Miroslav Sova

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ON THE EQUIVALENCE OF WIDDER-MIYADERA'S AND LEVIATAN'S REPRESENTABILITY CONDITIONS FOR THE LAPLACE TRANSFORM OF INTEGRABLE VECTOR-VALUED FUNCTIONS

MIROSLAV SOVA, Praha

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There are two theories of representability of functions as Laplace transforms of vector-valued functions integrable with a power greater than one: that of Widder-Miyadera [1], [2] based on the behaviour of certain integrals of derivatives and the other of Leviatan [3] based on the behaviour of certain sums of derivatives. The purpose of this paper is to give a direct proof of equivalence of these conditions which is desirable because both theories use different technical tools.

The main result is given in Proposition 8 which is new also in the simplest, i.e. numerical, case. This result is then applied in Theorem 9.

1. We shall denote: (1) $\mathbb{R}$ – the real number field, (2) (ω, ∞) – the set of all real numbers greater than ω where $\omega \in \mathbb{R}$, (3) $M_1 \to M_2$ – the set of all mappings of the whole set M_1 into the set M_2.

2. By E we denote an arbitrary Banach space over $\mathbb{R}$ with the norm $\|\cdot\|$.

3. We need only the most elementary properties of Banach spaces and of functions with values in a Banach space.

4. Proposition. *Let* $F \in (0, \infty) \to E$. *If*

(α) *the function* F *is infinitely differentiable on* $(0, \infty)$,

(β) $F(\lambda) \to 0$ $(\lambda \to \infty)$,

(γ) *there exist constants* $M \geqq 0$ *and* $\vartheta > 1$ *so that*

$$\int_0^\infty \mu^{\vartheta p + \vartheta - 2} \|F^{(q)}(\mu)\|^\vartheta \, \mathrm{d}\mu \leqq M \frac{(p!)^\vartheta}{p} \text{ for every } p \in \{1, 2, \ldots\},$$

then

(a) *the functions* $e^{-\lambda t}t^p((q+1)/t)^{q+1}F^{(q)}((q+1)/t)$ *are integrable over* $(0, \infty)$ *for every* $\lambda > 0$ *and* $p, q \in \{0, 1, \ldots\}$,

(b) $$\frac{(-1)^{p+q}}{q!}\int_0^\infty e^{-\lambda\tau}\tau^p\left(\frac{q+1}{\tau}\right)^{q+1}F^{(q)}\left(\frac{q+1}{\tau}\right)\mathrm{d}\tau \underset{q\to\infty}{\to^{\text{weakly}}} F^{(p)}(\lambda)$$

for every $\lambda > 0$ *and* $p \in \{0, 1, \ldots\}$.

Proof. We obtain from the assumptions (α) and (γ) by means of Hölder's inequality and substitution that for every $\lambda > 0$ and $p, q \in \{0, 1, \ldots\}$,

$$\int_0^\infty e^{-\lambda\tau}\tau^p\left(\frac{q+1}{\tau}\right)^{q+1}\left\|F^{(q)}\left(\frac{q+1}{\tau}\right)\right\|\mathrm{d}\tau \leqq$$

$$\leqq\left[\int_0^\infty\left(\left(\frac{q+1}{\tau}\right)^{q+1}\left\|F^{(q+1)}\left(\frac{q+1}{\tau}\right)\right\|\right)^\vartheta\mathrm{d}\tau\right]^{1/\vartheta}\left[\int_0^\infty(e^{-\lambda\tau}\tau^p)^{\vartheta/(\vartheta-1)}\right]^{(\vartheta-1)/\vartheta} =$$

$$=\left[(q+1)\int_0^\infty\mu^{\vartheta q+\vartheta-2}\|F^{(q)}(\mu)\|^\vartheta\,\mathrm{d}\mu\right]^{1/\vartheta}\left[\int_0^\infty(e^{-\lambda\tau}\tau^p)^{\vartheta/(\vartheta-1)}\right]^{(\vartheta-1)/\vartheta} \leqq$$

$$\leqq M^{1/\vartheta}(q+1)!\left[\int_0^\infty(e^{-\lambda\tau}\tau^p)^{\vartheta/(\vartheta-1)}\right]^{\vartheta/(\vartheta-1)}$$

which immediately gives the property (a).

Now we turn to the proof of the property (b).

We obtain again from the assumptions (α) and (γ) by means of Hölder's inequality and substitution that

(1) $$\left\|\int_0^t\frac{(-1)^k}{k!}\left(\frac{k}{\tau}\right)^{k+1}F^{(k)}\left(\frac{k}{\tau}\right)\mathrm{d}\tau\right\| \leqq \int_0^t\left\|\frac{1}{k!}\left(\frac{k}{\tau}\right)^{k+1}F^{(k)}\left(\frac{k}{\tau}\right)\right\|\mathrm{d}\tau \leqq$$

$$\leqq\left[\int_0^t\left\|\frac{1}{k!}\left(\frac{k}{\tau}\right)^{k+1}F^{(k)}\left(\frac{k}{\tau}\right)\right\|^\vartheta\mathrm{d}\tau\right]^{1/\vartheta}t^{(\vartheta-1)/\vartheta} =$$

$$=\left[\int_0^t\frac{1}{(k!)^\vartheta}\left(\frac{k}{\tau}\right)^{(k+1)\vartheta}\left\|F^{(k)}\left(\frac{k}{\tau}\right)\right\|^\vartheta\mathrm{d}\tau\right]^{1/\vartheta}t^{(\vartheta-1)/\vartheta} =$$

$$=\left[\int_{1/t}^\infty\frac{k}{(k!)^\vartheta}\mu^{k\vartheta+\vartheta-2}\|F^{(k)}(\mu)\|^\vartheta\,\mathrm{d}\mu\right]^{1/\vartheta}t^{(\vartheta-1)/\vartheta} \leqq M^{1/\vartheta}t^{(\vartheta-1)/\vartheta} \text{ for every } t > 0$$

and $k \in \{1, 2, \ldots\}$.

Let E^* be the set of all continuous linear functionals on E.

We see from (1) that

$$(2)\qquad \left|\int_0^t \frac{(-1)^k}{k!}\left(\frac{k}{\tau}\right)^{k+1} (lF)^{(k)}\left(\frac{k}{\tau}\right) \mathrm{d}\tau\right| \leqq \|l\|\, M^{1/\vartheta}\, t^{(\vartheta-1)/\vartheta} \text{ for every } t > 0,$$

$k \in \{1, 2, \ldots\}$ and $l \in E^*$.

The inequality (2) enables us to apply Theorem 11b, Chap. VII of [1] to every function lF, $l \in E^*$, and, taking into account the assumption (β), we obtain immediately that (b) holds for $p = 0$. The general validity of (b) can be then proved by induction on p analogously as in the proof of Lemma 4.15 in [5].

Remark. A simple and direct proof of a more general version of Proposition 3 (under weaker assumptions and with strong convergence) will be given in [4].

5. Lemma. *Let $F \in (0, \infty) \to E$. If the function F is infinitely differentiable on $(0, \infty)$, then for every $\lambda > 0$ and $p \in \{0, 1, \ldots\}$,*

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} F(s - s\mathrm{e}^{-\lambda/s}) \to_{s\to\infty} F^{(p)}(\lambda)\,.$$

Proof. We first need to prove that
(1) for every $p \in \{1, 2, \ldots\}$, there exist constants $a_1, a_2, \ldots, a_p$ such that $a_p = 1$ and

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} F(s - s\mathrm{e}^{-\lambda/s}) = (-1)^p \sum_{i=1}^{p} (-1)^i a_i \frac{\mathrm{e}^{-i\lambda/s}}{s^{p-i}} F^{(i)}(s - s\mathrm{e}^{-\lambda/s})$$

for every $\lambda > 0$ and $s > 0$.

In proving (1) we proceed by induction on p. The case $p = 1$ being clearly in order, we pass to the verification of the induction step.

First, we get

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}} F(s - s\mathrm{e}^{-\lambda/s}) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[(-1)^p \sum_{i=1}^{p} (-1)^i a_i \frac{\mathrm{e}^{-i\lambda/s}}{s^{p-i}} F^{(i)}(s - s\mathrm{e}^{-\lambda/s})\right] =$$

$$= (-1)^p \sum_{i=1}^{p} (-1)^{i+1}\, i a_i \frac{\mathrm{e}^{-i\lambda/s}}{s^{p+1-i}} F^{(i)}(s - s\mathrm{e}^{-\lambda/s}) +$$

$$+ (-1)^p \sum_{i=1}^{p} (-1)^i a_i \frac{\mathrm{e}^{-(i+1)\lambda/s}}{s^{p-i}} F^{(i+1)}(s - s\mathrm{e}^{-\lambda/s}) =$$

$$= (-1)^p \sum_{i=1}^{p} (-1)^{i+1}\, i a_i \frac{\mathrm{e}^{-i\lambda/s}}{s^{p+1-i}} F^{(i)}(s - s\mathrm{e}^{-\lambda/s}) +$$

$$+ (-1)^p \sum_{i=2}^{p+1} (-1)^{i+1} a_{i-1} \frac{\mathrm{e}^{-i\lambda/s}}{s^{p+1-i}} F^{(i)}(s - s\mathrm{e}^{-\lambda/s})\,.$$

Further, we put

$$a'_1 = a_1\,, \quad a'_{p+1} = 1\,, \quad a'_j = a_j + (j-1)\,a_{j-1}$$

for every $j \in \{2, 3, \ldots, p\}$.

Then we get from the preceding considerations

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}\lambda^{p+1}}\,F(s - s\mathrm{e}^{-\lambda/s}) = (-1)^{p+1}\sum_{j=1}^{p+1}(-1)^j\,a'_j\,\frac{\mathrm{e}^{-j\lambda/s}}{s^{p+1-j}}\,F^{(j)}(s - s\mathrm{e}^{-\lambda/s})$$

which confirms the validity of the induction step. Hence (1) is proved.

Now the statement of our Proposition is an immediate consequence of (1).

6. Lemma. *Let $\varphi, \psi \in (0, \infty) \to \mathbb{R}$. If the functions φ, ψ are continuous and nonnegative, then for every $\vartheta > 1$,*

$$\left[\int_0^\infty \varphi(\eta)\,\psi(\eta)\,\mathrm{d}\eta\right]^\vartheta \leqq \left[\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta\right]^{\vartheta-1}\int_0^\infty \varphi(\eta)\,(\psi(\eta))^\vartheta\,\mathrm{d}\eta\,.$$

Proof. It is clear that we can suppose $\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta > 0$.

Let now $\vartheta > 1$ be fixed. For $a_0 \geqq 0$, $a \geqq 0$ we get

$$a^\vartheta - a_0^\vartheta - \vartheta a_0^{\vartheta-1}(a - a_0) = \int_{a_0}^a \frac{\mathrm{d}}{\mathrm{d}\eta}\,(\eta^\vartheta - a_0^\vartheta - \vartheta a_0^{\vartheta-1}(\eta - a_0))\,\mathrm{d}\eta =$$

$$= \vartheta\int_{a_0}^a (\eta^{\vartheta-1} - a_0^{\vartheta-1})\,\mathrm{d}\eta \geqq 0$$

and consequently

$$a^\vartheta - a_0^\vartheta \geqq \vartheta a_0^{\vartheta-1}(a - a_0)\,.$$

Taking $a_0 = (\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta)^{-1}\int_0^\infty \varphi(\eta)\,\psi(\eta)\,\mathrm{d}\eta$ and $a = \psi(\eta)$ we get

$$(\psi(\eta))^\vartheta - \left[\left(\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta\right)^{-1}\int_0^\infty \varphi(\eta)\,\psi(\eta)\,\mathrm{d}\eta\right]^\vartheta \geqq$$

$$\geqq \vartheta a_0^{\vartheta-1}\left[\psi(\eta) - \left(\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta\right)^{-1}\int_0^\infty \varphi(\eta)\,\psi(\eta)\,\mathrm{d}\eta\right].$$

Multiplying this inequality by $\varphi(\eta)$ and integrating over $(0, \infty)$ we get evidently

$$\int_0^\infty \varphi(\eta)\,(\psi(\eta))^\vartheta\,\mathrm{d}\eta - \left(\int_0^\infty \varphi(\eta)\,\mathrm{d}\eta\right)^{\vartheta-1}\left(\int_0^\infty \varphi(\eta)\,\psi(\eta)\,\mathrm{d}\eta\right)^\vartheta \geqq 0$$

which gives the desired inequality.

Remark. The above inequality is a special case of the Jensen inequality. Since this inequality is still infrequent in standard text-books of advanced calculus (at least in the above simple form), we give its proof.

7. Lemma. $e^{-a\xi}\xi^p \leqq e^{-p}p^p/a^p$ *for every* $a > 0$, $\xi > 0$ *and* $p \in \{1, 2, \ldots\}$.

Proof. It suffices to find the maximum of the function $e^{-a\xi}\xi^p$, $\xi > 0$, by standard methods.

8. Proposition. *Let* $F \in (0, \infty) \to E$, $M \geqq 0$ *and* $1 < \vartheta < \infty$. *If the function* F *is infinitely differentiable on* $(0, \infty)$, *then the following two statements are equivalent*:

(W) (I) $F(\lambda) \to 0 \ (\lambda \to \infty)$,

(II) $\displaystyle\int_0^\infty \mu^{\vartheta p + \vartheta - 2} \|F^{(p)}(\mu)\|^\vartheta \, d\mu \leqq \frac{M(p!)^\vartheta}{p}$ *for every* $p \in \{1, 2, \ldots\}$,

(L) $$\sum_{p=0}^\infty \left[\frac{\lambda^p}{p!}\|F^{(p)}(\lambda)\|\right]^\vartheta \leqq \frac{M}{\lambda^{\vartheta - 1}} \text{ for every } \lambda > 0.$$

Proof. (W) $\Rightarrow$ (L). For the sake of simplicity, let us define

(1) $$f_q(t) = \frac{(-1)^q}{q!}\left(\frac{q+1}{t}\right)^{q+1} F^{(q)}\left(\frac{q+1}{t}\right) \text{ for every } t > 0 \text{ and } q \in \{0, 1, \ldots\}.$$

According to Proposition 4 we get from (W)(I) and (1) that

(2) the functions $e^{-\lambda t}t^p f_q(t)$ are integrable over $(0, \infty)$ for every $\lambda > 0$ and $p, q \in \{0, 1, \ldots\}$,

(3) $$(-1)^p \int_0^\infty e^{-\lambda\tau}\tau^p f_q(\tau) \, d\tau \xrightarrow[q \to \infty]{\text{weakly}} F^{(p)}(\lambda) \text{ for every } \lambda > 0 \text{ and } p \in \{0, 1, \ldots\}.$$

It follows from (W) (II) after a simple substitution that

(4) $$\int_0^\infty \|f_q(\tau)\|^\vartheta \, d\tau = \frac{1}{(q!)^\vartheta}\int_0^\infty \left(\frac{q+1}{\tau}\right)^{\vartheta q + \vartheta} \left\|F^{(q)}\left(\frac{q+1}{\tau}\right)\right\|^\vartheta d\tau =$$
$$= \frac{q+1}{(q!)^\vartheta}\int_0^\infty \mu^{\vartheta q + \vartheta - 2}\|F^{(q)}(\mu)\|^\vartheta \, d\mu \leqq M\frac{q+1}{q} \text{ for every } q \in \{1, 2, \ldots\}.$$

On the other hand, using the identity

$$\int_0^\infty \frac{(\lambda\tau)^p}{p!} e^{-\lambda\tau} \, d\tau = \frac{1}{\lambda}$$

for every $\lambda > 0$ and $p \in \{0, 1, \ldots\}$ we obtain from Lemma 6 that

(5) $$\left(\int_0^\infty \frac{(\lambda\tau)^p}{p!} e^{-\lambda\tau} \|f_q(\tau)\| \, d\tau\right)^\vartheta \leqq \frac{1}{\lambda^{\vartheta-1}} \int_0^\infty \frac{(\lambda\tau)^p}{p!} e^{-\lambda\tau} \|f_q(\tau)\|^\vartheta \, d\tau \text{ for every } \lambda > 0,$$

$p \in \{0, 1, \ldots\}$ and $q \in \{0, 1, \ldots\}$.

It follows from (4) and (5) that

(6) $$\sum_{p=0}^{\infty} \left\| \frac{\lambda^p}{p!} \int_0^\infty e^{-\lambda\tau} \tau^p f_q(\tau) \, d\tau \right\|^\vartheta \leqq \sum_{p=0}^{\infty} \left(\int_0^\infty \frac{(\lambda\tau)^p}{p!} e^{-\lambda\tau} \|f_q(\tau)\| \, d\tau\right)^\vartheta \leqq$$

$$\leqq \frac{1}{\lambda^{\vartheta-1}} \sum_{p=0}^{\infty} \int_0^\infty \frac{(\lambda\tau)^p}{p!} e^{-\lambda\tau} \|f_q(\tau)\|^\vartheta \, d\tau = \frac{1}{\lambda^{\vartheta-1}} \int_0^\infty \left(\sum_{p=0}^{\infty} \frac{(\lambda\tau)^p}{p!}\right) e^{-\lambda\tau} \|f_q(\tau)\|^\vartheta \, d\tau =$$

$$= \frac{1}{\lambda^{\vartheta-1}} \int_0^\infty \|f_q(\tau)\|^\vartheta \, d\tau \leqq \frac{q+1}{q} M \frac{1}{\lambda^{\vartheta-1}} \text{ for every } q \in 1\{, 2, \ldots\}.$$

Letting now q tend to infinity we obtain easily from (3) and (6) that (L) holds.

(L) $\Rightarrow$ (W). First we prove

(1) the series $\sum_{k=0}^{\infty} (\xi - \alpha)^k / k! \, F^{(k)}(\alpha)$ is absolutely convergent for every $\alpha > 0$ and $|\xi - \alpha| < \alpha$.

Indeed, it follows from (L) by means of Hölder's inequality that

$$\sum_{k=0}^{\infty} \frac{|\xi - \alpha|^k}{k!} \|F^{(k)}(\alpha)\| = \sum_{k=0}^{\infty} \left(\frac{|\xi - \alpha|}{\alpha}\right)^k \left(\frac{\alpha^k}{k!} \|F^{(k)}(\alpha)\|\right) \leqq$$

$$\leqq \left[\sum_{k=0}^{\infty} \left(\frac{\xi - \alpha}{\alpha}\right)^{k(\vartheta/(\vartheta-1))}\right]^{(\vartheta-1)/\vartheta} \left[\sum_{k=0}^{\infty} \left(\frac{\alpha^k}{k!} \|F^{(k)}(\alpha)\|\right)^\vartheta\right]^{1/\vartheta} \leqq$$

$$\leqq M \left[\sum_{k=0}^{\infty} \left\{\left(\frac{|\xi - \alpha|}{\alpha}\right)^{\vartheta/(\vartheta-1)}\right\}^k\right]^{(\vartheta-1)/\vartheta}.$$

Since the last series is a geometrical series with quotient less than 1 the desired property follows.

It follows from (1) by means of Taylor's theorem that

(2) $$F(\xi) = \sum_{k=0}^{\infty} \frac{(\xi - \alpha)^k}{k!} F^{(k)}(\alpha) \text{ for every } \alpha > 0 \text{ and } |\xi - \alpha| < \alpha.$$

Taking $\alpha = s$ and $\xi = s - s e^{-\lambda/s}$ we get from (1) and (2) that

(3) the series $\sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} F^{(k)}(s)$ is absolutely convergent for every $\lambda > 0$ and $s > 0$,

(4) $F(s - se^{-\lambda/s}) = \sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} F^{(k)}(s)$ for every $\lambda > 0$ and $s > 0$.

On the other hand, we can prove that

(5) the series $\sum_{k=1}^{\infty} e^{-\lambda k/s} \left(-\frac{s}{k}\right)^p \frac{(-s)^k}{k!} F^{(k)}(s)$ is absolutely convergent for every $\lambda > 0$, $s > 0$ and $p \in \{1, 2, \ldots\}$,

(6) the series $\sum_{k=1}^{\infty} e^{-\lambda k/s} \left(-\frac{s}{k}\right)^p \frac{(-s)^k}{k!} F^{(k)}(s)$ is uniformly convergent in $\lambda > \lambda_0 > 0$ for every $s > 0$ and $p \in \{1, 2, \ldots\}$.

Indeed, by Hölder's inequality we obtain from (L) that

(7) $$\sum_{k=i}^{j} e^{-\lambda k/s} \left(\frac{k}{s}\right)^p \frac{s^k}{k!} \|F^{(k)}(s)\| \leqq$$
$$\leqq \left[\sum_{k=i}^{j} \left(e^{-\lambda k/s} \left(\frac{k}{s}\right)^p\right)^{\vartheta/(\vartheta-1)}\right]^{(\vartheta-1)/\vartheta} \left[\sum_{k=i}^{j} \left(\frac{s^k}{k!} \|F^{(k)}(s)\|\right)^{\vartheta}\right]^{1/\vartheta} \leqq$$
$$\leqq \left[\frac{M}{s^{\vartheta-1}}\right]^{1/\vartheta} \left[\sum_{k=i}^{j} \left(e^{-\lambda k/s} \left(\frac{k}{s}\right)^p\right)^{\vartheta/(\vartheta-1)}\right]^{(\vartheta-1)/\vartheta} \quad \text{for every } \lambda > 0,\ s > 0,$$

$p \in \{1, 2, \ldots\}$ and $i, j \in \{1, 2, \ldots\}$, $i < j$.

Further, by Lemma 7 with $a = \lambda/2s$ we get

(8) $$\sum_{k=i}^{j} \left(e^{-\lambda k/s} \left(\frac{k}{s}\right)^p\right)^{\vartheta/(\vartheta-1)} = \frac{1}{s^{p(\vartheta/(\vartheta-1))}} \sum_{k=i}^{j} \left(e^{-(\lambda/s)k} k^p\right)^{\vartheta/(\vartheta-1)} \leqq$$
$$\leqq \frac{1}{s^{p(\vartheta/(\vartheta-1))}} \left(\frac{e^{-p} p^p (2s)^p}{\lambda^p}\right)^{\vartheta/(\vartheta-1)} \sum_{k=i}^{j} e^{-(\lambda/2s)k(\vartheta/(\vartheta-1))} =$$
$$= \left(\frac{2^p e^{-p} p^p}{\lambda^p}\right)^{\vartheta/(\vartheta-1)} \sum_{k=i}^{j} \left(e^{-(\lambda/2s)(\vartheta/(\vartheta-1))}\right)^k \quad \text{for every } \lambda > 0,\ s > 0,\ p \in \{1, 2, \ldots\}$$

and $i, j \in \{1, 2, \ldots\}$, $i < j$.

The statements (5) and (6) now follow immediately from (7) and (8).
We obtain from (3) and (6) that

(9) $$\frac{d^p}{d\lambda^p} \sum_{k=0}^{\infty} e^{-\lambda k/s} \frac{(-s)^k}{k!} F^{(k)}(s) = \sum_{k=1}^{\infty} e^{-\lambda k/s} \left(-\frac{k}{s}\right)^p \frac{(-s)^k}{k!} F^{(k)}(s) \quad \text{for every } \lambda > 0,$$

$s > 0$ and $p \in \{1, 2. \ldots\}$.

Consequently, by (4) and (9),

$$(10)\ \frac{d^p}{d\lambda^p} F(s - se^{-\lambda/s}) = \sum_{k=1}^{\infty} e^{-\lambda k/s}\left(-\frac{k}{s}\right)^p \frac{(-s)^k}{k!} F^{(k)}(s) \text{ for every } \lambda > 0,\ s > 0$$

$p \in \{1, 2, \dots\}$.

It is easy to see that

$$(11)\ \sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p = se^{-\mu s}\sum_{k=1}^{\infty}\frac{1}{s}e^{-\mu(k+1)/s}\left(\frac{k}{s}\right)^p \leqq$$

$$\leqq se^{-\mu s}\int_0^{\infty} e^{-\mu\tau}\tau^p\, d\tau = se^{-\mu s}\frac{p!}{\mu^{p+1}} \text{ for every } \mu > 0,\ s > 0 \text{ and } p \in \{0, 1, \dots\}.$$

It follows from (L) by means of (11) and of Hölder's inequality that

$$(12)\ \left\|\sum_{k=1}^{\infty} e^{-\mu k/s}\left(-\frac{k}{s}\right)^p \frac{(-s)^k}{k!} F^{(k)}(s)\right\|^{\vartheta} \leqq \left\{\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p \frac{s^k}{k!}\|F^{(k)}(s)\|\right\}^{\vartheta} =$$

$$= \left\{\sum_{k=1}^{\infty}\left[e^{-\mu k/s}\left(\frac{k}{s}\right)^p\right]^{(\vartheta-1)/\vartheta}\left[\left(e^{-\mu k/s}\left(\frac{k}{s}\right)^p\right)^{1/\vartheta}\frac{s^k}{k!}\|F^{(k)}(s)\|\right]\right\}^{\vartheta} \leqq$$

$$\leqq \left[\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p\right]^{\vartheta-1}\left[\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta}\right] \leqq$$

$$\leqq \left[se^{-\mu s}\frac{p!}{\mu^{p+1}}\right]^{\vartheta-1}\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta} \text{ for every } \mu > 0,\ s > 0 \text{ and}$$

$p \in \{0, 1, \dots\}$.

Now (12) together with (L) implies the following important estimate:

$$(13)\ \int_0^{\infty}\mu^{p\vartheta+\vartheta-2}\left\|\sum_{k=1}^{\infty} e^{-\mu k/s}\left(-\frac{k}{s}\right)^p\frac{(-s)^k}{k!}F^{(k)}(s)\right\|^{\vartheta} d\mu \leqq$$

$$\leqq \int_0^{\infty}\mu^{p\vartheta+\vartheta-2}\left[se^{-\mu s}\frac{p!}{\mu^{p+1}}\right]^{\vartheta-1}\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta} d\mu =$$

$$= \frac{(p!)^{\vartheta}}{p!}s^{\vartheta-1}\int_0^{\infty} e^{-\mu s(\vartheta-1)}\mu^{p-1}\sum_{k=1}^{\infty} e^{-\mu k/s}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta} d\mu =$$

$$= \frac{(p!)^{\vartheta}}{p!}s^{\vartheta-1}\sum_{k=1}^{\infty}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta}\int_0^{\infty} e^{-\mu((k/s)+s(\vartheta-1))}\mu^{p-1}\, d\mu =$$

$$= \frac{(p!)^{\vartheta}}{p!}s^{\vartheta-1}\sum_{k=1}^{\infty}\left(\frac{k}{s}\right)^p\left(\frac{s^k}{k!}\|F^{(k)}(s)\|\right)^{\vartheta}\frac{(p-1)!}{\left(\frac{k}{s}+s(\vartheta-1)\right)^p} \leqq$$

$$\leqq \frac{(p!)^{\vartheta}}{p!} s^{\vartheta-1} \sum_{k=1}^{\infty} \left(\frac{k}{s}\right)^{p} \left(\frac{s^k}{k!} \|F^{(k)}(s)\|\right)^{\vartheta} \frac{(p-1)!}{\left(\frac{k}{s}\right)^{p}} \leqq$$

$$\leqq \frac{(p!)^{\vartheta}}{p} s^{\vartheta-1} \sum_{k=1}^{\infty} \left(\frac{s^k}{k!} \|F^{(k)}(s)\|\right)^{\vartheta} \leqq M \frac{(p!)^{\vartheta}}{p} \text{ for every } s > 0 \text{ and } p \in \{1, 2, \ldots\}.$$

The above preparatory considerations enable us to conclude the proof of (L) ⇒ ⇒ (W).

It follows from (10) and (13) that

(14) $\int_0^{\infty} \mu^{\vartheta p+\vartheta-2} \left\| \frac{d^p}{d\mu^p} F(s - se^{-\mu/s}) \right\|^{\vartheta} d\mu \leqq M \frac{(p!)^{\vartheta}}{p}$ for every $s > 0$ and $p \in \{1, 2, \ldots\}$.

By Fatou's lemma and Lemma 5 we get from (14), letting s tend to infinity, that

(15) $\int_0^{\infty} \mu^{\vartheta p+\vartheta-2} \|F^{(p)}(\mu)\|^{\vartheta} d\mu \leqq M \frac{(p!)^{\vartheta}}{p}$ for every $p \in \{1, 2, ..\}$.

On the other hand, as an immediate consequence of (L) we have

(16) $F(\lambda) \to 0 \ (\lambda \to \infty)$.

From (15) and (16) we see that (W) is proved.

9. Theorem. *Let $\vartheta > 1$, $M \geqq 0$, $\omega \geqq 0$ and $F \in (\omega, \infty) \to E$. If the space E is reflexive, then the following three statements* (A), (B) *and* (C) *are equivalent*:

(A) (I) *the function F is infinitely differentiable on (ω, ∞),*

(II) $F(\lambda) \to 0 \ (\lambda \to \infty)$,

(III) $\int_{\omega}^{\infty} (\mu - \omega)^{\vartheta p+\vartheta-2} \|F^{(p)}(\mu)\|^{\vartheta} d\mu \leqq \frac{M(p!)^{\vartheta}}{p}$ *for every* $p \in \{1, 2, \ldots\}$,

(B) (I) *the function F is infinitely differentiable on (ω, ∞),*

(II) $\sum_{p=0}^{\infty} \left[\frac{(\lambda - \omega)^p}{p!} \|F^{(p)}(\lambda)\|\right]^{\vartheta} \leqq \frac{M}{(\lambda - \omega)^{\vartheta-1}}$ *for every* $\lambda > \omega$,

(C) *there exists a function $f \in (0, \infty) \to E$ such that*

(I) *f is measurable on* $(0, \infty)$,

(II) $\int_0^{\infty} [e^{-\omega\tau} \|f(\tau)\|]^{\vartheta} d\tau < \infty$,

(III) $\int_0^{\infty} e^{-\lambda\tau} f(\tau) d\tau = F(\lambda)$ *for every* $\lambda > \omega$.

Proof. First, we find easily that we can restrict ourselves to the case $\omega = 0$. In this case, the equivalence (A) $\Leftrightarrow$ (B) is proved by Proposition 8.

The remaining equivalences may be proved in two ways. We can use Widder-Miyadera's theorem [2], i.e. the equivalence (A) $\Leftrightarrow$ (C) and we get at once the equivalence (B) $\Leftrightarrow$ C; or we can start with Leviatan's theorem [3], i.e. with the equivalence (B) $\Leftrightarrow$ (C) and the equivalence (A) $\Leftrightarrow$ (C) follows.

Remark. It may be useful to draw attention to the fact that the theories of Widder-Miyadera [1], [2] and of Leviatan [3] are technically strongly different and therefore Proposition 8 represents a useful bridge to pass from one to the other. Moreover, naturally, if both theories are supposed to be proved, Proposition 8 is, in reflexive spaces, their simple consequence (in terms of the preceding Theorem 9, (A) $\Leftrightarrow$ (C) and (B) $\Leftrightarrow$ C imply (A) $\Leftrightarrow$ (B)).

Remark. In Proposition 8, we omitted the case $\vartheta = 1$ because we are interested only in the case of Laplace transforms of functions and not of measures. Moreover, this case was solved essentially (for numerical valued functions) by Widder [1].

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Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).