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Bohdan Zelinka

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ON TWO PROBLEMS OF THE GRAPH THEORY

BOHDAN ZELINKA, Liberec

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This paper concerns two problems of the graph theory. One of them was suggested by E. J. Cockayne and S. T. Hedetniemi [1], the other by S. Poljak [2]. In both these problems finite undirected graphs without loops and multiple edges are considered. Even if not solving these problems completely, we shall present some results concerning them.

§ 1 UNIQUELY DOMATIC REGULAR DOMATICALLY FULL GRAPHS

The problem of E. J. Cockayne and S. T. Hedetniemi is the following:

Characterize uniquely domatic graphs.

The domatic number of a graph was introduced by the authors of this problem. A subset D of the vertex set $V(G)$ of an undirected graph G is called a dominating set in G, if to each vertex $x \in V(G) - D$ there exists at least one vertex $y \in D$ adjacent to it. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximal number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

A graph G is called uniquely domatic, if there exists exactly one domatic partition of G with $d(G)$ classes.

We restrict our considerations to regular domatically full graphs. For the domatic number of a graph G the inequality $d(G) \leqq \delta(G) + 1$ holds, where $\delta(G)$ is the minimal degree of a vertex of G. A graph G for which $d(G) = \delta(G) + 1$ holds is called domatically full. If it is regular, then the degree of each of its vertices is equal to $d(G) - 1$. Regular domatically full graphs were characterized in [3].

Instead of a domatic partition we may speak about a domatic colouring. A colouring of vertices of a graph G is called domatic, if for each vertex u and for each colour of this colouring distinct from that of u there exists a vertex of this colour which is adjacent to u. Then the domatic number of G can be defined as the maximal number of colours of a domatic colouring of G. (The reader may verify the equivalence of this definition with the previous one.) Note that, when a domatic colouring is concerned, two adjacent vertices may have the same colour. A colouring in the usual sense, i.e. a colouring at which adjacent vertices have always distinct colours, will be called a chromatic colouring. The chromatic number of a graph G will be denoted by $\chi(G)$.

Now we shall determine which regular domatically full graphs are uniquely domatic. First we shall prove an auxiliary result.

Theorem 1. *Let G be a finite regular domatically full graph, let G^2 be the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if their distance in G is at most 2. Then $d(G) = \chi(G^2)$ and each chromatic colouring of G^2 with $\chi(G^2)$ colours is a domatic colouring of G with $d(G)$ colours and vice versa.*

Proof. Let a domatic colouring of G with $d(G)$ colours be given. Suppose that there exist two distinct vertices u, v which have the same colour in this colouring and are adjacent in G^2. Then the distance between u and v in G is 1 or 2. If it is 1, then u, v are adjacent in G. As G is regular and domatically full, the degree of each vertex of G is $d(G) - 1$. As u is adjacent to the vertex v of the same colour, it is adjacent to at most $d(G) - 2$ vertices of colours distinct from its own one. But there are $d(G) - 1$ colours distinct from that of u, which is a contradiction. If the distance between u and v in G is 2, then there exists a vertex w which is adjacent in G to both u and v. Then w is adjacent to two vertices with equal colours, therefore it is adjacent to at most $d(G) - 3$ vertices of the colours distinct from its own one and from that of u and v; as there are $d(G) - 2$ such colours, we have again a contradiction. Hence the considered domatic colouring of G is a chromatic colouring of G^2 and $\chi(G^2) \leqq d(G)$.

Now let a chromatic colouring of G^2 with $\chi(G^2)$ colours be given. Let u be a vertex of G. Any vertex v adjacent to u in G is adjacent to u also in G^2 and therefore it has a colour distinct from that of u. If v and w are two vertices both adjacent to u in G, then they are adjacent in G^2 and therefore they have distinct colours. As $\chi(G^2) \leqq d(G)$, there are at most $d(G)$ colours; as the degree of each vertex of G is $d(G) - 1$, to each colour distinct from that of u there exists a vertex of this colour which is adjacent to u and the considered colouring is a domatic colouring of G. Then also $d(G) = \chi(G^2)$.

Now we formulate a corollary which gives the characterization of such regular domatically full graphs which are uniquely domatic.

Corollary 1. *Le G be a finite regular domatically full graph, let G^2 be the graph with the same vertex set as G in which two distinct vertices are adjacent if and only if their distance in G is at most 2. Then G is uniquely domatic if and only if G^2 is uniquely colourable in the sense of the chromatic number.*

For each positive integer $d \geqq 3$ we can show an example of a regular domatically full graph with the domatic number d which is uniquely domatic. This is the graph obtained from the complete bipartite graph $K_{d,d}$ by deleting edges of a complete matching. For this graph G the graph G^2 is a complete d-partite graph; it is uniquely colourable in the sense of the chromatic number.

§ 2 COVERING OF A DISJOINT SUM OF ISOMORPHIC COMPLETE GRAPHS BY INDEPENDENT SETS

At the Czechoslovak Conference on Graph Theory at Zemplínska Šírava in 1978 S. Poljak proposed the following problem:

Let a graph consist of t disjoint copies of the graph K_p. Find the minimal number of independent sets of this graph so that each "non-edge" might be covered by at least one set.

(By a non-edge, a pair of non-adjacent vertices is meant.)

We shall give some results on the asymptotic behaviour of the minimal cardinality of the covering required.

By tK_p we denote a graph having t connected components, each of which is a complete graph with p vertices. (Such a graph can be called a disjoint sum of isomorphic complete graphs.)

The minimal number of independent sets in tK_p with the property that each non-edge of tK_p is a subset of some of these sets will be denoted by $\mu(p, t)$. We shall prove some assertions on $\mu(p, t)$.

First we prove a proposition.

Proposition. *For any two positive integers p, t, where $t \geqq 2$, we have*

$$\mu(p, t) \geqq p^2 .$$

Proof. Let $\mathcal{S}$ be the required covering of tK_p. The vertices of tK_p will be denoted by ordered pairs of positive integers so that the connected components of tK_p are denoted by $C_1, \ldots, C_t$ and the vertices of C_i (for $i = 1, \ldots, t$) are $[i, 1], \ldots, [i, p]$. Consider the vertices of the connected components C_1, C_2. Each set $S \in \mathcal{S}$ can cover at most one vertex from C_1 and at most one vertex from C_2. Therefore each set $S \in \mathcal{S}$ can cover at most one non-edge which has one vertex in C_1 and one in C_2. There are p^2 such non-edges, hence $\mathcal{S}$ must contain at least p^2 sets.

Now we prove a theorem.

Theorem 2. *Let p, t be two positive integers such that $2 \leqq t \leqq p$ and p is a power of a prime number. Then*

$$\mu(p, t) = p^2 .$$

Proof. According to Proposition, $\mu(p, t) \geqq p^2$. Therefore it suffices to prove that there exists a covering with the required properties consisting of p^2 sets. It is wellknown that if p is a power of a prime number, then there exists a finite projective geometry with the property that each line contains $p + 1$ points and each point is contained in $p + 1$ lines. Consider such a geometry and choose a point o in it. Let $P_0, P_1, \ldots, P_p$ be the lines containing the point o. Put $t = p$ and consider the graph pK_p; let is connected components be $C_1, \ldots, C_p$. Choose a bijection φ of the vertex set of pK_p onto the set of all points of our geometry which lie on the lines $P_1, \ldots, P_p$ and are different from o; this bijection will be chosen so that each vertex of C_i is mapped onto a point of P_i for $i = 1, \ldots, p$. Each line of the geometry which does not contain o has exactly one common point with each P_i for $i = 1, \ldots, p$ and for any two points lying on two different lines from the family $P_1, \ldots, P_p$ and different from o there exists exactly one line not containing o which contains them both. Therefore the system of images of these lines in φ^{-1} covers all non-edges of pK_p and these images are independent sets in pK_p. There are exactly p^2 such lines and hence also p^2 such sets in pK_p. We have

$$\mu(p, p) = p^2 .$$

If $t < p$, then tK_p can be embedded into pK_p. If we have a covering with the required properties of pK_p, the intersections of the sets of this covering with the vertex set of tK_p form a covering with the required properties of tK_p. Therefore $\mu(p, t) \leqq \mu(p, p)$. But, as $\mu(p, t) \geqq p^2$, we have

$$\mu(p, t) = p^2 .$$

Corollary 2. *For each positive integer $t \geqq 2$ there exists a positive integer p_0 with the property that*

$$\mu(p, t) = p^2$$

for each $p \geqq p_0$.

For p_0 we can choose the least integer which is greater than or equal to t and is a power of a prime number.

Theorem 3. *The function $\mu(p, t)$ for a constant $p \geqq 2$ can be majorized by a logarithmic function of t.*

Proof. As for $t_1 < t_2$ the graph t_1K_p can be embedded into t_2K_p, we have $\mu(p, t_1) \leqq \mu(p, t_2)$. Hence the function $\mu(p, t)$ for a constant p is non-decreasing. Consider the number $\mu(p, t)$ and $\mu(p, 2t)$ for some p and t. The connected components of $2tK_p$ will be denoted by $C_1, \ldots, C_{2t}$. Let G_1 (or G_2) be the union of the connected components $C_1, \ldots, C_t$ (or $C_{t+1}, \ldots, C_{2t}$, respectively). The graphs G_1, G_2 are both isomorphic to tK_p. Let $\mathscr{S}_1$ (or $\mathscr{S}_2$) be a covering with the required properties of G_1 (or G_2 respectively) with the cardinality $\mu(p, t)$. The vertices of the connected component C_i (for $i = 1, \ldots, 2t$) will be denoted by ordered pairs of numbers $[i, 1], \ldots, [i, p]$. We denote $T_1(j) = \{[i, j] \mid i = 1, \ldots, t\}$, $T_2(j) = \{[i, j] \mid i = t + 1, \ldots$

$\ldots, 2t\}$ for $j = 1, \ldots, p$. Further, we put $T(j, k) = T_1(j) \cup T_2(k)$ and denote the system of all sets $T(j, k)$ by $\mathcal{T}$; the cardinality of $\mathcal{T}$ is p^2. We choose a bijection $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and denote $\mathcal{S} = \{S \cup \varphi(S) \mid S \in \mathcal{S}_1\}$. As no vertex of G_1 is adjacent to a vertex of G_2, each set from $\mathcal{S}$ is independent. Each non-edge of G_1 or G_2 is covered by a set from $\mathcal{S}$, each non-edge having one vertex in G_1 and the other in G_2 is covered by a set from $\mathcal{T}$. Therefore $\mathcal{S} \cup \mathcal{T}$ is a covering with the required properties of $2tK_p$; it has the cardinality at most $\mu(p, t) + p^2$. Hence

$$\mu(p, 2t) \leqq \mu(p, t) + p^2 .$$

By induction

$$\mu(p, 2^n t) \leqq \mu(p, t) + np^2$$

for each positive integer n. If we denote $2^n = m$, we obtain

$$\mu(p, mt) \leqq \mu(p, t) + p^2 \log_2 m$$

for each m which is a power of 2. As $\mu(p, t)$ is non-decreasing, we can write

$$\mu(p, mt) \leqq \mu(p, t) + p^2 \,]\log_2 m[\, \leqq \mu(p, t) + p^2 \log_2 m + 1$$

for each positive integer m.

References

[1] *E. J. Cockayne, S. T. Hedetniemi:* Towards a theory of domination in graphs. Networks *7* (1977), 247–261.

[2] *S. Poljak:* Problem 6. Presented at the Czechoslovak Conference on Graph Theory at Zemplínska Šírava in 1978.

[3] *B. Zelinka:* Domatically critical graphs. Czech. Math. J. *30* (1980), 486–489.

Author's address: 460 01 Liberec 1, Komenského 2 (katedra matematiky VŠST).

A REMARK TO MY PAPER "FINITE SPHERICAL GEOMETRIES"*)

BOHDAN ZELINKA, Liberec

The assertion on the equivalence of the existence of $SG(n)$ with the existence of $PG(n)$ is false. This was pointed out by E. Gonin in his review of this paper in RŽMat.

*) Čas. pěst. mat. *106* (1981), 207–209.