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NEW UPPER BOUNDS FOR THE CROSSING NUMBER OF K_n ON THE KLEIN BOTTLE

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1. INTRODUCTION

The crossing number of a graph G for an orientable as well as for a nonorientable surface with the genus g is defined as the least possible number of crossings in a drawing of G on the mentioned surface. For an orientable surface we use the notation $cr_g^*(G)$, for the nonorientable one $cr_g(G)$, $g = 1, 2, 3, \ldots$.

The crossing numbers are most frequently investigated for the complete graphs K_n and the complete bipartite graphs $K_{m,n}$. Most papers deal with the plane. The other surfaces mentioned are the torus, the projective plane and the Klein bottle.

We shall give briefly the most important results in the chronological order.

1. ZARANKIEWICZ 1954 [1] and URBANIK 1955 [2];

$$cr_0^*(K_{m,n}) \leqq \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right].$$

2. GUY 1960 [3], HARARY and HILL 1962 [4], BLAŽEK and KOMAN 1963 [5], SAATY 1964 [6]:

$$cr_0^*(K_n) \leqq \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right].$$

3. GUY, JENKYNS and SCHAER 1967 [7]:

$$cr_1^*(K_n) \leqq \frac{59}{216}\binom{n-1}{4}, \quad n \geqq 10 .$$

4. BLAŽEK and KOMAN 1967 [8, 9], HARBORTH 1971 [10]:

$$cr_0^*(K_{n_1 n_2 \ldots n_k}) \leqq \sum_i a_i b_i A_i B_i - \sum_{i<j} a_i b_i a_j b_j +$$

$$+ \sum_{r<s<t<u} (a_r a_s a_t a_u + a_r a_s c_t c_u + a_r c_s c_t a_u + c_r c_s c_t c_u + c_r c_s a_t a_u + c_r a_s a_t c_u) ,$$

where

$$a_i = \left[\frac{n_i}{2}\right], \quad b_i = \left[\frac{n_i - 1}{2}\right], \quad c_i = \left[\frac{n_i + 1}{2}\right],$$

$$A_i = [(\sum_j n_j - n_i)/2], \quad B_i = [(\sum_j n_j - n_i - 1)/2].$$

5. Guy and Jenkyns 1969 [11]:

$$cr_1^*(K_{m,n}) \leqq \frac{1}{6}\binom{m-1}{2}\binom{n-1}{2}, \quad m, n \geqq 45.$$

6. Koman 1969 [12, 13]:

$$cr_1(K_n) \leqq \frac{39}{128}\binom{n-1}{4}, \quad n \geqq 10;$$

$$cr_2(K_n) \leqq \frac{37}{128}\binom{n-1}{4}, \quad n \geqq 10.$$

7. Guy and Hill 1973 [14]:

$$cr_0^*(\bar{C}_n) \leqq \frac{1}{64}(n-3)^2(n-5)^2, \quad n \text{ odd};$$

$$\leqq \frac{1}{64} n(n-4)(n-6)^2, \quad n \text{ even},$$

where $\bar{C}_n$ is the complement of a circuit of the length n.

8. Koman 1974 [13]:

$$cr_2(K_{m,n}) \leqq \frac{1}{6}\binom{m-1}{2}\binom{n-1}{2}$$

for infinite integers m and n. It is a hiatus that the inequality holds for all $m, n \geqq 45$.

From the given survey it is seen among other that on the torus as well as on the Klein bottle the same upper bounds for the crossing numbers of the complete bipartite graphs $K_{m,n}$ hold.

In this paper we shall show that for the complete graph K_n the known value

$$\frac{59}{216}\binom{n-1}{4}$$

is the upper bound not only for the torus but also for the Klein bottle.

2. PRECISE VALUES AND BOUNDS FOR $n \leqq 15$

The results for the Klein bottle and for the torus are given simultaneously (see [7, 12]):

$$cr_2(K_7) = 1\,, \qquad cr_1^*(K_7) = 0\,,$$

$$cr_2(K_8) = 4\,, \qquad cr_1^*(K_8) = 4\,,$$

$$cr_2(K_9) = 9\,, \qquad cr_1^*(K_9) = 9\,.$$

For $n = 10$ in contradistinction to the torus, where the precise value $cr_1^*(K_{10})$ is known, only the upper and lower estimates are known for the Klein bottle:

$$22 \leqq cr_2(K_{10}) \leqq 24\,, \qquad cr_1^*(K_{10}) = 23\,.$$

The upper bounds for the both surfaces follow from Figs. 1 and 2.

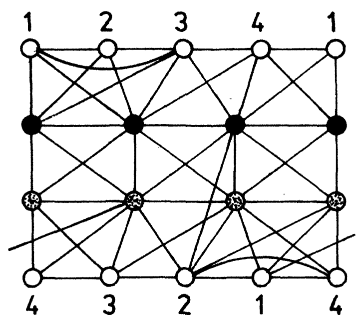

Fig. 1 ($cr_2(K_{10}) \leqq 24$)

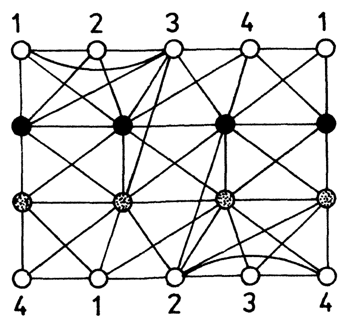

Fig. 2 ($cr_1^*(K_{10}) \leqq 23$)

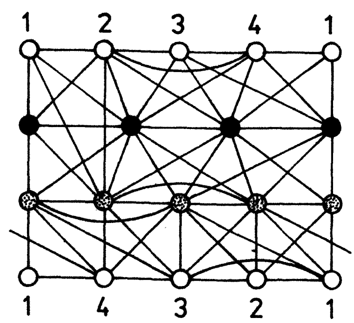

Fig. 3

For $11 \leqq n \leqq 15$ we can give for the Klein bottle as well as for the torus the following estimates:

$$35 \leqq cr_2(K_{11}) \leqq 43\,, \qquad 37 \leqq cr_1^*(K_{11}) \leqq 42\,,$$
$$53 \leqq cr_2(K_{12}) \leqq 72\,, \qquad 56 \leqq cr_1^*(K_{12}) \leqq 70\,,$$
$$77 \leqq cr_2(K_{13}) \leqq 109\,, \qquad 81 \leqq cr_1^*(K_{13}) \leqq 105\,,$$
$$108 \leqq cr_2(K_{14}) \leqq 161\,, \qquad 114 \leqq cr_1^*(K_{14}) \leqq 154\,,$$
$$148 \leqq cr_2(K_{15}) \leqq 225\,, \qquad 156 \leqq cr_1^*(K_{15}) \leqq 225\,.$$

The inequality $cr_2(K_{11}) \leqq 43$ gives Fig. 3. The upper bound is here lower by 1 than that given in [12]. The inequality $cr_2(K_{15}) \leqq 225$ is a consequence of the construction which will be presented in Part 3. For $n = 15$, the upper bound for $cr_2(K_{15})$ decreases from 239 (see [12]) to 225.

3. BOUNDS FOR $n \geqq 16$

The drawing of the graphs K_n, $n \geqq 3$ on the Klein bottle, which gives the upper bound for the crossing number $cr_2(K_n)$, arises by a simple modification of an analogous Guy-Jenkyns's drawing on the torus [17]. For $n = 12$ the both drawings are obvious from Figs. 4 and 5.

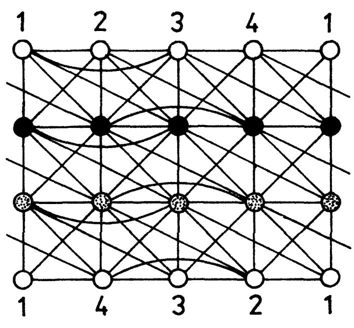

Fig. 4 ($cr_2(K_{12}) \leqq 72$)

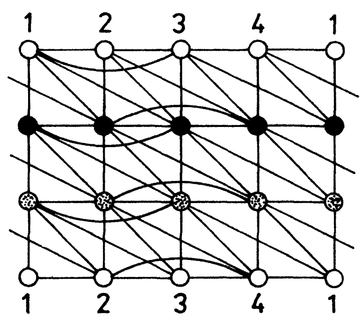

Fig. 5 ($cr_1^*(K_{12}) \leqq 72$)

For other n's the drawings for K_n are generalizations of Figs. 4, 5. All n vertices are divided into three approximately equal parts: white, black and grey. We dislocate the vertices according to Fgs. 6, 7. After sticking the appropriate pairs of vertices we obtain in the first case the Klein bottle, in the second the torus.

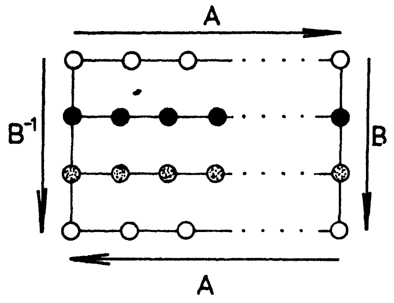

Fig. 6 (Klein bottle)

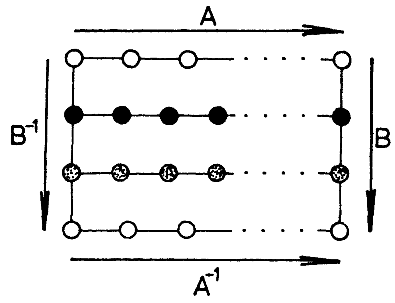

Fig. 7 (torus)

On the constructed surface we join all the vertices having the same colour with the identically coloured circuit. In this way three disjoined circuits arise: white, black and grey, which form already a crossingfree subdrawing of the sought drawing. The edges joining two vertices with different colours are determined by these crossingfree circuits. The other edges are constructed according to the following rule.

First of all we label all the vertices with cyclical orders on each of the crossingfree coloured circuits with the integers

$$1, 2, 3, \ldots, n_i$$

(n_i is the length of the white, black and grey circuit C_i respectively). We join two identically coloured vertices u, v coinciding with a crossingfree circuit C_i (with length n_i) on one side iff the edge uv is parallel to some of the edges

$$1w\,, \quad w = 2, 3, \ldots, [n_i/2]\,.$$

Other remaining joins of the vertices belonging to C_i are constructed on the opposite side of the circuit C_i. Two edges uv, wt, whose endpoints belong to the same crossingfree circuits C_i of length n_i are called parallel iff

$$u + v \equiv w + t \pmod{n_i}$$

holds.

The just obtained drawing of the graph K_n on the Klein bottle as well as that on the torus contains three subdrawings of complete graphs generated by white, black or grey vertices respectively. These subdrawings are homeomorphical with the drawing given in [5] by means of the construction B.

From the table and from [12, 13] we obtain the inequalities

$$\frac{1}{14}\binom{n}{4} \leqq cr_2(K_n) \leqq \frac{59}{216}\binom{n-1}{4}.$$

As in [7] we obtain the following upper bounds:

$n =$	Common upper bound for $cr_2(K_n)$ and $cr_1(K_n)$
$6k = 3u$	$u(u-2)(59u^2-98u+24)/64 = n(n-6)(59n^2-294n+216)/5184$
$6k+1 = 3u+1$	$u(177u^3-412u^2+180u+64)/192 =$ $= (n-1)(59n^3-589n^2+1541n-435)/5184$
$6k+2 = 3u-1$	$(u-1)(177u^3-707u^2+727u-117)/192 =$ $= (n-2)(59n^3-530n^2+944n+480)/5184$
$6k+3 = 3u$	$(u-1)(59u^3-157u^2+45u-27)/64 =$ $= (n-3)(59n^3-471n^2+405n-243)/5184$
$6k+4 = 3u+1$	$(u-1)(177u^3-235u^2-97u+27)/192 =$ $= (n-4)(59n^3-412n^2+356n+240)/5184$
$6k+5 = 3u-1$	$(u-2)(177u^3-530u^2+416u-96)/192 =$ $= (n-5)(59n^3-353n^2+365n-87)/5184$

To compare the upper bounds for the same graph (class of graphs) but for different surfaces it is useful to investigate the coefficients by the leading members:

	K_n	$K_{m,n}$	$\bar{C}_n$
Euclidean plane	$\frac{1}{64} = 0{\cdot}0156\ldots$	$\frac{1}{16} = 0{\cdot}0625\ldots$	$\frac{1}{64} = 0{\cdot}0156\ldots$
Projective plane	$\frac{13}{1024} = 0{\cdot}0126\ldots$	?	?
Torus	$\frac{59}{5184} = 0{\cdot}0113\ldots$	$\frac{1}{24} = 0{\cdot}0416\ldots$	?
Klein bottle	$\frac{59}{5184} = 0{\cdot}0113\ldots$	$\frac{1}{24} = 0{\cdot}0416\ldots$	?

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