Jiří Jarník

Constructing the minimal differential relation with prescribed solutions

Časopis pro pěstování matematiky, Vol. 105 (1980), No. 3, 311--315

Persistent URL: `http://dml.cz/dmlcz/118074`

CONSTRUCTING THE MINIMAL DIFFERENTIAL RELATION WITH PRESCRIBED SOLUTIONS

JIŘÍ JARNÍK, Praha

(Received May 16, 1978)

Let R^n be the n-dimensional Euclidean space, $\mathcal{K}_n$ the system of its nonempty compact convex subsets, $\mathcal{K}_n^0 = \mathcal{K}_n \cup \{\emptyset\}$.

Let us denote by $B(x, r)$, $\bar{B}(x, r)$ respectively the open and the closed ball in R^n with a centre x and a radius r.

Given $M \subset R^n$, then $\Omega(M, \varepsilon)$ is the ε-neighbourhood of the set M, $\bar{\Omega}(M, \varepsilon)$ the closure of the neighbourhood. The symbol $\overline{\text{conv}}\, M$ stands for the closed convex hull of a set $M \subset R^n$, $m(A)$ is the (one-dimensional) Lebesgue measure of a set $A \subset R$. If J is an interval, $M \subset R^n$, then the upper semicontinuity of a mapping $F : J \times M \to \mathcal{K}_n$ or $F : J \times M \to \mathcal{K}_n^0$ is defined in the usual way.

Our aim is to prove the following theorem.

Theorem. *Let $\alpha < \beta$. Let Ξ denote a set of functions $x : J_x \to R^n$ with the following properties:*

(i) *for each $x \in \Xi$, J_x is a closed subinterval of $J = [\alpha, \beta]$;*

(ii) *x is absolutely continuous;*

(iii) *there exists a function $\xi : [\alpha, \beta] \to R^+ = [0, +\infty)$ with $\int_\alpha^\beta \xi(t)\, dt \leqq 1$ such that $|\dot{x}(t)| \leqq \xi(t)$ holds for almost all $t \in J_x$;*

(iv) *to each $x \in \Xi$ there is $\tau_x \in J_x$ such that $|x(\tau_x)| \leqq 1$.*

Then there exists a mapping $Q : H \to \mathcal{K}_n^0$, where $H = [\alpha, \beta] \times \bar{B}(0, 2)$, such that $Q(t, \cdot)$ is upper semicontinuous for almost all $t \in [\alpha, \beta]$, each $x \in \Xi$ is a solution of the relation

$$\dot{x} \in Q(t, x) \tag{1}$$

and Q is minimal in the following sense: if $S : H \to \mathcal{K}_n^0$, $S(t, \cdot)$ is upper semicontinuous for almost all $t \in [\alpha, \beta]$ and each $x \in \Xi$ is a solution (on J_x) of the relation

$$\dot{x} \in S(t, x),$$

then

$$Q(t, x) \subset S(t, x)$$

for almost all $t \in [\alpha, \beta]$ and all $x \in \bar{B}(0, 2)$.

Remarks. 1. Let us notice that the minimality property of Q guarantees its uniqueness.

2. In addition to the upper semicontinuity of Q, it will be clear from the proof that $Q(t, x) \subset \bar{B}(0, \xi(t))$ (cf. condition (iii) of Theorem). Hence Q satisfies assumptions for the existence of solutions of (1).

3. According to [1, Definition 1.4], a mapping $F : H \to \mathscr{K}_n^0$ belongs to the class $\mathscr{S}\mathscr{D}^*(H \to \mathscr{K}_n^0)$ if it satisfies the condition: to every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset R$ such that $m(R - A_\varepsilon) < \varepsilon$ and the function $F|_{H \cap (A_\varepsilon \times R^n)}$ is upper semicontinuous; mappings from $\mathscr{S}\mathscr{D}^*(H \to \mathscr{K}_n^0)$ may be called Scorza-Dragonian mappings as Scorza-Dragoni introduced the corresponding class of functions $f : H \to R$.

The main result [1, Theorem 1.5] applied to the mapping $Q : H \to \mathscr{K}_n^0$ with the properties specified in Theorem yields that there exists a Scorza-Dragonian mapping $Q_0 : H \to \mathscr{K}_n^0$ which fulfils $Q_0(t, x) \subset Q(t, x)$ for almost all $t \in [\alpha, \beta]$ and all $x \in \bar{B}(0, 2)$, and each $u \in \Xi$ is a solution of the relation

$$\dot{x} \in Q_0(t, x).$$

Hence necessarily $Q \equiv Q_0$, i.e. Q is Scorza-Dragonian.

Proof of Theorem. If x, y are two functions satisfying conditions (i)–(iv), let us introduce the distance $\varrho(x, y)$ in the following way:

Denote by $J_x = [a_x, b_x]$, $J_y = [a_y, b_y]$ the definition intervals of x, y, respectively, and set

$$\bar{x}(t) = \begin{cases} x(t) & \text{for} \quad t \in J_x, \\ x(a_x) & \text{for} \quad \alpha \leqq t < a_x, \\ x(b_x) & \text{for} \quad b_x < t \leqq \beta; \end{cases}$$

then $\bar{x} : J \to R^n$. Introducing $\bar{y} : J \to R^n$ analogously, we define

$$\varrho(x, y) = \max_{t \in J} |\bar{x}(t) - \bar{y}(t)| + |a_x - a_y| + |b_x - b_y|.$$

It is easily verified that this formula defines a metric on the set of functions satisfying (i)–(iv). We shall show that the set Ξ has an at most countable dense (with respect to ϱ) subset. Indeed, set

$$\Gamma = \{x : J \to R^n \mid x \text{ satisfies (ii), (iii), (iv)}\}.$$

The set Ξ with the above defined metric ϱ is naturally imbedded into the Cartesian product $\Gamma \times J \times J$. As Γ is separable in virtue of (ii)–(iv), we conclude that Ξ is separable as well.

Consequently, there is an at most countable dense subset of Ξ, say

$$V = \{v_1, v_2, \dots\} \subset \Xi .$$

Let us denote

$$\Lambda_i = \{t \in J_{v_i} \mid \dot{v}_i(t) \text{ does not exist}\} , \quad i = 1, 2, \dots ,$$

$$\Lambda = J - \bigcup_{i=1}^{\infty} \Lambda_i .$$

Then $m(\Lambda) = \beta - \alpha$.

Let us define functions $Q_i : [\alpha, \beta] \times \bar{B}(0, 2) \to \mathscr{K}_n^0$, $i = 1, 2, \dots$ by

$$Q_i(t, x) = \begin{cases} \{0\} & \text{for } t \in [\alpha, \beta] - \Lambda \\ \overline{\text{conv}} \{\dot{v}_p(t) \mid v_p(t) \in \bar{B}(x, i^{-1})\} & \text{for } t \in \Lambda \end{cases} \tag{2}$$

and put

$$Q(t, x) = \bigcap_{i=1}^{\infty} Q_i(t, x) .$$

We shall prove that the mapping Q has the properties from Theorem. First, let us introduce an auxiliary result.

Lemma. *Let* $x_j : [\alpha, \beta] \to R^n$ *satisfy the assumptions* (ii), (iii) *of Theorem* (*with* x *replaced by* x_j). *Let there exist* $x : [\alpha, \beta] \to R^n$,

$$x(t) = \lim_{j \to \infty} x_j(t)$$

for all $t \in [\alpha, \beta]$.

Then

$$\dot{x}(t) \in \bigcap_{j=1}^{\infty} \overline{\text{conv}} \{\dot{x}_j(t), \dot{x}_{j+1}(t), \dots\}$$

for almost all $t \in [\alpha, \beta]$.

For this lemma, see [2, p. 395, Theorem D 18.3.10] or [3, Lemma 2].

Now we shall prove that each $u \in \Xi$ satisfies the relation

$$\dot{u}(t) \in Q(t, u(t)) \tag{3}$$

for almost all $t \in J_u$.

Indeed, since V is a set dense in Ξ, there exists a sequence $w_j = v_{k_j} \in V, j = 1, 2, \dots$, such that

$$u(t) = \lim_{j \to \infty} w_j(t) . \tag{4}$$

According to Lemma there is a set $A \subset [\alpha, \beta]$, $m(A) = \beta - \alpha$, such that

$$\dot{u}(t) \in \bigcap_{j=1}^{\infty} \overline{\text{conv}} \{\dot{w}_j(t), \dot{w}_{j+1}(t), \dots\}$$

for all $t \in A \cap J_u$.

Given $t \in A \cap \Lambda$, there exists for every positive integer i a positive integer j such that

$$\overline{\text{conv}} \{\dot{w}_j(t), \dot{w}_{j+1}(t), \dots\} \subset Q_i(t, u(t)) .$$

(To this aim it is sufficient to choose j large enough to satisfy $|w_q(t) - u(t)| \leqq i^{-1}$ for all $q \geqq j$.)

Hence

$$\dot{u}(t) \in Q_i(t, u(t)), \quad i = 1, 2, \ldots$$

for almost all t which implies (3) immediately.

Further, we shall prove that the mapping $Q(t, \cdot)$ is upper semicontinuous for almost all $t \in [\alpha, \beta]$.

Let us first mention an elementary assertion which is an immediate consequence of the compactness of the sets $Q_i(t, x)$, $i = 1, 2, \ldots$. For every $\varepsilon > 0$ there is a positive integer $i(\varepsilon)$ such that

$$Q_i(t, x) \subset \Omega(Q(t, x), \varepsilon) \tag{5}$$

for all $i \geqq i(\varepsilon)$. Indeed, if this were not the case and if $Q(t, x) \neq \emptyset$ then we could choose $\eta > 0$ and a sequence $z_i \in Q_i(t, x)$, $|z_i - y| \geqq \eta > 0$ for $y \in Q(t, x)$. However, passing to a convergent subsequence if necessary we obtain $z_0 \in Q(t, x)$ for $z_0 = \lim z_i$, a contradiction. On the other hand, if $Q(t, x) = \emptyset$ then $Q_i(t, x) = \emptyset$ for i sufficiently large and (5) is obvious.

Now let $(t, x_0) \in H$ and $\varepsilon > 0$. Find $i(\varepsilon)$ so that (5) holds for $i \geqq i(\varepsilon)$ and suppose $|x - x_0| < (2i(\varepsilon))^{-1}$, $z \in Q(t, x)$. Then also $z \in Q_{2i(\varepsilon)}(t, x)$, i.e. for every $\eta > 0$ there exists a convex combination

$$\sum_{j=1}^{p} \beta_j \dot{v}_j(t), \quad \sum_{j=1}^{p} \beta_j = 1, \quad \beta_j > 0$$

with $v_j \in V$ so that

$$\left|z - \sum_{j=1}^{p} \beta_j \dot{v}_j(t)\right| < \eta$$

and simultaneously

$$\left|x - \sum_{j=1}^{p} \beta_j v_j(t)\right| \leqq \frac{1}{2i(\varepsilon)},$$

hence

$$\left|x_0 - \sum_{j=1}^{p} \beta_j v_j(t)\right| < \frac{1}{i(\varepsilon)}.$$

This means $z \in Q_{i(\varepsilon)}(t, x_0)$. Now we conclude from (5) that

$$Q(t, x) \subset Q_{2i(\varepsilon)}(t, x) \subset Q_{i(\varepsilon)}(t, x_0) \subset \Omega(Q(t, x), \varepsilon)$$

provided $|x - x_0| < \delta = (2i(\varepsilon))^{-1}$ which proves the upper semicontinuity of the map Q.

It remains to prove that Q is minimal in the sense mentioned in the theorem. Let us suppose that S has the properties from the theorem, i.e. $S : H \to \mathscr{K}_n^0$, $S(t, \cdot)$ is upper semicontinuous for almost all $t \in [\alpha, \beta]$ and each $u \in \Xi$ is a solution of the relation

$$\dot{x} = S(t, x). \tag{6}$$

Let $\varepsilon > 0$, $t \in [\alpha, \beta]$. Then there exists a positive integer i with the following property: if $y \in B(x, i^{-1})$ then

$$S(t, y) \subset \Omega(S(t, x), \varepsilon) . \tag{7}$$

On the other hand, as the set V is at most countable and all $v_j \in V$ are solutions of (6), there exists a set $D \subset [\alpha, \beta]$ with $m(D) = \beta - \alpha$ such that

$$\dot{v}_j(t) \in S(t, v_j(t)) \quad \text{for} \quad t \in D \cap J_{v_j}, \quad j = 1, 2, \ldots . \tag{8}$$

Let $x \in \bar{B}(0, 1)$, $t \in D \cap A$. Then we have in virtue of the definition of Q_i (see (2))

$$Q_i(t, x) = \overline{\text{conv}} \{\dot{v}_p(t) \mid v_p(t) \in \bar{B}(x, i^{-1})\} \subset \overline{\text{conv}} \bigcup_p S(t, v_p(t)) \tag{9}$$

where the union is taken over all p such that

$$v_p(t) \in \bar{B}(x, i^{-1}) .$$

Consequently, (7) and (9) together imply

$$Q(t, x) = \bigcap_{i=1}^{\infty} Q_i(t, x) \subset \bar{\Omega}(S(t, x), \varepsilon) .$$

The number $\varepsilon > 0$ has been arbitrary, hence the last inclusion holds for all $\varepsilon > 0$. This implies immediately $Q(t, x) \subset S(t, x)$ for all $t \subset D \cap A$, i.e. for almost all $t \in [\alpha, \beta]$ which completes the proof of the theorem.

References

[1] *Jarník, J.* and *Kurzweil, J.*: On conditions on right hand sides of differential relations. Časopis pěst. mat. *102* (1977), 334—349.

[2] *Kurzweil, J.*: Ordinary differential equations. SNTL - Publishing House of Technical Literature, Praha 1978 (Czech).

[3] *Krbec, P.* and *Kurzweil, J.*: Kneser's theorem for multivalued differential delay equations. Časopis pěst. mat. *104* (1979), 1—8.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV) Czechoslovakia.