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ON OVALOIDS IN  $E^4$

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One of the main tools used for characterizations of hyperspheres in  $E^n$  is the integral formula (1.14.1) of [1]. For a simple situation to be described below, we are going to rewrite this formula and to prove several more profound consequences of it.

Let  $M^3 \subset E^4$  be an hypersurface satisfying: (i) on  $M^3$ , there is a system of lines of curvature, (ii) the principal curvatures are positive, (iii) the boundary  $\partial M^3$  of  $M^3$  consists of umbilical points.

In a suitable domain of  $M^3$ , consider the moving orthonormal frames  $\{M, v_1, v_2, v_3, v_4\}$  such that  $v_1, v_2, v_3$  are tangent to the lines of curvature. Then

$$\begin{aligned}
 (1) \quad dM &= \omega^1 v_1 + \omega^2 v_2 + \omega^3 v_3, \\
 dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4, \\
 dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4, \\
 dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4, \\
 dv_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3
 \end{aligned}$$

with the usual integrability conditions, and we may write

$$(2) \quad \omega_1^4 = a\omega^1, \quad \omega_2^4 = b\omega^2, \quad \omega_3^4 = c\omega^3,$$

$a > 0, b > 0, c > 0$  being the principal curvatures. From (2), we get

$$\begin{aligned}
 (3) \quad da \wedge \omega^1 + (a - b)\omega_1^2 \wedge \omega^2 + (a - c)\omega_1^3 \wedge \omega^3 &= 0, \\
 (a - b)\omega_1^2 \wedge \omega^1 + db \wedge \omega^2 + (b - c)\omega_2^3 \wedge \omega^3 &= 0, \\
 (a - c)\omega_1^3 \wedge \omega^1 + (b - c)\omega_2^3 \wedge \omega^2 + dc \wedge \omega^3 &= 0
 \end{aligned}$$

and the existence of functions  $a_1, \dots, c_3, e$  such that

$$\begin{aligned}
 (4) \quad da &= a_1\omega^1 + a_2\omega^2 + a_3\omega^3, \quad (a - b)\omega_1^2 = a_2\omega^1 + b_1\omega^2 + e\omega^3, \\
 db &= b_1\omega^1 + b_2\omega^2 + b_3\omega^3, \quad (a - c)\omega_1^3 = a_3\omega^1 + e\omega^2 + c_1\omega^3, \\
 dc &= c_1\omega^1 + c_2\omega^2 + c_3\omega^3, \quad (b - c)\omega_2^3 = e\omega^1 + b_3\omega^2 + c_2\omega^3.
 \end{aligned}$$

The curvatures of  $M^3$  be defined by

$$(5) \quad H = a + b + c, \quad L = ab + ac + bc, \quad K = abc.$$

On  $M^3$ , consider the invariant 2-form

$$(6) \quad \tau = (b - c)^2 \omega^1 \wedge \omega_2^3 - (a - c)^2 \omega^2 \wedge \omega_1^3 + (a - b)^2 \omega^3 \wedge \omega_1^2;$$

we have

$$(7) \quad d\tau = \{2\varphi(b_1, c_1, a_1) + 2\varphi(a_2, c_2, b_2) + 2\varphi(a_3, b_3, c_3) + 6e^2 + (a - b)^2 ab + (a - c)^2 ac + (b - c)^2 bc\} \omega^1 \wedge \omega^2 \wedge \omega^3,$$

where

$$(8) \quad \varphi(x, y, z) = x^2 + y^2 - xy - xz - yz.$$

**Theorem 1.** Let  $M^3 \subset E^4$  satisfy (i)–(iii) and (iv): there is, on  $M^3$ ,

$$(9) \quad A da + B db + C dc = 0,$$

$A, B, C$  being functions such that

$$(10) \quad A > 0, \quad B > 0, \quad C > 0; \quad \varrho(A, B, C) \geq 0, \quad \varrho(A, C, B) \geq 0, \\ \varrho(B, C, A) \geq 0; \quad \varrho(X, Y, Z) := 4(X + Z)(Y + Z) - (X + Y - Z)^2.$$

Then  $M^3$  is a (part of a) hypersphere.

**Proof.** From (9), we get

$$(11) \quad Aa_1 = -Bb_1 - Cc_1, \quad Bb_2 = -Aa_2 - Cc_2, \quad Cc_3 = -Aa_3 - Bb_3.$$

Then

$$(12) \quad A\varphi(b_1, c_1, a_1) = (A + B)b_1^2 + (B + C - A)b_1c_1 + (A + C)c_1^2 \geq 0, \\ B\varphi(a_2, c_2, b_2) = (A + B)a_2^2 + (A + C - B)a_2c_2 + (B + C)c_2^2 \geq 0, \\ C\varphi(a_3, b_3, c_3) = (A + C)a_3^2 + (A + B - C)a_3b_3 + (B + C)b_3^2 \geq 0$$

as a consequence of (10). From the Stokes formula  $\int_{\partial M} \tau = \int_M d\tau$ , we get  $a = b = c$ . QED.

**Theorem 2.** Let  $M^3 \subset E^4$  satisfy (i)–(iii) and (iv): there is a function  $F(H, L, K)$  such that, on  $M^3$ ,

$$(14) \quad \sigma(b, c) > 0, \quad \sigma(a, c) > 0; \quad \sigma(a, b) > 0; \\ \sigma(\xi, \eta) := F_H + (\xi + \eta)F_L + \xi\eta F_K;$$

$$\begin{aligned}
(15) \quad & \kappa(a, b, c) \geq 0, \quad \kappa(a, c, b) \geq 0, \quad \kappa(b, c, a) \geq 0, \\
& \kappa(u, v, w) := 15F_H^2 + 4(2u^2 + 2v^2 + 5uv + 3uw + 3vw)F_L^2 + \\
& + (3u^2v^2 - u^2w^2 - v^2w^2 + 6u^2vw + 6uv^2w + 2uvw^2)F_K^2 + \\
& + 12(2u + 2v + w)F_HF_L + 6(3uv + uw + vw)F_HF_K + \\
& + 4(3u^2v + u^2w + 3uv^2 + v^2w + 7uvw)F_LF_K.
\end{aligned}$$

Then  $M^3$  is a (part of a) hypersphere.

Proof. From (13), we get (9) with

$$(16) \quad A = \sigma(b, c), \quad B = \sigma(a, c), \quad C = \sigma(a, b)$$

and the conditions (10) turn out to be exactly (14) and (15). QED.

Let us prove just two consequences of our Theorem 2.

**Corollary 1.** Let  $M^3 \subset E^4$  satisfy (i)–(iii) and (iv): we have, on  $M^3$ ,

$$(17) \quad f(H, L, rHL + K) = 0,$$

$r \in \mathbb{R}$  satisfying  $83r \geq 6\sqrt{(3) - 5}$  and  $f(\alpha, \beta, \gamma)$  being a function with one of its derivatives positive and the other two non-negative. Then  $M^3$  is a (part of a) hypersphere.

Proof. Set

$$(18) \quad F(H, L, K) = f(H, L, rHL + K).$$

Then

$$(19) \quad F_H = f_\alpha + rLf_\gamma, \quad F_L = f_\beta + rHf_\gamma, \quad F_K = f_\gamma,$$

$$(20) \quad \sigma(\xi, \eta) = f_\alpha + (\xi + \eta)f_\beta + \{r(\xi + \eta)(2H - \xi - \eta) + (r + 1)\xi\eta\}f_\gamma,$$

and we have (14). Further,

$$(21) \quad \kappa(u, v, w) = \mu_1f_\alpha^2 + \mu_2f_\beta^2 + \mu_3f_\alpha f_\beta + \mu_4f_\alpha f_\gamma + \mu_5f_\beta f_\gamma + \\ + \{\mu_6 + (83r^2 + 10r - 1)w^2(u^2 + v^2)\}f_\gamma^2$$

with  $\mu_i = \mu_i(u, v, w) \geq 0$  for  $u \geq 0, v \geq 0, w \geq 0$ , and (15) follow easily. QED.

**Corollary 2.** Let  $M^3 \subset E^4$  satisfy (i)–(iii) and (iv): we have, on  $M^3$ ,

$$(22) \quad K = \text{const.}, \quad 4HK \geq L^2.$$

Then  $M^3$  is a (part of a) hypersphere.

**Proof.** For  $F(H, L, K) = K - \text{const.}$ , we get

$$\begin{aligned}\kappa(u, v, w) &= 3u^2v^2 - u^2w^2 - v^2w^2 + 6u^2vw + 6uv^2w + 2uvw^2 = \\ &= 4(u + v + w)uvw - (uv + uw + vw)^2 + 4uv(uv + uw + vw).\end{aligned}$$

Thus (22<sub>2</sub>) implies (15). QED.

#### *Bibliography*

- [1] *U. Simon*: A further method in global differential geometry. *Abh. Math. Sem. Hamburg*, 44 (1975), 52–69.

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