y^{(i)}(t) > 0 \quad \text{for} \quad t \in [t_1, \infty) \quad (i = l, \ldots, n - 1) .$$

An analogous statement can be made if $y(t) < 0$ and $y^{(n)}(t) \geqq 0$ for $t \in [t_0, \infty)$.

EQUATION (1)

We consider equation (1) where

a) $p(t)$ is continuous and nonnegative on $[t_0, \infty)$;

b) $g(t)$ is a continuous and nondecreasing function on $[t_0, \infty)$ such that $\lim_{t\to\infty} g(t) = \infty$;

c) $f(u)$ is a continuous function on $R = (-\infty, \infty)$ such that $u f(u) > 0$ for $u \neq 0$.

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some interval $[T_y, \infty)$ and satisfy

$$\sup \{|y(t)| : t_0 \leqq t < \infty\} > 0 \quad \text{for any} \quad t_0 \in [T_y, \infty) .$$

We introduce the notation:

$$g_0(t) = \min \{g(t), t\} ,$$

$$M_f = \max \left\{ \limsup_{y\to\infty} \frac{y}{f(y)} , \ \limsup_{y\to-\infty} \frac{y}{f(y)} \right\} .$$

The next lemma characterizes the oscillatory behavior of bounded solutions.

Lemma 2. *Suppose that the conditions* a)–c) *are satisfied and in addition,*

$$\int^{\infty} t^{n-1} p(t) \, dt = \infty . \tag{4}$$

Then every bounded of equation (1) *is oscillatory, if n is even, and every bounded solution of equation* (1) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$, *if n is odd.*

Proof. Let $y(t)$ be a bounded and positive solution of equation (1) on $[t_0, \infty)$ and let $y(g(t)) > 0$ for $t \geqq t_1 \geqq t_0$. From the equality

$$y^{(j)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(i)}(s) + \frac{(-1)^{n-j}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} y^{(n)}(u) \, du ,$$

$s \geqq t \geqq t_1$, with regard to equation (1) we get

$$y^{(j)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(i)}(s) + \\ + \frac{(-1)^{n-j+1}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} p(u) f(y(g(u))) \, du . \tag{5}$$

Let n be even. Because $y(t)$ is positive and bounded solution of equation (1), in view of Lemma 1 we have $l = 1$ and for $j = 1$, (5) implies

$$y'(t) \geqq \frac{1}{(n-2)!} \int_t^{\infty} (u-t)^{n-2} p(u) f(y(g(u))) \, du .$$

Integrating the last inequality from T to t, $t > T \geqq t_1$, we obtain

$$y(t) \geqq \frac{1}{(n-1)!} \int_T^t (u - T)^{n-1} p(u) f(y(g(u)))\, du .$$

Since $y(t)$ is nondecreasing and bounded we have $\frac{1}{2}c \leqq y(g(t)) < c$ for $t \geqq t_2 \geqq T$, where c is a suitable positive constant. Hence there exist positive constants c_1, c_2 such that $c_1 \leqq f(y(g(t))) \leqq c_2$, $t \geqq t_2$, since the interval $[\frac{1}{2}c, c)$ is bounded. As $t \to \infty$ we have

$$c > \frac{c_1}{(n-1)!} \int_{t_2}^{\infty} (u - T)^{n-1} p(u)\, du ,$$

which contradicts (4).

Let n be odd. In view of the fact that $y(t)$ is bounded, $l = 0$ and from the equality (5) for $j = 0$ we get

$$y(T) - y(t) \geqq \frac{1}{(n-1)!} \int_T^t (u - T)^{n-1} p(u) f(y(g(u)))\, du , \quad t \geqq T \geqq t_1 .$$

Since $y(t)$ is a nonincreasing solution of (1) we have $y(t) \to L > 0$ or $y(t) \to 0$ as $t \to \infty$. Let $y(t) \to L > 0$, then $L < y(t) \leqq 2L$ for $t \geqq t_2 \geqq T$. Then there exist positive constants L_1, L_2 such that $L_1 \leqq f(y(g(t))) \leqq L_2$, $t \geqq t_2$. As $t \to \infty$ we get

$$y(t) > y(T) - L > \frac{L_1}{(n-1)!} \int_{t_2}^{\infty} (u - T)^{n-1} p(u)\, du ,$$

which contradicts (4), so $\lim_{t \to \infty} y(t) = 0$. The proof of Lemma 2 is complete.

Theorem 1. *Suppose that the conditions* a)–c) *are satisfied,* $M_f < \infty$ *and in addition,*

$$\limsup_{t \to \infty} [g_0(t)]^{n-1} \int_t^{\infty} p(s)\, ds > M_f (n-1)! . \tag{6}$$

Then every solution of equation (1) *is oscillatory, if n is even, and every solution of equation* (1) *is oscillatory or* $\lim_{t \to \infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$, *if n is odd.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we may suppose that $y(t)$ is eventually positive on $[t_0, \infty)$. Let $y(g(t)) > 0$ for $t \geqq t_1 \geqq t_0$.

Suppose that n is even and $l = 1$. From (5) with regard to Lemma 1 for $j = 1$ we obtain

$$y'(t) \geqq \frac{1}{(n-2)!} \int_t^{\infty} (u - t)^{n-2} p(u) f(y(g(u)))\, du , \quad t \geqq t_1 .$$

Integration of the last inequality from T to t, $t > T \geqq t_1$, yields

(7)

$$y(t) \geqq \frac{1}{(n-1)!} \int_T^t (u - T)^{n-1} p(u) f(y(g(u))) \, du + \frac{(t - T)^{n-1}}{(n-1)!} \int_t^\infty p(u) f(y(g(u))) \, du ,$$

which implies

$$y(g_0(t)) \geqq \frac{[g_0(t) - T]^{n-1}}{(n-1)!} \int_{g_0(t)}^\infty p(u) f(y(g(u))) \, du$$

for $t \geqq t_2 \geqq T$, where t_2 is sufficiently large. From the last inequality we have

$$(8) \qquad y(g(t)) \geqq \frac{[g_0(t) - T]^{n-1}}{(n-1)!} \int_t^\infty p(u) f(y(g(u))) \, du , \quad t \geqq t_2 .$$

Notice that the condition (6) implies (4). Otherwise if

$$\int^\infty t^{n-1} p(t) \, dt < \infty ,$$

then

$$0 < \limsup_{t \to \infty} \, [g_0(t)]^{n-1} \int_t^\infty p(s) \, ds \leqq \limsup_{t \to \infty} \int_t^\infty s^{n-1} p(s) \, ds = 0 ,$$

which is a contradiction.

If $y(t)$ increases to a finite limit as $t \to \infty$, then similarly as in the proof of Lemma 2 we get a contradiction with (4).

Let $y(t)$ increase to infinity as $t \to \infty$. From (8) we get

$$y(g(t)) \geqq y(g(t)) \frac{[g_0(t) - T]^{n-1}}{(n-1)!} \int_t^\infty p(u) \frac{f(y(g(u)))}{y(g(u))} \, du ,$$

$$1 \geqq \inf_{u \geqq t} \frac{f(y(g(u)))}{y(g(u))} \frac{[g_0(t) - T]^{n-1}}{(n-1)!} \int_t^\infty p(u) \, du ,$$

$$\sup_{z \geqq y(g(t))} \frac{z}{f(z)} \geqq \frac{[g_0(t) - T]^{n-1}}{(n-1)!} \int_t^\infty p(u) \, du ,$$

$$(n-1)! \limsup_{z \to \infty} \frac{u}{f(z)} \geqq \limsup_{t \to \infty} \, [g_0(t) - T]^{n-1} \int_t^\infty p(u) \, du ,$$

which contradicts the condition (6).

Let n be odd and $l = 0$. In view of Lemma 1, from (5) for $j = 0$, $t > T \geqq t_1$, we have

$$y(T) - y(t) \geqq \frac{1}{(n-1)!} \int_T^t (u - T)^{n-1} p(u) f(y(g(u))) \, du .$$

Since $y'(t) \leqq 0$ for $t > T$, $y(t)$ decreases to a limit $L \geqq 0$ as $t \to \infty$. Let $L > 0$. Then similarly as in the proof of Lemma 2 we get a contradiction with (4), so $\lim_{t \to \infty} y(t) = 0$.

Let $l \in \{2, \ldots, n - 1\}$. With regard to Lemma 1, from (5) for $j = l$, $t > T \geqq t_1$, we have

$$y^{(l)}(t) \geqq \frac{1}{(n - l - 1)!} \int_t^\infty (u - t)^{n-l-1} p(u) f(y(g(u))) \, du \, .$$

Integrating the inequality from T to t, we obtain

$$y^{(l-1)}(t) \geqq \frac{(t - T)^{n-l}}{(n - l)!} \int_t^\infty p(u) f(y(g(u))) \, du \, .$$

Repeating this proceoure we get

$$y'(t) \geqq \frac{(t - T)^{n-2}}{(n - 2)!} \int_t^\infty p(u) f(y(g(u))) \, du \, ,$$

which leads to the inequality

$$y(t) \geqq \int_T^t \frac{(u - T)^{n-1}}{(n - 1)!} p(u) f(y(g(u))) \, du + \frac{(t - T)^{n-1}}{(n - 1)!} \int_t^\infty p(u) f(y(g(u))) \, du \, ,$$

which is the inequality (7). The proof now proceeds as above, when $y(t)$ increases to infinity. This completes the proof.

Example [5]. For the equation with a delayed argument

$$y^{(4)}(t) + \frac{\ln t}{t^4} y(kt) = 0 \, , \quad 0 < k < 1 \, , \quad t > 0 \, ,$$

the well-known sufficient condition for oscillation of every solution

$$\int^\infty [g(t)]^{3-\varepsilon} p(t) \, dt = \infty \quad (0 < \varepsilon) \, ,$$

is not satisfied, but the condition (6) is. So every solution of this equation is oscillatory.

EQUATION (2)

We consider equation (2) where

a$_1$) $p(t)$ is continuous and nonnegative on $[t_0, \infty)$;

b$_1$) $g(t)$ is a continuously differentiable nondecreasing function on $[t_0, \infty)$ such that $g(t) \leqq t$ and $\lim_{t \to \infty} g(t) = \infty$, $g'(t) \leqq 1$.

Lemma 3. *Suppose that the conditions* a$_1$), b$_1$) *are satisfied and let* $l \in \{1, \ldots$ $\ldots, n-1\}$, $l + n$ *odd and let a solution* $y(t)$ *of equation* (2) *satisfy the condition* (3). *Then*

$$\int_{t_0}^{\infty} [g(t)]^{n-2} p(t) \, \mathrm{d}t < \infty ,$$

$$y^{(l-1)}(t) \geqq y^{(l-1)}(t_1) + \frac{1}{(n-1-l)!} \int_{t_1}^{t} \int_{s}^{\infty} (u-s)^{n-1-l} p(u) \, y(g(u)) \, \mathrm{d}u \, \mathrm{d}s ,$$

$$y(t) \geqq \frac{(t-t_1)^{l-1}}{l!} \, y^{(l-1)}(t) , \quad t \in [t_1, \infty) .$$

This lemma is a consequence of Lemma 1.4 in [3] and Lemma 2 in [2].

Lemma 4. *Suppose that the conditions* a$_1$), b$_1$) *are satisfied and let* $l \in \{1, \ldots$ $\ldots, n-1\}$, $l + n$ *odd and let a solution* $y(t)$ *of equation* (2) *satisfy the condition* (3). *Then the integral equation*

$$v''(t) + \frac{g'(t)}{(n-3)!} \int_{t}^{\infty} [g(u) - g(t)]^{n-3} p(u) \, v(g(u)) \, \mathrm{d}u = 0 \tag{9}$$

has a nonoscillatory solution.

Proof. With regard to Lemma 3 we get

$$y(g(t)) \geqq \frac{[g(t) - t_1]^{l-1}}{l!} \, y^{(l-1)}(g(t)) \geqq \frac{[g(t) - g(s)]^{l-1}}{l!} \, y^{(l-1)}(g(t)) ,$$

$t \geqq s \geqq t_2$, where $t_2 \geqq t_1$ is a sufficiently large number. The condition b$_1$) implies $g(t) - g(s) \leqq t - s$, $t \geqq s \geqq t_2$, and in view of Lemma 3 we have

$$y^{(l-1)}(t) \geqq y^{(l-1)}(t_2) + \frac{1}{(n-2)!} \int_{t_2}^{t} \int_{s}^{\infty} [g(u) - g(s)]^{n-2} p(u) \, y^{(l-1)}(g(u)) \, \mathrm{d}u \, \mathrm{d}s .$$

Now the method of successive approximations asserts that there is a continuous function $v(t)$ on $[t_2, \infty)$ such that

$$y^{(l-1)}(t_2) \leqq v(t) \leqq y^{(l-1)}(t) , \quad t \geqq t_2 ,$$

$$v(t) = y^{(l-1)}(t_2) + \frac{1}{(n-2)!} \int_{t_2}^{t} \int_{s}^{\infty} [g(u) - g(s)]^{n-2} p(u) \, v(g(u)) \, \mathrm{d}u \, \mathrm{d}s ,$$

which is a solution of equation (9).

Theorem 2. *Suppose that the conditions* a$_1$), b$_1$) *are satisfied and let*

$$\limsup_{t \to \infty} g(t) \int_{t}^{\infty} [g(s)]^{n-2} p(s) \, \mathrm{d}s > (n-1)! . \tag{10}$$

Then every solution of equation (2) *is oscillatory, if n is even, and every solution of equation* (2) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n - 1$, *if n is odd.*

Proof. Let $y(t)$ be a positive solution of equation (2). Choose t_0 so that $y(g(t)) > 0$ for $t \geqq t_0$. Then in view of Lemma 1 there is a number $t_1 \in [t_0, \infty)$ and $l \in \{0, 1, \ldots \ldots, n - 1\}$ such that $l + n$ is odd and (3) is satisfied.

Let $l = 0$, then $y(t)$ is bounded and in view of Lemma 2 (which holds for equation (2) as well) we have $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n - 1$.

Let $l \in \{1, \ldots, n - 1\}$. Then with regard to Lemma 4 equation (9) has a nonoscillatory solution $v(t) > 0$, $t \geqq t_2 \geqq t_1$. Integrating equation (9) from t to z, $z > t \geqq t_2$, we have

$$v'(z) - v'(t) = -\frac{1}{(n-3)!}\int_t^z g'(u)\int_u^\infty [g(s) - g(u)]^{n-3}\, p(s)\, v(g(s))\, \mathrm{d}s\, \mathrm{d}u\,.$$

Let $z \to \infty$, then

$$v'(t) \geqq \frac{1}{(n-3)!}\int_t^\infty g'(u)\int_u^\infty [g(s) - g(u)]^{n-3}\, p(s)\, v(g(s))\, \mathrm{d}s\, \mathrm{d}u =$$

$$= \frac{1}{(n-3)!}\int_t^\infty p(s)\, v(g(s))\int_t^s g'(u)\,[g(s) - g(u)]^{n-3}\, \mathrm{d}u\, \mathrm{d}s\,.$$

Since $v'(t)$ is noninscreasing, we get

$$v'(g(t)) \geqq \frac{1}{(n-2)!}\int_t^\infty [g(s) - g(t)]^{n-2}\, p(s)\, v(g(s))\, \mathrm{d}s\,.$$

Multiplying the last inequality by $g'(t)$ and integrating from T to t, $t > T \geqq t_2$, we obtain

$$v(g(t)) \geqq \frac{1}{(n-2)!}\left\{\int_T^t p(s)\, v(g(s))\int_T^s g'(u)\,[g(s) - g(u)]^{n-2}\, \mathrm{d}u\, \mathrm{d}s\, +\right.$$

$$\left. + \int_t^\infty p(s)\, v(g(s))\int_T^t g'(u)\,[g(s) - g(u)]^{n-2}\, \mathrm{d}u\, \mathrm{d}s\right.,$$

$$v(g(t)) \geqq \frac{1}{(n-2)!}\int_t^\infty p(s)\, v(g(s))\int_T^t g'(u)\,[g(s) - g(u)]^{n-2}\, \mathrm{d}u\, \mathrm{d}s\,,$$

$$(n-1)! \geqq [g(t) - g(T)]\int_t^\infty [g(s) - g(T)]^{n-2}\, p(s)\, \mathrm{d}s\,,$$

and from the last inequality we obtain a contradiction with (10). This completes the proof.

Corollary. *Suppose that the contitions* a_1), b_1) *are satisfied and there exists a non-decreasing function* $\omega \in C[[t_0, \infty), (0, \infty)]$ *such that*

$$(11) \qquad \int_{t_1}^{\infty} \frac{dt}{t\,\omega(t)} < \infty \quad \text{and} \quad \int_{t_1}^{\infty} \frac{[g(t)]^{n-1}\,p(t)}{\omega(g(t))}\,dt = \infty\,, \quad t_1 \in [t_0, \infty)\,.$$

Then every solution of equation (2) *is oscillatory, if n is even, and every solution of equation* (2) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$, *if n is odd.*

Proof. We shall prove that if (11) holds, then

$$(12) \qquad \limsup_{t\to\infty} g(t) \int_t^{\infty} [g(s)]^{n-2}\,p(s)\,ds = \infty\,.$$

Suppose that

$$g(t) \int_t^{\infty} [g(s)]^{n-2}\,p(s)\,ds \leqq c_0 \quad \text{for} \quad t \in [t_1, \infty)\,.$$

Then

$$\int_{t_1}^{t} \frac{[g(s)]^{n-1}\,p(s)}{\omega(g(s))}\,ds = -g(t)\int_t^{\infty} \frac{[g(u)]^{n-2}\,p(u)}{\omega(g(u))}\,du + g(t_1)\int_{t_1}^{\infty} \frac{[g(u)]^{n-2}\,p(u)}{\omega(g(u))}\,du +$$

$$+ \int_{t_1}^{t} g'(s) \int_s^{\infty} \frac{[g(u)]^{n-2}\,p(u)}{\omega(g(u))}\,du\,ds \leqq \frac{c_0}{\omega(g(t_1))} +$$

$$+ \int_{t_1}^{t} \frac{g'(s)}{g(s)\,\omega(g(s))}\, g(s) \int_s^{\infty} [g(u)]^{n-2}\,p(u)\,du\,ds \leqq c_0 \left[\frac{1}{\omega(g(t_1))}\right] +$$

$$+ \int_{t_1}^{t} \frac{g'(s)}{g(s)\,\omega(g(s))}\,ds\,,$$

which contradicts (11). So the condition (12) holds and we can apply Theorem 2.

References

[1] *Т. А. Чантурия:* Интегральные признаки колеблемости решений линейных дифференциальных уравнений высших порядков, Дифференц. уравнения, 1980, т. *16*, № 3.

[2] *И. Т. Кигурадзе:* О колеблемости решений уравнения $u^{(m)} + a(t)\,|u|^n \operatorname{sig} nu = 0$, Мат. сб., 1964, *65*, № 2, с. 172—187.

[3] *Р. Г. Коплатадзе, Т. А. Чантурия:* Об осцилляционных свойствах дифференциальных уравнений с отклоняющимся аргументом, Тбилиси, Изд-во Тбил. гос. ун-та, 1977.

[4] *Ch. G. Philos:* Some results on the oscillatory and asymptotic behavior of the solutions of defferential equations with deviating arguments, Archivum Math. *1* (1979), 19—36.

[5] *В. Н. Шевело:* Осцилляция решений дифференциальных уравнений с отклоняющимся аргументом, Киев „Наукова думка" 1978.

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