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SOME PROPERTIES OF SEMIBASE PFAFFIAN FORMS ON THE TANGENT BUNDLE

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Let M be a differentiable manifold. Let TM, or T^*M denote the tangent, or the co-tangent bundle of M. In the theory of the mechanical structures (see [1] p. 173) the semibase forms on the bundle TM are of particular interest. In this paper we shall describe some properties of these forms and of the related structures.

1. Let (x^i), (x^i, y^i), (x^i, z_i), $(x^i, y^i, \xi^i, \eta^i)$, $(x^i, z_i, \sigma^i, \tau_i)$ be local charts on M, TM, T^*M, TTM, TT^*M, respectively. Let $\Lambda(TM)$ denote the graded algebra of exterior differential forms on TM. Denote $\mathscr{B}(TM)$ the subalgebra of all semibase forms on TM (see [1] p. 167). If $\omega \in \Lambda(TM)$ is a 1-form, then $\omega \in \mathscr{B}(TM)$ if and only if, with respect to a local coordinate system, we have

$$\omega = f_i(x, y)\,\mathrm{d}x^i\,. \tag{1}$$

There is a bijection between the vector space of all semibase 1-forms on TM and the vector space of all morphisms $TM \to T^*M$. The morphism p determined by the form (1) can be written locally

$$p : (x^i, y^i) \longmapsto (x^i, z_i = f_i(x, y))\,.$$

Then the morphism

$$p_* : TTM \to TT^*M$$

will be written locally in the form

$$p_* : \begin{cases} x^i = x^i\,, \quad z_i = f_i(x, y)\,, \\ \sigma^i = \xi^i\,, \quad \tau_i = \dfrac{\partial f_i}{\partial x^j}\,\xi^j + \dfrac{\partial f_i}{\partial y^j}\,\eta^j\,. \end{cases} \tag{2}$$

Definition 1. A semibase 1-form $\omega \in \Lambda(TM)$ is called an *L-form* iff the corresponding morphism p is linear.

Locally, ω is an L-form if and only if

$$\omega = f_{ij}(x)\, y^j\, \mathrm{d}x^i .$$

2. Let V or V^* be a Liouville vector field on TM or T^*M, respectively. Locally, we can writte

$$V = y^i\, \partial/\partial y^i\,, \quad V^* = z_i\, \partial/\partial z_i\,.$$

Using (2) we get

$$p_*(x^i, y^i, 0, y^i) = \left(x^i,\, z_i = f_i(x, y),\, 0,\, \frac{\partial f_i}{\partial y^j}\, y^j\right). \tag{3}$$

Theorem 1. *The morphism p_* maps a Liouville vector field V on TM into a Liouville vector field V^* on T^*M if and only if the form ω is homogeneous of the 1-st order.*

Proof. A semibase form ω is homogeneous of the *1*-st order iff its Lie derivative $L_V\omega = \omega$, which is equivalent to

$$\frac{\partial f_i}{\partial y^j}\, y^j = f_i\,.$$

Hence and from (3) the theorem follows.

Corollary. *If ω is an L-form then $p_*(V) = V^*$ (see* [2]*).*
Let

$$X = a^i(x, y)\, \partial/\partial x^i + b^i(x, y)\, \partial/\partial y^i$$

be a vector field on TM, ω a semibase form (1) and p_* the corresponding morphism (2). We ask under which conditions we have

$$p_*(X) = V^*\,. \tag{4}$$

We can see easily that (4) holds iff

$$a^i = 0\,, \quad z_i = \frac{\partial f_i}{\partial y^j}\, b^j\,,$$

or equivalently, iff

$$f_i = \frac{\partial f_i}{\partial y^j}\, b^j\,. \tag{5}$$

Definition 2. The vector fields X on TM which are mapped into a Liouville vector field V^* on T^*M we shall call *Z-fields*.

Theorem 2. *For the Z-fields from Definition* 2 *and for the form* ω *from* (1)

$$L_Z\omega = \omega\,, \quad i_Z\omega = 0\,, \quad i_Z\,\mathrm{d}\omega = \omega\,, \quad L_Z\,\mathrm{d}\omega = \mathrm{d}\omega$$

hold.

Proof.

$$L_Z\omega = \sum_i [Z(f_i)\,\mathrm{d}x^i + f_i\,\mathrm{d}(Z(x^i))] = \sum_{i,j}\left[\frac{\partial f_i}{\partial x^j}\,b^j\,\mathrm{d}x^i\right] = f_i\,\mathrm{d}x^i = \omega$$

if we use (5).

$i_Z\omega = \omega(Z) = 0$, because Z is a vertical field. From the relation

$$L_X\omega = i_X\,\mathrm{d}\omega + \mathrm{d}i_X\omega \tag{6}$$

(see [1] p. 92) we get

$$i_Z\,\mathrm{d}\omega = \omega \tag{7}$$

if we use last relations.

Relation (6) can also be written as follows

$$L_Z\,\mathrm{d}\omega = i_Z\,\mathrm{dd}\omega + \mathrm{d}i_Z\,\mathrm{d}\omega\,. \tag{8}$$

However $\mathrm{dd}\omega = 0$, so $i_Z\,\mathrm{dd}\omega = 0$. By the (7) $i_Z\,\mathrm{d}\omega = \omega$, therefore (8) implies $L_Z\,\mathrm{d}\omega = \mathrm{d}\omega$, q.e.d.

Definition 3. The form ω from (1) will be called *regular* or *singular* at $u \in TM$, if the map p_* is regular or singular at u.

3. Let ω be the singular form and dim Ker p_* be the constant function on TM. In such a case the tangent spaces Ker p_* form distribution $\mathcal{V}$. The distribution is known to be integrable. As can be seen from (2) the distribution is vertical. The equations (2) also imply that the vector field

$$Y = b^i\,\partial/\partial y^i$$

is a subfield of vertical distribution $\mathcal{V}$ if and only if

$$\frac{\partial f_i}{\partial y^j}\,b^j = 0\,. \tag{9}$$

Theorem 3. *Vertical vector Y is a vector of distribution* $\mathcal{V}$ *if and only if* $i_Y\,\mathrm{d}\omega = 0$.

Proof. The exterior differentiation of ω from (1) is

$$\mathrm{d}\omega = \frac{\partial f_i}{\partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i + \frac{\partial f_i}{\partial y^j}\,\mathrm{d}y^j \wedge \mathrm{d}x^i\,. \tag{10}$$

Then

$$i_Y \,\mathrm{d}\omega = \frac{\partial f_i}{\partial y^j} b^j \,\mathrm{d}x^i$$

which with respect to (9) demonstrates Theorem 3.

Corollary. *Denote by $A_h(\omega)$ the set of all such tangent vectors $Y \in T_h TM$ that $i_Y \,\mathrm{d}\omega = 0$. Then*

$$\operatorname{Ker} p_*(h) = A_h(\omega) \cap T_h T_{\pi h} M\,,$$

where $\pi : TM \to M$ is a fiber projection.

Theorem 4. *Let Y be a vector subfield of distribution ∇. Then the form ω from (1) is invariant with respect to vector field Y, i.e. $L_Y \omega = 0$.*

Proof. According to Theorem 3 $i_Y \,\mathrm{d}\omega = 0$. The form ω is semibase, the vector field Y is vertical and therefore $i_Y\omega = \omega(Y) = 0$; moreover, according to (6) also $L_Y \omega = 0$, q.e.d.

Theorem 5. *Let ω be a closed form, M be connected manifold and X be a vector field on TM. Then the form ω is invariant with to respect to vector field X if and only if $i_X\omega$ is a constant function.*

Proof. If ω is a closed form then $\mathrm{d}\omega = 0$. Relation (6) implies that $L_X\omega = \mathrm{d} i_X \omega$. This further implies that $L_X \omega = 0$ (the form ω is invariant) iff $\mathrm{d} i_X \omega = 0$, i.e. $i_X \omega$ is a constant function and vice versa.

Corollary. *If Y is a vertical vector field and ω is a closed form then ω is variant with respect to the vector field Y.*

Theorem 6. *Let ω be a semibase 1-form on TM. Let*

$$X = a^i(x)\, \partial/\partial x^i$$

be a vector field on M. Let ${}^1X$, or ${}^1X^*$ respectively, be a prolongation of vector field X on TM, or T^*M respectively. Then

$$p_*({}^1X_h) = {}^1X^*_{p(h)} \quad \textit{iff} \quad [L_{{}^1X}(\omega)]_h = 0\,,$$

where $h \in TM$ and ${}^1X_h \in T_h TM$.

Proof. In local coordinates we get

$$\tag{11} {}^1X_h = a^i\, \partial/\partial x^i + \frac{\partial a^i}{\partial x^j} y^j\, \partial/\partial y^i\,,$$

$${}^1X^*_{p(h)} = a^i\, \partial/\partial x^i - \frac{\partial a^j}{\partial x^i} f_j\, \partial/\partial z_i\,.$$

The following expression is obtained by calculation

(12) $$[L_{1\,X}(\omega)]_h = \sum_{i,j}\left[\frac{\partial f_i}{\partial x^j}\,a^j + \frac{\partial f_i}{\partial y^j}\cdot\frac{\partial a^j}{\partial x^k}\,y^k + f_j\,\frac{\partial a^j}{\partial x^i}\right]\mathrm{d}x^i\,.$$

From (2) we have

(13) $$p_*({}^1X_h) = a^i\,\partial/\partial x^i + \left(\frac{\partial f_i}{\partial x^j}\,a^j + \frac{\partial f_i}{\partial y^j}\cdot\frac{\partial a^j}{\partial x^k}\,y^k\right)\partial/\partial z_i\,.$$

Comparing (11), (12), (13) the statement of Theorem 6 is confirmed.

4. The equations (2) imply that

$$p_*(T_h T_{\pi h} M) \subset T_{p(h)} T^*_{\pi h} M\,.$$

Let us consider a vector field

$$X = a^i(x)\,\partial/\partial x^i\,,$$

i.e. a section $M \to TM$. Let $X_m \equiv X(m) \in T_m M$. Let us denote map

$$p_* : T_{X_m} T_m M \to T_{p(X_m)} T^*_m M$$

by p_*/X_m. Using canonic identification

$$T_{X_m} T_m M \equiv T_m M\,,\quad T_{p(X_m)} T^*_m M \equiv T^*_m M$$

we obtain the linear morphism

$$p_*/X_m : T_m M \to T^*_m M\,,$$

which can be locally expressed according to (2) as follows

(14) $$p_*/X : x^i = x^i\,,\quad z_i = \frac{\partial f_i(x, a(x))}{\partial y^j}\,y^j\,.$$

The linear map (14) determines the semibase L-form on TM

(15) $$\beta = (\omega/X) = \frac{\partial f_i(x, a(x))}{\partial y^j}\,y^j\,\mathrm{d}x^i\,.$$

Theorem 7. *Let $V = y^i\,\partial/\partial y^i$ be the Liouville vector field on TM. Let X be a vector field by means of which the form (15) was formed. Then the following is true for any $m \in M$:*

$$(i_V\,\mathrm{d}\omega)_{X_m} = \beta_{X_m}\,.$$

Proof. By contraction of form (10) we obtain

(16) $$i_V\,\mathrm{d}\omega = \frac{\partial f_i(x, y)}{\partial y^j}\,y^j\,\mathrm{d}x^i\,.$$

Comparing (15) and (16) the statement of Theorem 7 is confirmed.

By exterior differentiation of the form (15) we obtain

$$(17)\quad \mathrm{d}\beta = \left(\frac{\partial^2 f_i(x, a)}{\partial y^j\, \partial x^k} + \frac{\partial^2 f_i(x, a)}{\partial y^j\, \partial y^l} \cdot \frac{\partial a^l}{\partial x^k}\right) y^j\, \mathrm{d}x^k \wedge \mathrm{d}x^i + \frac{\partial f_i(x, a)}{\partial y^j}\, \mathrm{d}y^j \wedge \mathrm{d}x^i .$$

From (10) and (17) we get:

Theorem 8. *Form* $\mathrm{d}\beta$ *belongs to class* $2n$ *on* TM *if and only if form* $\mathrm{d}\omega$ *is a* 2*-form of class* $2n$ *along the section* $X : M \to TM$. *The form* $\mathrm{d}\omega - \mathrm{d}\beta$ *is semibase along the field* X.

Corollary. *Let us recall that symplectic structure on* TM *(see [1] p. 123) is determined by a closed differential 2-form* $\delta \in \Lambda^2(TM)$ *of a constant class* $2n$. *In our case the symplectic structure on* TM *is determined by form* $\mathrm{d}\beta$ *iff* $\mathrm{d}\omega$ *is the symplectic form along section* $X : M \to TM$.

Theorem 9. *Let* $Y = c^i\, \partial/\partial x^i + b^i\, \partial/\partial y^i \in T_{X_m}TM$. *Let* i_β *or* i_ω *be the map* $Y \longmapsto i_Y\, \mathrm{d}\beta$ *or* $Y \longmapsto i_Y\, \mathrm{d}\omega$. *Then* $i_\beta(Y) - i_\omega(Y)$ *is a semibase form.*

Proof.

$$(18)\qquad i_\beta : Y \longmapsto \left(\left(\frac{\partial^2 f_i(x, a)}{\partial y^j\, \partial x^k} + \frac{\partial^2 f_i(x, a)}{\partial y^j\, \partial y^l} \cdot \frac{\partial a^l}{\partial x^k}\right) a^j (c^k\, \mathrm{d}x^i - c^i\, \mathrm{d}x^j) + \right.$$

$$\left. + \frac{\partial f_i(x, a)}{\partial y^j}\, b^j\, \mathrm{d}x^i - \frac{\partial f_i(x, a)}{\partial y^j}\, c^i\, \mathrm{d}y^j\right),$$

$$(19)\qquad i_\omega : Y \longmapsto \left(\frac{\partial f_i(x, a)}{\partial x^j} (c^j\, \mathrm{d}x^i - c^i\, \mathrm{d}x^j) + \frac{\partial f_i(x, a)}{\partial y^j} (b^j\, \mathrm{d}x^i - c^i\, \mathrm{d}y^j)\right).$$

Comparing (18) and (19) we obtain confirmation of the statement of Theorem 9.

Theorem 10. *Let* X *be a projectable vector field on* TM. *Then* $\mathrm{d}i_X\beta$ *is a semibase form if and only if* $\mathrm{d}i_X\beta = 0$.

Proof. Let us remember that vector field X on TM is projectable iff $\pi_* X$ is a vector field on M, i.e. locally

$$(20)\qquad X = a^i(x)\, \partial/\partial x^i + b^i(x, y)\, \partial/\partial y^i .$$

By contraction of form (15) by the vector field (20) we obtain

$$i_X\beta = \frac{\partial f_i}{\partial y^j}\, y^j a^i .$$

Therefore

$$(21)\quad \mathrm{d}i_X\beta = \left[\left(\frac{\partial^2 f_i}{\partial y^j\,\partial y^k} + \frac{\partial^2 f_i}{\partial y^j\,\partial y^l}\cdot\frac{\partial a^l}{\partial x^k}\right) y^j a^i + \frac{\partial f_i}{\partial y^j}\cdot\frac{\partial a^i}{\partial x^k}\, y^j\right]\mathrm{d}x^k + \frac{\partial f_i}{\partial y^k}\, a^i\,\mathrm{d}y^k\,.$$

Form $\mathrm{d}i_X\beta$ is semibase iff

$$(22)\qquad \frac{\partial f_i}{\partial y^k}\, a^i = 0\,.$$

By differentiation (22) we obtain

$$(23)\qquad \left(\frac{\partial^2 f_i}{\partial y^j\,\partial x^k} + \frac{\partial^2 f_i}{\partial y^j\,\partial y^l}\cdot\frac{\partial a^l}{\partial x^k}\right) a^i + \frac{\partial f_i}{\partial y^j}\cdot\frac{\partial a^i}{\partial x^k} = 0\,.$$

By comparing (21) and (23) the statement of Theorem 10 is obtained.

Theorem 11. *If ω is a semibase form and X is a projectable vector field on TM then $L_X\omega$ is a semibase form.*

Proof. For the form ω from (1) and vector field X from (20) the following is true:

$$(24)\qquad \mathrm{d}i_X\omega = \left(\frac{\partial f_i}{\partial x^j}\, a^i + f_i\,\frac{\partial a^i}{\partial x^j}\right)\mathrm{d}x^j + \frac{\partial f_i}{\partial y^j}\, a^i\,\mathrm{d}y^j$$

and

$$(25)\qquad i_X\,\mathrm{d}\omega = \left(\frac{\partial f_i}{\partial x^j}\, a^j - \frac{\partial f_j}{\partial x^i}\, a^j + \frac{\partial f_i}{\partial y^j}\, b^j\right)\mathrm{d}x^i - \frac{\partial f_i}{\partial y^j}\, a^i\,\mathrm{d}y^j\,.$$

By substituting from (24) and (25) into (6) we get the result that $L_X\omega$ is semibase form, q.e.d.

Theorem 12. *Let X be the vector field on TM. Then $L_X\omega$ is a semibase form for any semibase form ω if and only if X is a projectable vector field.*

Proof. The contraction of any form ω from (1) along a vector field

$$X = a^i(x, y)\,\partial/\partial x^i + b^i(x, y)\,\partial/\partial y^i \quad \text{on} \quad TM$$

is

$$i_X\omega = f_i(x, y)\, a^i(x, y)\,.$$

By exterior differentiation we obtain

$$(26)\qquad \mathrm{d}i_X\omega = \left(\frac{\partial f_i}{\partial x^j}\, a^i + f_i\,\frac{\partial a^i}{\partial x^j}\right)\mathrm{d}x^j + \left(\frac{\partial f_i}{\partial y^j}\, a^i + f_i\,\frac{\partial a^i}{\partial y^j}\right)\mathrm{d}y^j\,.$$

The form $i_X\,\mathrm{d}\omega$ for any vector field X on TM can be expressed in form (25). From

(6) and from the addition of (25) and (26) we get that the form $L_X\omega$ is semibase on *TM* iff

$$f_i \frac{\partial a^i}{\partial y^j} = 0 .$$

This is possible for all f_i iff a^i are functions of x only, i.e. if the vector field X on *TM* is projectable, q.e.d.

Literature

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