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A NOTE ON TOLERANCE LATTICES OF PRODUCTS OF LATTICES

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It is shown that the tolerance lattice of the product of a finite number of lattices is isomorphic to the product of their tolerance lattices.

Lemma 1. *Let L, L_1, L_2 be lattices, $L = L_1 \times L_2$. Then for any compatible tolerance T on L the following conditions are equivalent for each pair $a, b \in L_1$:*

(i) *there exist $u, v \in L_2$ such that $[a, u]\, T[b, v]$;*
(ii) *there exists $x \in L_2$ such that $[a, x]\, T[b, x]$;*
(iii) *$[a, y]\, T[b, y]$ for each $y \in L_2$.*

Proof. The proof will be omitted.

Lemma 2. *Let L, L_1, L_2 be lattices, $L = L_1 \times L_2$. If T is a compatible tolerance on the lattice L, then $f_1(T)$ defined by*

$$a\, f_1(T)\, b :\Leftrightarrow [a, x]\, T[b, x] \quad \text{for each} \quad x \in L_2$$

and $f_2(T)$ defined by

$$c\, f_2(T)\, d :\Leftrightarrow [y, c]\, T[y, d] \quad \text{for each} \quad y \in L_1$$

are compatible tolerances on L_1 and L_2, respectively. The maps $f_1 : TL(L) \to TL(L_1)$ and $f_2 : TL(L) \to TL(L_2)$ are lattice homomorphisms.

Proof. The proof will be done for f_1. $f_1(T)$ is obviously a tolerance relation. Let $a_1 f_1(T)\, b_1$ and $a_2 f_1(T)\, b_2$. Then $[a_1, x]\, T[b_1, x]$ and $[a_2, x]\, T[b_2, x]$ for each $x \in L_2$. It follows that $[a_1 \wedge a_2, x]\, T[b_1 \wedge b_2, x]$ and $[a_1 \vee a_2, x]\, T[b_1 \vee b_2, x]$, hence $(a_1 \wedge a_2) f_1(T) (b_1 \wedge b_2)$ and $(a_1 \vee a_2) f_1(T) (b_1 \vee b_2)$. Thus $f_1(T)$ is a compatible tolerance on L_1. Now, let $S, T \in TL(L)$. Obviously $f_1(S \wedge T) = f_1(S) \wedge f_1(T)$. $a f_1(S \vee T)\, b \Leftrightarrow [a, x]\, (S \vee T)\, [b, x]$ for each $x \in L_2 \Leftrightarrow$ there exists $y \in L_2$ such that $[a, y]\, (S \vee T)\, [b, y] \Leftrightarrow$ there exist $y \in L_2$, a lattice polynomial p and ordered pairs $[a_1, u_1], \ldots, [a_n, u_n], [b_1, v_1], \ldots, [b_n, v_n] \in L$ such that $[a_i, u_i]\, S[b_i, v_i]$ or $[a_i, u_i]\, T[b_i, v_i]$ and $[a, y] = p([a_1, u_1], \ldots, [a_n, u_n])$ and $[b, y] = p([b_1, v_1], \ldots, [b_n, v_n]) \Leftrightarrow$ there exist $y \in L_2$, a lattice polynomial p and $a_1, \ldots, a_n, b_1, \ldots, b_n \in L_1$, $y_1, \ldots, y_n \in L_2$ such that $[a_i, y_i]\, S[b_i, y_i]$ or $[a_i, y_i] \,.$ $.\, T[b_i, y_i]$ and $[a, y] = p([a_1, y_1], \ldots, [a_n, y_n])$ and $[b, y] = p([b_1, y_1], \ldots$ $\ldots, [b_n, y_n] \Leftrightarrow$ there exists a lattice polynominal p and $a_1, \ldots, a_n, b_1,, \ldots b_n \in L_1$ such that $a_i f_1(S)\, b_i$ or $a_i f_1(T)\, b_i$ and $a = p(a_1, \ldots, a_n)$ and $b = p(b_1, \ldots, b_n) \Leftrightarrow$ $\Leftrightarrow a(f_1(S) \vee f_1(T))\, b$. Q.E.D.

Proposition. *For lattices L, L_1, L_2, $L = L_1 \times L_2$ implies $TL(L) \cong TL(L_1) \times$ $\times\, TL(L_2)$.*

Proof. Define a map $f : TL(L) \to TL(L_1) \times TL(L_2)$ by the rule $f(T) =$ $= [f_1(T), f_2(T)]$. The map f is obviously a lattice homomorphism. Let $[T_1, T_2]$ be an arbitrary element of $TL(L_1) \times TL(L_2)$. Construct a relation T on L by $[a, b]\, T[c, d] :\Leftrightarrow aT_1c$ and bT_2d. Clearly, T is a compatible tolerance on L. We have $f(T) = [T_1, T_2]$, and so f is onto. Now, let $f(S) = f(T)$. Then $[a, b]\, S[c, d]$ implies $[a, x]\, T[c, x]$ for each $x \in L_2$ and $[y, b]\, T[y, d]$ for each $y \in L_1$. Hence $[a, b \wedge d]\, T[c, b \wedge d]$ and $[a \wedge c, b]\, T[a \wedge c, d]$ and so $[a, b]\, T[c, d]$. Thus $S \leqq T$. Analogously $T \leqq S$, hence $S = T$. The lattice homomorphism f is onto and injective and so a lattice isomorphism. Q.E.D.

Corollary. *Let $L, L_1, \ldots, L_n$ be lattices, $n \in N$, $L = L_1 \times \ldots \times L_n$. Then $TL(L) \cong$ $\cong TL(L_1) \times \ldots \times TL(L_n)$.*

Remark. The finiteness of number of direct factors is substantial. If their number is infinite f is not injective.

Reference

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