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CONGRUENCES OF SURFACES IN $\mathbb{C}^2$

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A. ŠVEC [1] studied transitive layers of hypersurfaces in $\mathbb{C}^2$; in what follows, I am going to describe transitive two-parametric systems of surfaces in $\mathbb{C}^2$.

As usually, let us write $\mathbb{C}^2 = (\mathbb{R}^4, J)$, $J : \mathbb{R}^4 \to \mathbb{R}^4$ being the endomorphism satisfying $J^2 = -\mathrm{id}$ and defined by $J(v) = iv$. In $\mathbb{C}^2 = (\mathbb{R}^4, J)$, consider a congruence, i.e.. a 2-parametric system of 2-dimensional surfaces $\mathscr{L}$ such that through each point $m \in \mathbb{R}^4$ there passes exactly one surface $V^2 \in \mathscr{L}$. Let $T_m(V_m^2)$ be the tangent plane of V_m^2 at m; in what follows, we restrict ourselves to congruences $\mathscr{L}$ such that

$$T_m(V_m^2) \cap J\, T_m(V_m^2) = \{0\} \tag{1}$$

for each point $m \in \mathbb{R}^4$. At each point $m \in \mathbb{R}^4$, let us choose a frame $\{v_1, v_2, v_3, v_4\}$ such that

$$v_1, v_2 \in T_m(V_m^2)\,, \quad v_3 = Jv_1\,, \quad v_4 = Jv_2\,. \tag{2}$$

Choosing another frames $\{w_1, w_2, w_3, w_4\}$ satisfying equations analogous to (2), we have

$$\begin{aligned} w_1 &= \alpha v_1 + \beta v_2\,, \qquad w_2 = \gamma v_1 + \delta v_2\,,\\ w_3 &= \alpha v_3 + \beta v_4\,, \qquad w_4 = \gamma v_3 + \delta v_4\,,\\ &\alpha\delta - \beta\gamma \neq 0\,. \end{aligned} \tag{3}$$

Thus the congruence $\mathscr{L}$ induces on $\mathbb{R}^4$ a G-structure, being given by (3).

Let us write

$$\begin{aligned} [v_1, v_2] &= a_1v_1 + a_2v_2\,,\\ [v_1, v_3] &= b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4\,,\\ [v_1, v_4] &= c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4\,,\\ [v_2, v_3] &= d_1v_1 + d_2v_2 + d_3v_3 + d_4v_4\,,\\ [v_2, v_4] &= e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4\,,\\ [v_3, v_4] &= f_1v_1 + f_2v_2 + f_3v_3 + f_4v_4\,. \end{aligned} \tag{4}$$

From the Jacobi identities

$$(5) \qquad [v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0 ;$$
$$i, j, k = 1, \ldots, 4 ;$$

we get

$$\begin{aligned}
(6) \quad & v_1 d_1 + d_2 a_1 + d_3 b_1 + d_4 c_1 - v_2 b_1 - b_3 d_1 - b_4 e_1 + v_3 a_1 - a_2 d_1 = 0 , \\
& v_1 d_2 + d_3 b_2 + d_4 c_2 - v_2 b_2 + b_1 a_2 - b_3 d_2 - b_4 e_2 + v_3 a_2 - a_1 b_2 = 0 , \\
& v_1 d_3 + d_4 c_3 - v_2 b_3 - b_4 e_3 - a_1 b_3 - a_2 d_3 = 0 , \\
& v_1 d_4 + d_3 b_4 + d_4 c_4 - v_2 b_4 - b_3 d_4 - b_4 e_4 - a_1 b_4 - a_2 d_4 = 0 ;
\end{aligned}$$

$$\begin{aligned}
(7) \quad & v_1 e_1 + e_2 a_1 + e_3 b_1 + e_4 c_1 - v_2 c_1 - c_3 d_1 - c_4 e_1 + v_4 a_1 - a_2 e_1 = 0 , \\
& v_1 e_2 + e_3 b_2 + e_4 c_2 - v_2 c_2 + c_1 a_2 - c_3 d_2 - c_4 e_2 + v_4 a_2 - a_1 c_2 = 0 , \\
& v_1 e_3 + e_3 b_3 + e_4 c_3 - v_2 c_3 - c_3 d_3 - c_4 e_3 - a_1 c_3 - a_2 e_3 = 0 , \\
& v_1 e_4 + e_3 b_4 - v_2 c_4 - c_3 d_4 - a_1 c_4 - a_2 e_4 = 0 ;
\end{aligned}$$

$$\begin{aligned}
(8) \quad & v_1 f_1 + f_2 a_1 + f_3 b_1 + f_4 c_1 - v_3 c_1 + c_2 d_1 - c_4 f_1 + v_4 b_1 - b_2 e_1 - \\
& - b_3 f_1 = 0 , \\
& v_1 f_2 + f_2 a_2 + f_3 b_2 + f_4 c_2 - v_3 c_2 + c_1 b_2 + c_2 d_2 - c_4 f_2 + v_4 b_2 - \\
& - b_1 c_2 - b_2 e_2 - b_3 f_2 = 0 , \\
& v_1 f_3 + f_4 c_3 + c_1 b_3 + c_2 d_3 - c_4 f_3 + v_4 b_3 - b_1 c_3 - b_2 e_3 - v_3 c_3 = 0 , \\
& v_1 f_4 + f_3 b_4 - v_3 c_4 + c_1 b_4 + c_2 d_4 + v_4 b_4 - b_1 c_4 - b_2 e_4 - b_3 f_4 = 0 ;
\end{aligned}$$

$$\begin{aligned}
(9) \quad & v_2 f_1 - f_1 a_1 + f_3 d_1 + f_4 e_1 - v_3 e_1 + e_1 b_1 + e_2 d_1 - e_4 f_1 + v_4 d_1 - \\
& - d_1 c_1 - d_2 e_1 - d_3 f_1 = 0 , \\
& v_2 f_2 - f_1 a_2 + f_3 d_2 + f_4 e_2 - v_3 e_2 + e_1 b_2 - e_4 f_2 + v_4 d_2 - d_1 c_2 - \\
& - d_3 f_2 = 0 , \\
& v_2 f_3 + f_4 e_3 - v_3 e_3 + e_1 b_3 + e_2 d_3 - e_4 f_3 + v_4 d_3 - d_1 c_3 - d_2 e_3 = 0 , \\
& v_2 f_4 + f_3 d_4 - v_3 e_4 + e_1 b_4 + e_2 d_4 + v_4 d_4 - d_1 c_4 - d_2 e_4 - d_3 e_3 = 0 .
\end{aligned}$$

Analoguously, we have

$$\begin{aligned}
(10) \qquad & [w_1, w_2] = A_1 w_1 + A_2 w_2 , \\
& [w_1, w_3] = B_1 w_1 + B_2 w_2 + B_3 w_3 + B_4 w_4 , \\
& [w_1, w_4] = C_1 w_1 + C_2 w_2 + C_3 w_3 + C_4 w_4 ,
\end{aligned}$$

$$[w_2, w_3] = D_1 w_1 + D_2 w_2 + D_3 w_3 + D_4 w_4 ,$$
$$[w_2, w_4] = E_1 w_1 + E_2 w_2 + E_3 w_3 + E_4 w_4 ,$$
$$[w_3, w_4] = F_1 w_1 + F_2 w_2 + F_3 w_3 + F_4 w_4$$

with the integrability conditions

(11) $$w_1 D_1 + D_2 A_2 + D_3 B_1 + D_4 C_1 - w_2 B_1 - B_3 D_1 - B_4 E_1 + w_3 A_1 - $$
$$- A_2 D_1 = 0 ,$$
$$w_1 D_2 + D_3 B_2 + D_4 C_2 - w_2 B_2 + B_1 A_2 - B_3 D_2 - B_4 E_2 + w_3 A_2 -$$
$$- A_1 B_2 = 0 ,$$
$$w_1 D_3 + D_4 C_3 - w_3 B_3 - B_4 E_3 - A_1 B_3 - A_2 D_3 = 0 ,$$
$$w_1 D_4 + D_3 B_4 + D_4 C_4 - w_2 B_4 - B_3 D_4 - B_4 E_4 + A_1 B_4 - A_2 D_4 = 0 ,$$
$$w_1 E_1 + E_2 A_1 + E_3 B_1 + E_4 C_1 - w_2 C_1 - C_3 D_1 - C_4 E_1 +$$
$$+ w_4 A_1 - A_2 E_1 = 0 ,$$
$$w_1 E_2 + E_3 B_2 + E_4 C_2 - w_2 C_2 + C_1 A_2 - C_3 D_2 - C_4 E_2 +$$
$$+ w_4 A_2 - A_1 C_2 = 0 ,$$
$$w_1 E_3 + E_3 B_3 + E_4 C_3 - w_2 C_3 - C_3 D_3 - C_4 E_3 - A_1 C_3 - A_2 C_3 = 0 ,$$
$$w_1 E_4 + E_3 B_4 + E_4 C_4 - w_2 C_4 - C_3 D_4 - C_4 E_4 - A_1 C_4 - A_2 C_4 = 0 ,$$
$$w_1 F_1 + F_2 A_1 + F_3 B_1 + F_4 C_1 - w_3 C_1 + C_2 D_1 - C_4 F_1 +$$
$$+ w_4 B_1 - B_2 E_1 - B_3 F_1 = 0 ,$$
$$w_1 F_2 + F_2 A_2 + F_3 B_2 + F_4 C_2 - w_3 C_2 + C_2 D_2 - C_4 F_2 +$$
$$+ C_1 B_2 + w_4 B_2 - B_1 C_2 - B_2 E_2 - B_3 F_2 = 0 ,$$
$$w_1 F_3 + F_4 C_3 + C_1 B_3 + C_2 D_3 - C_4 F_3 + w_4 B_3 - B_1 C_3 - B_2 E_3 -$$
$$- w_3 C_3 = 0 ,$$
$$w_1 F_4 + F_3 B_4 - w_3 C_4 + C_1 B_4 + C_2 D_4 + w_4 B_4 - B_1 C_4 -$$
$$- B_2 E_4 - B_3 E_4 = 0 ,$$
$$w_2 F_1 - F_1 A_1 + F_3 D_1 + F_4 E_1 - w_3 E_1 + E_1 B_1 + E_2 D_1 - E_4 F_1 +$$
$$+ w_4 D_1 - D_1 C_1 - D_2 E_1 - D_3 F_1 = 0 ,$$
$$w_2 F_2 - F_1 A_2 + F_3 D_2 + F_4 E_2 - w_3 E_2 + E_1 B_2 - E_4 E_2 +$$
$$+ w_4 D_2 - D_1 C_2 - D_3 F_2 = 0 ,$$

$$w_2F_3 + F_4E_3 - V_3E_3 + E_1B_3 + E_2D_3 - E_4F_3 + V_4D_3 -$$
$$- D_1C_3 - D_2E_3 = 0\,,$$
$$w_2F_4 + F_3D_4 - w_3E_4 + E_1B_4 + E_2D_4 + w_4D_4 - D_1C_4 -$$
$$- D_2E_4 - D_3F_3 = 0\,.$$

Comparing (4) with (10), we get

$$(12)\quad [w_1, w_2] = [\alpha v_1 + \beta v_2, \gamma v_1 + \delta v_2] = w_1\gamma \,.\, v_1 + w_2\delta \,.\, v_2 - w_2\alpha \,.\, v_1 -$$
$$- w_2\beta \,.\, v_2 - \gamma\beta(a_1v_1 + a_2v_2) + \alpha\delta(a_1v_1 + a_2v_2) =$$
$$= A_1w_1 + A_2w_2 = A_1(\alpha v_1 + \beta v_2) + A_2(\gamma v_1 + \delta v_2)\,,$$

$$(13)\quad [w_1, w_3] = [\alpha v_1 + \beta v_2, \alpha v_3 + \beta v_4] = w_1\alpha \,.\, v_3 + w_1\beta \,.\, v_4 +$$
$$+ \alpha^2(b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4) + \alpha\beta(d_1v_1 + d_2v_2 +$$
$$+ d_3v_3 + d_4v_4) + \beta\alpha(c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4) +$$
$$+ \beta^2(e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4) - w_3\alpha \,.\, v_1 - w_3\beta \,.\, v_2 =$$
$$= B_1w_1 + B_2w_2 + B_3w_3 + B_4w_4 = B_1(\alpha v_1 + \beta v_2) +$$
$$+ B_2(\gamma v_1 + \delta v_2) + B_3(\alpha v_3 + \beta v_4) + B_4(\gamma v_3 + \delta v_4)\,,$$

$$(14)\quad [w_1, w_4] = [\alpha v_1 + \beta v_2, \gamma v_3 + \delta v_4] = w_1\gamma \,.\, v_3 + w_1\delta \,.\, v_4 +$$
$$+ \alpha\gamma(b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4) + \beta\gamma(d_1v_1 + d_2v_2 +$$
$$+ d_3v_3 + d_4v_4) - w_4\alpha \,.\, v_1 - w_4\beta \,.\, v_2 + \alpha\delta(c_1v_1 + c_2v_2 +$$
$$+ c_3v_3 + c_4v_4) + \beta\delta(e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4) =$$
$$= C_1w_1 + C_2w_2 + C_3w_3 + C_4w_4 = C_1(\alpha v_1 + \beta v_2) +$$
$$+ C_2(\gamma v_1 + \delta v_2) + C_3(\alpha v_3 + \beta v_4) + C_4(\gamma v_3 + \delta v_4)\,,$$

$$(15)\quad [w_2, w_3] = [\gamma v_1 + \delta v_2, \alpha v_3 + \beta v_4] = w_2\alpha \,.\, v_3 + w_2\beta \,.\, v_4 +$$
$$+ \alpha\gamma(b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4) + \alpha\delta(d_1v_1 + d_2v_2 +$$
$$+ d_3v_3 + d_4v_4) - w_3\delta \,.\, v_2 + \beta\gamma(c_1v_1 + c_2v_2 + c_3v_3 +$$
$$+ c_4v_4) + \beta\delta(e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4) = D_1w_1 +$$
$$+ D_2w_2 + D_3w_3 + D_4w_4 = D_1(\alpha v_1 + \beta v_2) +$$
$$+ D_2(\gamma v_1 + \delta v_2) + D_3(\alpha v_3 + \beta v_4) + D_4(\gamma v_3 + \delta v_4)\,,$$

$$(16)\quad [w_2, w_4] = [\gamma v_1 + \delta v_2, \gamma v_3 + \delta v_4] = w_2\gamma \,.\, v_3 + w_2\delta \,.\, v_4 + \\ + \gamma^2(b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4) + \gamma\delta(d_1v_1 + d_2v_2 + \\ + d_3v_3 + d_4v_4) + \gamma\delta(c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4) + \\ + \delta^2(e_1v_1 + e_2v_2 + e_3v_3 + e_4v_4) - w_4\gamma \,.\, v_1 - w_4\delta \,.\, v_2 = \\ = E_1w_1 + E_2w_2 + E_3w_3 + E_4w_4 = E_1(\alpha v_1 + \beta v_2) + \\ + E_2(\gamma v_1 + \delta v_2) + E_3(\alpha v_3 + \beta v_4) + E_4(\gamma v_3 + \delta v_4),$$

$$(17)\quad [w_3, w_4] = [\alpha v_3 + \beta v_4, \gamma v_3 + \delta v_4] = w_3\gamma \,.\, v_3 + w_3\delta \,.\, v_4 - \\ - \gamma\beta(f_1v_1 + f_2v_2 + f_3v_3 + f_4v_4) + \alpha\delta(f_1v_1 + f_2v_2 + \\ + f_3v_3 + f_4v_4) - w_4\alpha \,.\, v_3 - w_4\beta \,.\, v_4 = F_1w_1 + F_2w_2 + \\ + F_3w_3 + F_4w_4 = F_1(\alpha v_1 + \beta v_2) + F_2(\gamma v_1 + \delta v_2) + \\ + F_3(\alpha v_3 + \beta v_4) + F_4(\gamma v_3 + \delta v_4).$$

From (12)–(17),

$$(18)\quad \begin{aligned} w_1\alpha &= -\alpha^2b_3 - \alpha\beta d_3 - \beta\alpha c_3 - \beta^2e_3 + B_3\alpha + B_4\gamma, \\ w_2\alpha &= -\alpha\gamma b_3 - \alpha\delta d_3 - \beta\gamma c_3 - \beta\delta e_3 + D_3\alpha + D_4\gamma, \\ w_3\alpha &= \alpha^2b_1 + \alpha\beta d_1 + \beta\alpha c_1 + \beta^2e_1 - B_1\alpha - B_2\gamma, \\ w_4\alpha &= \alpha\gamma b_1 + \beta\gamma d_1 + \alpha\delta c_1 + \delta\beta e_1 - C_1\alpha - C_2\gamma, \\ w_1\beta &= -\alpha^2b_4 - \alpha\beta d_4 - \beta\alpha c_4 - \beta^2e_4 + B_3\beta + B_4\delta, \\ w_2\beta &= -\alpha\gamma b_4 - \alpha\delta d_4 - \beta\gamma c_4 - \beta\delta e_4 + D_3\beta + D_4\delta, \\ w_3\beta &= \alpha^2b_2 + \alpha\beta d_2 + \beta\alpha c_2 + \beta^2e_2 - B_1\beta - B_2\delta, \\ w_4\beta &= \alpha\gamma b_2 + \beta\gamma d_2 + \alpha\delta c_2 + \beta\delta e_2 - C_1\beta - C_2\delta, \\ w_1\gamma &= -\alpha\gamma b_3 - \beta\gamma d_3 - \alpha\delta c_3 - \beta\delta e_3 + C_3\alpha + C_4\gamma, \\ w_2\gamma &= -\gamma^2b_3 - \gamma\delta d_3 - \gamma\delta c_3 - \delta^2e_3 + E_3\alpha + E_4\gamma, \\ w_3\gamma &= \alpha\gamma b_1 + \alpha\delta d_1 + \beta\gamma c_1 + \beta\delta e_1 - D_1\alpha - D_2\gamma, \\ w_4\gamma &= \gamma^2b_1 + \gamma\delta d_1 + \gamma\delta c_1 + \delta^2e_1 - E_1\alpha - E_2\gamma, \\ w_1\delta &= -\alpha\gamma b_4 - \beta\gamma d_4 - \alpha\delta c_4 - \beta\delta e_4 + C_3\beta + C_4\delta, \\ w_2\delta &= -\gamma^2b_4 - \gamma\delta d_4 - \gamma\delta c_4 - \delta^2e_4 + E_3\beta + E_4\delta, \\ w_3\delta &= \alpha\gamma b_2 + \alpha\delta d_2 + \beta\gamma c_2 + \beta\delta e_2 - D_1\beta - D_2\delta, \\ w_4\delta &= \gamma^2b_2 + \gamma\delta d_2 + \gamma\delta c_2 + \delta^2e_2 - E_1\beta - E_2\delta; \end{aligned}$$

$$
\begin{aligned}
(19)\qquad & w_1\gamma - w_2\alpha = \beta\gamma a_1 - \alpha\delta a_1 + A_1\alpha + A_2\gamma\,,\\
& w_1\delta - w_2\beta = \beta\gamma a_2 - \alpha\delta a_2 + A_1\beta + A_2\delta\,,\\
& w_3\gamma - w_4\alpha = \beta\gamma f_3 - \alpha\delta f_3 + F_3\alpha + F_4\gamma\,,\\
& w_3\delta - w_4\beta = \beta\gamma f_4 - \alpha\delta f_4 + F_3\beta + F_4\delta\,;
\end{aligned}
$$

$$
\begin{aligned}
(20)\qquad & (\alpha\delta - \beta\gamma) f_1 = \alpha F_1 + \gamma F_2\,,\\
& (\alpha\delta - \beta\gamma) f_2 = \beta F_1 + \delta F_2\,.
\end{aligned}
$$

From (18), we obtain

$$
\begin{aligned}
(21)\qquad & w_1\gamma - w_2\alpha = -\beta\gamma d_3 + \alpha\delta d_3 - \alpha\delta c_3 + \beta\gamma c_3 + c_3\alpha -\\
& \qquad - D_3\alpha + C_4\gamma - D_4\gamma\,,\\
& w_1\delta - w_2\beta = -\beta\gamma d_4 + \alpha\delta d_4 - \alpha\delta c_4 + \beta\gamma c_4 + c_3\beta -\\
& \qquad - D_3\beta + C_4\delta - D_4\delta\,,\\
& w_3\gamma - w_4\alpha = \alpha\delta d_1 - \beta\gamma d_1 + \beta\gamma c_1 - \alpha\delta c_1 - D_1\alpha +\\
& \qquad + C_1\alpha - D_2\gamma + C_2\gamma\,,\\
& w_3\delta - w_4\beta = \alpha\delta d_2 - \beta\gamma d_2 + \beta\gamma c_2 - \alpha\delta c_2 - D_1\beta +\\
& \qquad + C_1\beta - D_2\delta + C_2\delta\,.
\end{aligned}
$$

Multiplying the first equations in (19) and (21) by δ, the second ones by γ, the third ones by β and the fourth ones by α resp., we get

$$
\begin{aligned}
(22)\qquad & \delta(\beta\gamma - \alpha\delta)\, a_1 + \alpha\delta A_1 + \gamma\delta A_2 = \delta(\alpha\delta - \beta\gamma)\, d_3 +\\
& + \delta(\beta\gamma - \alpha\delta)\, c_3 + \alpha\delta(C_3 - D_3) + \gamma\delta(C_4 - D_4)\,,\\
& \gamma(\beta\gamma - \alpha\delta)\, a_2 + \beta\gamma A_1 + \gamma\delta A_2 = \gamma(\alpha\delta - \beta\gamma)\, d_4 +\\
& + \gamma(\beta\gamma - \alpha\delta)\, c_4 + \beta\gamma(C_3 - D_3) + \gamma\delta(C_4 - D_4)\,,\\
& \beta(\beta\gamma - \alpha\delta) f_3 + \alpha\beta F_3 + \beta\gamma F_4 = \beta(\alpha\delta - \beta\gamma)\, d_1 +\\
& + \beta(\beta\gamma - \alpha\delta)\, c_1 - \alpha\beta(C_1 - D_1) + \beta\gamma(C_2 - D_2)\,,\\
& \alpha(\beta\gamma - \alpha\delta) f_4 + \alpha\beta F_3 + \alpha\delta F_4 = \alpha(\alpha\delta - \beta\gamma)\, d_2 +\\
& + \alpha(\beta\gamma - \alpha\delta)\, c_2 - \alpha\beta(C_1 - D_1) + \alpha\delta(C_2 - D_2)
\end{aligned}
$$

and

$$
\begin{aligned}
(23)\qquad & A_1 + D_3 - C_3 = \delta(a_1 + d_3 - \varepsilon_3) - \gamma(a_2 + d_4 - c_2)\,,\\
& F_4 + D_2 - C_2 = \alpha(f_4 + d_2 - c_2) - \beta(f_3 + d_1 - c_1)\,.
\end{aligned}
$$

Analoguously, multiplying the first equations in (19) and (21) by β, the second ones by α, the third ones by δ and the fourth ones by γ resp. we get

$$(24)\qquad \begin{aligned} &\beta(\beta\gamma - \alpha\delta)\, a_1 + \beta\alpha A_1 + \beta\gamma A_2 = \beta(\alpha\delta - \beta\gamma)\, d_3 + \\ &+ \beta(\beta\gamma - \alpha\delta)\, c_3 + \beta\alpha(C_3 - D_3) + \beta\gamma(C_4 - D_4), \\ &\alpha(\beta\gamma - \alpha\delta)\, a_2 + \alpha\beta A_1 + \alpha\delta A_2 = \alpha(\alpha\delta - \beta\gamma)\, d_4 + \\ &+ \alpha(\beta\gamma - \alpha\delta)\, c_4 + \alpha\beta(C_3 - D_3) + \alpha\delta(C_4 - D_4), \\ &\delta(\beta\gamma - \alpha\delta) f_3 + \alpha\delta F_3 + \gamma\delta F_4 = \delta(\alpha\delta - \beta\gamma)\, d_1 + \\ &+ \delta(\beta\gamma - \alpha\delta)\, c_1 + \alpha\delta(C_1 - D_1) + \gamma\delta(C_2 - D_2), \\ &\gamma(\beta\gamma - \alpha\delta) f_4 + \beta\gamma F_3 + \gamma\delta F_4 = \gamma(\alpha\delta - \beta\gamma)\, d_2 + \\ &+ \gamma(\beta\gamma - \alpha\delta)\, c_2 + \beta\gamma(C_1 - D_1) + \gamma\delta(C_2 - D_2) \end{aligned}$$

and

$$(25)\qquad \begin{aligned} A_2 + D_4 - C_4 &= \alpha(a_2 + d_4 - c_4) - \beta(a_1 + d_3 - c_3), \\ F_3 + D_1 - C_1 &= \delta(f_3 + d_1 - c_1) - \gamma(f_4 + d_2 - c_2). \end{aligned}$$

Thus we see that we may choose the frames $\{w_1, w_2, w_3, w_4\}$ in such a way that

$$(26)\qquad A_1 + D_3 - C_3 = F_4 + D_2 - C_2 = 0.$$

Suppose that the frames $\{v_1, v_2, v_3, v_4\}$ have been chosen in this way, too. Then

$$(27)\qquad a_1 + d_3 - c_3 = f_4 + d_2 - c_2 = 0$$

From (23) and (25),

$$(28)\qquad \begin{aligned} 0 &= -\gamma(a_2 + d_4 - c_4), \\ A_2 + D_4 - C_4 &= \alpha(a_2 + d_4 - c_4), \\ 0 &= -\beta(f_3 + d_1 - c_1), \\ F_3 + D_1 - C_1 &= \delta(f_3 + d_1 - c_1). \end{aligned}$$

The functions $a_2 + d_4 - c_4, f_3 + d_1 - c_1$ are thus relative invariants. Let us restrict ourselves to the "general" case

$$(29)\qquad a_2 + d_4 - c_4 \neq 0, \quad f_3 + d_1 - c_1 \neq 0.$$

Then it is possible to choose the frames in such a way that

$$(30)\qquad a_2 + d_4 - c_4 = 1, \quad f_3 + d_1 - c_1 = 1.$$

Restricting ourselves to the frames satisfying (27) and (30), we get

$$(31) \qquad \alpha = 1\,, \quad \beta = 0\,, \quad \gamma = 0\,, \quad \delta = 1\,,$$

i.e., the conditions (27) and (30) determine exactly one field of frames $\{v_i\}$. In other words, the conditions (27) and (30) reduce our original G-structure to an $\{e\}$-structure. For this $\{e\}$-structure, $a_1, a_2, b_1, \ldots, b_4, c_1, \ldots, c_4, e_1, \ldots, e_4, f_1, \ldots, f_4$ are the invariants; $d_1, \ldots, d_4$ are given by (27) and (30). Now, it is easy to see the following

Theorem. *In* $\mathbb{C}^2$, *be given a transitive congruence* $\mathscr{L}$ *of surfaces. Considering the associated G-structure, we may reduce it to an* $\{e\}$*-structure given by* (4) *with* $a_1, \ldots, f_4 = \text{const.}$, (27), (30) *and*

$$(32) \quad \begin{aligned}
& d_3b_1 + d_4c_1 - b_3d_1 - b_4e_1 - a_2d_1 + a_1e_2 = 0\,,\\
& d_3b_2 + d_4c_2 - b_3d_2 - b_4e_2 + b_1a_2 - a_1b_2 = 0\,,\\
& d_4c_3 - b_4e_3 - d_3e_2 - a_1b_3 = 0\,,\\
& d_3b_4 + d_4c_4 - b_3d_4 - b_4e_4 - d_4a_2 - a_1b_4 = 0\,,\\
& e_3b_1 + e_4c_1 - c_3d_1 - c_4e_1 - e_1a_2 + e_2a_1 = 0\,,\\
& e_3b_2 + e_4c_2 - c_3d_2 - c_4e_2 + c_1a_2 - a_1c_2 = 0\,,\\
& e_3b_3 + e_4c_3 - c_3d_3 - c_4e_3 - e_3a_2 - a_1c_3 = 0\,,\\
& e_3b_4 - c_3d_4 - e_4a_2 - a_1c_4 = 0\,,\\
& f_3b_1 + f_4c_1 + c_2d_1 - c_4f_1 - b_2e_1 - b_3f_1 + f_2a_1 = 0\,,\\
& f_3b_2 + f_4c_2 + c_1b_2 + c_2d_2 - c_4f_2 - b_1c_2 - b_2e_2 - b_3f_2 + f_2a_2 = 0\,,\\
& f_4c_3 + c_1b_3 + c_2d_3 - c_4f_3 - b_1c_3 - b_2e_3 = 0\,,\\
& f_3b_4 + c_1b_4 + c_2d_4 - b_1c_4 - b_2e_4 - b_3f_4 = 0\,,\\
& f_3d_1 + f_4e_1 + e_1b_1 + e_2d_1 - e_4f_1 - d_1c_1 - d_2e_1 - d_3f_1 - a_1f_1 = 0\,,\\
& f_3d_2 + f_4e_2 + e_1b_2 - e_4f_2 - d_1c_2 - d_3f_2 - f_1a_2 = 0\,,\\
& f_4e_3 + e_1b_3 + e_2d_3 - e_4f_3 - d_1c_3 - d_2e_3 = 0\,,\\
& f_3d_4 + e_1b_4 + e_2d_4 - d_1c_4 - d_2e_4 + d_3f_3 = 0\,.
\end{aligned}$$

Bibliography

[1] *A. Švec:* On a group of holomorphic transformations in $\mathscr{C}^2$. Czech. Math. J., *24* (99), 1974, 97—106.

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