Josef Niederle
On atoms in tolerance lattices of distributive lattices

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 3, 311--315

Persistent URL: `http://dml.cz/dmlcz/118090`

Časopis pro pěstování matematiky, roč. 106 (1981), Praha

ON ATOMS IN TOLERANCE LATTICES OF DISTRIBUTIVE LATTICES

JOSEF NIEDERLE, Brno

(Received June 20, 1979)

Notation. For an algebra $\mathfrak{A} = (A, F)$, $TL(\mathfrak{A})$ will denote the tolerance lattice of $\mathfrak{A}$, $CL(\mathfrak{A})$ the congruence lattice of $\mathfrak{A}$.

For a lattice $\mathfrak{L} = (L, \wedge, \vee)$, $\mathscr{A}\ell(\mathfrak{L})$ will denote the set of all atoms in $\mathfrak{L}$.

For a compatible tolerance T on an algebra $\mathfrak{A}$, $\mathscr{C}(T)$ will denote the transitive hull of T.

It is known that $\mathscr{C}(T)$ is the minimal congruence including T ([5], Thm. 1). $\mathscr{C}$ can be regarded as a mapping of $TL(\mathfrak{A})$ into $CL(\mathfrak{A})$. $\mathscr{C}$ is evidently an order homomorphism (= isotone mapping). A little more will be shown.

Lemma 1. *Let $a = x_0 \leqq x_1 \leqq \ldots \leqq x_m = b$ and $a = y_0 \leqq y_1 \ldots \leqq y_n = b$ be two chains, not necessarily maximal, connecting the elements $a < b$ of a lattice $\mathfrak{L} = (L, \wedge, \vee)$. Let S and T be compatible tolerances on $\mathfrak{L}$ and let $[x_{i-1}, x_i] \in S$ for $i = 1, \ldots, m$ and $[y_{j-1}, y_j] \in T$ for $j = 1, \ldots, n$. Then there exists a chain z, $a = z_0 \leqq z_1 \leqq \ldots \leqq z_k = b$ such that $[z_{l-1}, z_l] \in S \wedge T$ for $l = 1, \ldots, k$.*

Proof. By induction with respect to m. For $m = 1$ the statement holds, because $[a, b] \in S$ implies $[y_{j-1}, y_j] \in S$ for $j = 1, \ldots, n$. Put $k = n$ and $z_j = y_j$. Suppose the statement holds for $1, \ldots, m - 1$.

Construct a chain y' from x_0 to x_{m-1} as follows: $y'_i = y_i \wedge x_{m-1}$ for $i = 0, \ldots, n$. Clearly $y'_0 = x_0 = a$, $y'_n = x_{m-1}$ and $[y'_{i-1}, y'_i] \in T$ for $i = 1, \ldots, n$. By the assumption, there exists a chain z', $a = z'_0 \leqq z'_1 \leqq \ldots \leqq z'_{k'} = x_{m-1}$ such that $[z'_{l-1}, z'_l] \in S \wedge T$ for $l = 1, \ldots, k'$. Denote $k = k' + n$, $z_l = z'_l$ for $l = 0, \ldots, k'$ and $z_l = x_{m-1} \vee y_{l-k'}$ for $l = k', \ldots, k$. Clearly $a = z_0 \leqq z_1 \leqq \ldots \leqq z_k = b$ and $[z_{l-1}, z_l] \in S \wedge T$ for $l = 1, \ldots, k$. Q.E.D.

Proposition 1. *For every algebra $\mathfrak{A}$ the operator $\mathscr{C}$ is a complete join-homomorphism of $TL(\mathfrak{A})$ onto $CL(\mathfrak{A})$.*

For every lattice $\mathfrak{L}$ the operator $\mathscr{C}$ is a lattice homomorphism of $TL(\mathfrak{L})$ onto $CL(\mathfrak{L})$, which need not be meet-complete.

Proof. $\mathscr{C}$ is a complete join-homomorphism: Let $T_i \in TL(\mathfrak{A})$, $i \in I$. Then $\mathscr{C}(\bigvee_{TL} T_i) \geqq \mathscr{C}(T_i)$, hence $\mathscr{C}(\bigvee_{TL} T_i) \geqq \bigvee_{CL} \mathscr{C}(T_i)$. Conversely, $\bigvee_{TL} T_i \leqq \bigvee_{TL} \mathscr{C}(T_i)$, thus $\mathscr{C}(\bigvee_{TL} T_i) \leqq \mathscr{C}(\bigvee_{TL} \mathscr{C}(T_i)) = \bigvee_{CL} \mathscr{C}(T_i)$. For a lattice, $\mathscr{C}$ is a meet-homomorphism: Clearly $\mathscr{C}(S) \wedge \mathscr{C}(T) \geqq \mathscr{C}(S \wedge T)$ (meet-operations both in TL and CL coincide with the set intersection). $\mathscr{C}(S \wedge T) \geqq \mathscr{C}(S) \wedge \mathscr{C}(T)$ is to be shown. Let $[a, b] \in \mathscr{C}(S) \wedge \mathscr{C}(T)$, $a \leqq b$. There exist elements $x'_0, \ldots, x'_m, y'_0, \ldots, y'_n$ such that $a = x'_0 = y'_0$, $b = x'_m = y'_n$ and $[x'_{i-1}, x'_i] \in S$, for $i = 1, \ldots, m$, $[y'_{j-1}, y'_j] \in T$ for $j = 1, \ldots, n$. Put $x_i = (x'_0 \vee \ldots \vee x'_i) \wedge b$ for $i = 0, \ldots, m$ and $y_j = (y'_0 \vee \ldots \vee y'_j) \wedge b$ for $j = 0, \ldots, n$. Then $a = x_0 \leqq x_1 \leqq \ldots \leqq x_m = b$, $a = y_0 \leqq y_1 \leqq \ldots \leqq y_n = b$ and $[x_{i-1}, x_i] \in S$ for $i = 1, \ldots, m$, $[y_{j-1}, y_j] \in T$ for $j = 1, \ldots, n$. By Lemma 1 a chain $a = z_0 \leqq z_1 \leqq \ldots \leqq z_k = b$ can be constructed such that $[z_{l-1}, z_l] \in S \wedge T$ for $l = 1, \ldots, k$. Thus $[a, b] \in \mathscr{C}(T \wedge S)$. Q.E.D.

Notation. Denote by $\mathscr{T}(\Theta)$ the set of all compatible tolerances the transitive hull of which is Θ, $\mathscr{T}(\Theta) = \{T \in TL(\mathfrak{A}) \mid \mathscr{C}(T) = \Theta\}$.

Corollary. *Let Θ be a congruence on a lattice $\mathfrak{L} = (L, \wedge, \vee)$. Then $\mathscr{T}(\Theta)$ is a convex sublattice of $TL(\mathfrak{L})$ with Θ as the greatest element.*

Remark. For every algebra $\mathfrak{A} = (A, F)$, $TL(\mathfrak{A})$ is a disjoint union of all $\mathscr{T}(\Theta)$, $TL(\mathfrak{A}) = \dot{\bigcup}_{\Theta \in CL(\mathfrak{A})} \mathscr{T}(\Theta)$.

Remark. If Θ is a congruence, then the infimum of $\mathscr{T}(\Theta)$ either belongs to $\mathscr{T}(\Theta)$ or not, both cases can occur.

Definition. A *principal tolerance* on the algebra $\mathfrak{A} = (A, F)$ is the least compatible tolerance on $\mathfrak{A}$ containing a given pair of elements $[a, b] \in A \times A$; it will be denoted by $T(a, b)$.

A *c-principal tolerance* on the lattice $\mathfrak{L} = (L, \wedge, \vee)$ is the least compatible tolerance on $\mathfrak{L}$ containing a given pair of elements $[a, b] \in L \times L$, $a \prec b$. Evidently, every c-principal tolerance on a lattice is principal.

As shown by Chajda and Zelinka ([3], Thm. 1), each principal tolerance $T(a, b)$ on a distributive lattice is identical with the principal congruence $\Theta(a, b)$. By [4] (Thm. 16 and Cor. 4), tolerance lattices of distributive lattices are complete, compactly generated and distributive. As every compactly generated lattice is upper continuous (cf. [1], 2.3.), they are upper continuous, and since every distributive upper continuous lattice is infinitely distributive (cf. [1], p. 35) they are infinitely distributive.

Lemma 2. *Let $\mathfrak{L} = (L, \wedge, \vee)$ be a distributive lattice, $a, b, c \in L$, $a < c < b$. Then $T(a, c) < T(a, b)$.*

Proof. Clearly $T(a, c) \leqq T(a, b)$. $[a, b] \in T(a, c)$ would imply that there exist $x, y, z \in L$ such that $(x \wedge a) \vee (y \wedge c) \vee z = a$, and $(x \wedge c) \vee (y \wedge a) \vee z = b$. Hence $z \leqq a < c$, $x \wedge c \leqq c$, $y \wedge a \leqq a < c$ and consequently $b \leqq c$, which contradicts the assumptions. Q.E.D.

Proposition 2. *Let T be a compatible tolerance on the distributive lattice $\mathfrak{L} = (L, \wedge, \vee)$, $[a, b] \in T$, $a < b$, $a \not\prec b$. Then T is not an atom in $TL(\mathfrak{L})$.*

Proof. $T \neq T(a, b)$ implies T is not an atom. Assume $T = T(a, b)$. There exists an element $c \in L$, $a < c < b$. By Lemma 2, $T(a, c) < T(a, b)$ and therefore T is not an atom. Q.E.D.

In other words, if T is an atom in $TL(\mathfrak{L})$ and $[a, b] \in T$, then $a = b$ or $a \prec b$ or $a \succ b$. This follows from the fact that $[x, y] \in T$ if and only if $[x \wedge y, x \vee y] \in T$ ([2], Thm. 1). The converse is not true.

Proposition 3. *Let $\mathfrak{L} = (L, \wedge, \vee)$ be a distributive lattice, T a compatible tolerance on $\mathfrak{L}$. The following assertions are equivalent:*

(i) *T is an atom in $TL(\mathfrak{L})$;*
(ii) *T is c-principal.*

Proof. (i) ⇒ (ii): Suppose T is an atom in $TL(\mathfrak{L})$, then there exist elements $a, b \in L$, $a \prec b$, $[a, b] \in T$. Then $T = T(a, b)$, consequently T is c-principal.

(ii) ⇒ (i): Let $T = T(a, b)$, $a \prec b$, and let S be a compatible tolerance on $\mathfrak{L}$, $\Delta \neq S \leqq T$. $[x, y] \in S$, $x < y$, implies that there exist elements $p, q, r \in L$ such that $x = (p \wedge a) \vee (q \wedge b) \vee r$ and $y = (p \wedge b) \vee (q \wedge a) \vee r$. But $q \wedge a \leqq q \wedge b \leqq x$, $r \leqq x$, so that $x = (p \wedge a) \vee x$ and $y = (p \wedge b) \vee x$. By the assumption $a \prec b$, the intervals $\langle a, b\rangle$ and $\langle p \wedge a, p \wedge b\rangle$ are transposed, consequently $p \wedge a \prec p \wedge b$. Analogously, intervals $\langle p \wedge a, p \wedge b\rangle$ and $\langle x, y\rangle$ are transposed and $x \prec y$. Now, $[x, y] \in S$ implies $a = a \vee (x \wedge (p \wedge b))$, $b = a \vee (y \wedge (p \wedge b))$ and consequently $[a, b] \in S$. Hence $T \leqq S$ and finally $T = S$. Q.E.D.

Remark. Atoms in $TL(\mathfrak{L})$ are exactly the same as in $CL(\mathfrak{L})$.

Proposition 4. *For a distributive lattice $\mathfrak{L} = (L, \wedge, \vee)$, the following assertions are equivalent:*

(i) *$\mathfrak{L}$ is locally finite;*
(ii) *$CL(\mathfrak{L})$ is a Boolean lattice;*
(iii) *every element in $CL(\mathfrak{L})$ is join of atoms;*
(iv) *the greatest element in $CL(\mathfrak{L})$ is join of atoms.*

Proof. (i) ⇔ (ii) by Hashimoto (cf. [1], p. 80).

(ii) ⇔ (iii) ⇔ (iv) by [1], Thm. 4.3, because $CL(\mathfrak{L})$ is always distributive, complete, compactly generated and upper continuous. Q.E.D.

Proposition 5. *Let $\mathfrak{L} = (L, \wedge, \vee)$ be a distributive lattice. If an element $x \in CL(\mathfrak{L})$ (or $x \in TL(\mathfrak{L})$) is join of a set A of atoms and $a \in \mathscr{At}(CL(\mathfrak{L}))$ (or $a \in \mathscr{At}(TL(\mathfrak{L}))$), then $a \leqq x$ implies $a \in A$. In other words, the set A is uniquely determined by the element x, i.e. $A = \{a \in \mathscr{At}(CL(\mathfrak{L})) \mid a \leqq x\}$ (or $A = \{a \in \mathscr{At}(TL(\mathfrak{L})) \mid a \leqq x\}$).*

Proof. Both $CL(\mathfrak{L})$ and $TL(\mathfrak{L})$ are infinitely distributive. Hence $a \leqq x = \bigvee_{i\in I} a_i$ implies $a = a \wedge x = a \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I}(a \wedge a_i)$. If a and a_i are atoms, $a \neq a_i$ implies $a \wedge a_i = \Delta$. Thus, there exists $i \in I$, $a = a_i$. Q.E.D.

Denote by $\langle \mathscr{T} \rangle_{\mathfrak{A}}$ the partition of $TL(\mathfrak{A})$ corresponding to $\mathscr{C} : TL(\mathfrak{A}) \to CL(\mathfrak{A})$. Obviously $\langle \mathscr{T} \rangle_{\mathfrak{A}} = \{\mathscr{T}(\Theta) \mid \Theta \in CL(\mathfrak{A})\}$. Another natural partition can be constructed on the tolerance lattice $TL(\mathfrak{A})$. Denote by $\mathscr{A} : TL(\mathfrak{A}) \to \mathscr{E}xp(\mathscr{A}t(TL(\mathfrak{A})))$ the mapping $T \mapsto \{a \in \mathscr{A}t(TL(\mathfrak{A})) \mid a \leqq T\}$. Put $\mathscr{S}(A) = \{T \in TL(\mathfrak{A}) \mid \mathscr{A}(T) = A\}$ for each $A \in \mathscr{E}xp(\mathscr{A}t(TL(\mathfrak{A})))$. The partition corresponding to $\mathscr{A}$ will be denoted by $\langle \mathscr{S} \rangle_{\mathfrak{A}}$. Clearly $\langle \mathscr{S} \rangle_{\mathfrak{A}} = \{\mathscr{S}(A) \mid A \in \mathscr{E}xp(\mathscr{A}t(TL(\mathfrak{A})))\}$. $\mathscr{E}xp(\mathscr{A}t(TL(\mathfrak{A})))$ can be regarded as a Boolean lattice.

Proposition 6. *The mapping $\mathscr{A}$ is a complete meet-homomorphism. If $TL(\mathfrak{A})$ is distributive, then $\mathscr{A}$ is also a complete join-homomorphism.*

Proof. Obviously, $\mathscr{A}$ is an order homomorphism. Consequently, $\bigwedge_{i \in I} \mathscr{A}(T_i) \leqq \leqq \mathscr{A}(\bigvee \bigwedge_{i \in I} \mathscr{A}(T_i)) \leqq \mathscr{A}(\bigwedge_{i \in I} T_i) \leqq \bigwedge_{i \in I} \mathscr{A}(T_i)$ for an arbitrary family of compatible tolerances $\{T_i\}_{i \in I}$. The first assertion is proved. Let $TL(\mathfrak{A})$ be distributive. It always holds $\bigvee_{i \in I} \mathscr{A}(T_i) \leqq \mathscr{A}(\bigvee_{i \in I} T_i)$. The tolerance lattice $TL(\mathfrak{A})$ is infinitely distributive and $a \in \mathscr{A}(\bigvee_{i \in I} T_i)$ implies $a = a \wedge \bigvee_{i \in I} T_i = \bigvee_{i \in I}(a \wedge T_i)$, hence there is an $i \in I$ such that $a \in \mathscr{A}(T_i)$. Thus $\bigvee_{i \in I} \mathscr{A}(T_i) = \mathscr{A}(\bigvee_{i \in I} T_i)$. Q.E.D.

Corollary. *Each block $\mathscr{S}(A)$ of $\langle \mathscr{S} \rangle_{\mathfrak{A}}$ contains its least element. If $TL(\mathfrak{A})$ is distributive, all $\mathscr{S}(A)$ contain their greatest elements.*

A natural question arises, what is the relation between the two partitions of $TL(\mathfrak{A})$ mentioned above.

Proposition 7. *Let $\mathfrak{L} = (L, \wedge, \vee)$ be a lattice. Then T and $\mathscr{C}(T)$ include the same atoms in $TL(\mathfrak{L})$ provided $T \in TL(\mathfrak{L})$.*

Proof. Obviously $\mathscr{A}(T) \leqq \mathscr{A}(\mathscr{C}(T))$. By Proposition 1, $a \in \mathscr{A}(\mathscr{C}(T))$ implies $\mathscr{C}(a \wedge T) = \mathscr{C}(a) \wedge \mathscr{C}(T) \geqq a$, thus $a \wedge T \neq \Delta$ and consequently $a \leqq T$, i.e. $a \in \mathscr{A}(T)$. Q.E.D.

Corollary. *For a lattice $\mathfrak{L}$, $\langle \mathscr{T} \rangle_{\mathfrak{L}}$ is a refinement of $\langle \mathscr{S} \rangle_{\mathfrak{L}}$.*

Proposition 8. *For a distributive lattice $\mathfrak{L} = (L, \wedge, \vee)$, the following assertions are equivalent:*

(i) *$\mathfrak{L}$ is locally finite;*

(ii) $\langle \mathscr{T} \rangle_{\mathfrak{L}} = \langle \mathscr{S} \rangle_{\mathfrak{L}}$;

(iii) *$T \in TL(\mathfrak{L})$ is a congruence if and only if each element of $TL(\mathfrak{L})$ including the same atoms as T is less than T or equal to T.*

Proof. (i) $\Rightarrow$ (ii): If $\mathfrak{L}$ is locally finite, then for any $A \in \mathscr{E}xp(\mathscr{A}t(TL(\mathfrak{L})))$, $\mathscr{S}(A)$ contains only a unique congruence, $\bigvee_{CL} A$. Hence $T \in \mathscr{S}(A)$ implies $\mathscr{C}(T) = \bigvee_{CL} A$ and consequently $\mathscr{S}(A) = \mathscr{T}(\bigvee_{CL} A)$, i.e. $\langle \mathscr{S} \rangle_{\mathfrak{L}} = \langle \mathscr{T} \rangle_{\mathfrak{L}}$.

(ii) $\Rightarrow$ (iii): Let $\langle \mathscr{T} \rangle_{\mathfrak{L}} = \langle \mathscr{S} \rangle_{\mathfrak{L}}$. If $T \in TL(\mathfrak{L})$ is a congruence, T is the greatest element in $\mathscr{T}(T) = \mathscr{S}(\mathscr{A}(T))$. On the other hand, if each element of $TL(\mathfrak{L})$ including the same atoms as T is less than T or equal to T, then T is the greatest element of $\mathcal{S}(\mathcal{A}(T)) = \mathcal{T}(\mathcal{C}(T))$, hence a congruence.

(iii) $\Rightarrow$ (i): If (iii) holds, the all-relation is the only congruence on $\mathfrak{L}$ including the set of all atoms in $TL(\mathfrak{L})$, so that it is the join of all atoms in $CL(\mathfrak{L})$. By Proposition 4, $\mathfrak{L}$ is locally finite. Q.E.D.

It was proved that if $\mathfrak{L} = (L, \wedge, \vee)$ is a locally finite distributive lattice, the least congruence $\mathcal{C}(T)$ including a given element T of the tolerance lattice $TL(\mathfrak{L})$ can be found without knowing the nature of elements; it is the greatest element in $TL(\mathfrak{L})$ including the same atoms as T.

The tolerance lattice of the four-element chain may serve as an illustration:

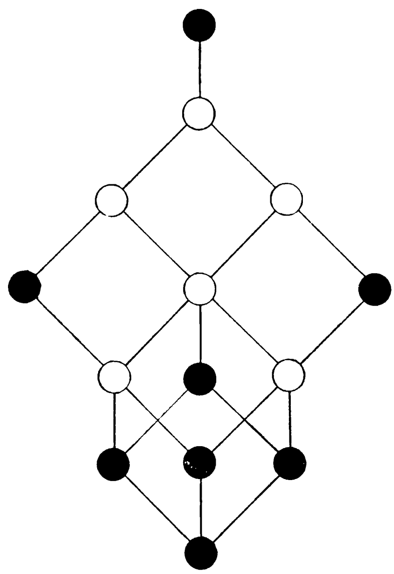

Fig. 1.

Remark. In this paper, infinitely distributive means satisfying the Join Infinite Distributive Identity $x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i)$.

References

[1] *P. Crawley* and *R. P. Dilworth:* Algebraic theory of lattices. Prentice Hall, Englewood Cliffs, N.J. 1973.

[2] *I. Chajda* and *B. Zelinka:* Tolerance relation on lattices. Časop. pěst. mat. *99* (1974), 394—399.

[3] *I. Chajda* and *B. Zelinka:* Minimal compatible tolerances on lattices. Czech. Math. J. *27* (1977), 452—459.

[4] *I. Chajda* and *B. Zelinka:* Lattices of tolerances. Časop. pěst. mat. *102* (1977), 10—24.

[5] *B. Zelinka:* Tolerance in algebraic structures II. Czech. Math. J. *25* (1975), 175—178.

Author's address: 615 00 Brno 15, Viniční 60.