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Časopis pro pěstování matematiky, Vol. 101 (1976), No. 3, 237--255

Persistent URL: `http://dml.cz/dmlcz/117917`

Časopis pro pěstování matematiky, roč. 101 (1976), Praha

APPROXIMATION OF DOMAINS WITH LIPSCHITZIAN BOUNDARY

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(Received May 21, 1975)

1. INTRODUCTION

In this paper we shall consider (bounded) domains with a lipschitzian boundary and their approximations. First, Theorem 4,1 is proved which states that if Ω has a lipschitzian boundary then "δ-neighbourhoods" of this domain have lipschitzian boundaries as well, and approximate Ω uniformly; tangent hyperplanes of these neighbourhoods approximate those of Ω in the integral sense. This theorem is of importance by itself. With the help of this theorem, it is possible to prove some assertions concerning perturbations of Sobolev spaces (see [1] where Theorem 4,1 is used without proof; our paper is a supplement to [1]). Moreover, it allows us to prove a theorem regarding an approximation of the boundary of a lipschitzian domain with the help of infinitely differentiable manifolds (Theorem 5,1). A theorem of this type was proved by NEČAS ([3]) in 1962 in a rather difficult and technical way. MASSARI and PEPE ([2]) proved a similar theorem in 1974; the proof is simpler but, on the other hand, the assertion is weaker then that of NEČAS.

The idea of our proof is similar as in [2], but by making use of Theorem 4,1 and by modifying the procedure we obtain an assertion which is more or less equivalent to the theorem of Nečas. In comparison with this theorem, we prove only the existence of a discrete sequence Ω_n of approximating domains instead of a system Ω_t depending on $t > 0$; on the other hand, we obtain in addition the point (iv) which reppresents a certain mode of "controlled convergence". Further, the theorem of Nečas deals with an approximation "from the interior" instead of our approximation "from the exterior", nonetheless, considering the domain $B - \Omega$ where B is a ball containing Ω we obtain one type of approximation from the other.

2. GEOMETRICAL PROPERTIES OF LIPSCHITZIAN FUNCTIONS

We shall consider $(N + 1)$-dimensional Euclidean space R^{N+1} with a fixed coordinate system $(x_1, x_2, \ldots, x_N, y)$ or, briefly, (X', y), where $X' = (x_1, x_2, \ldots, x_N) \in$

$\in R^N$; by $|X|$ or $|X'|$ we shall denote the $(N + 1)$-dimensional or N-dimensional Euclidean norm, respectively, and by (X, Y) or (X', Y') we denote the corresponding inner product.

Definition 2,1. Let a positive number L be given, and let $X = (X', y) \in R^{N+1}$. By $K_1(X; L)$, $K_2(X; L)$, $K_3(X; L)$ we shall denote the open cones

$$(2,1) \qquad \begin{aligned} K_1(X; L) &= \{Z = (Z', w) \in R^{N+1};\ (w - y) > L|(X' - Z')|\}\,, \\ K_2(X; L) &= \{Z \in R^{N+1};\ -L|X' - Z'| < (w - y) < L|X' - Z'|\}\,, \\ K_3(X; L) &= \{Z \in R^{N+1}; (w - y) < -L|X' - Z'|\}\,, \end{aligned}$$

(see Fig. 1).

These cones allow us to give a geometrical characterisation of the Lipschitz property of functions:

Lemma 2,1. *Let φ be a function defined on R^N, and let $[\varphi]$ be its graph.*

Then φ has the Lipschitz property with a Lipschitz constant L if and only if for every $Z' \in R^N$ the following inclusion holds:

$$(2,2) \qquad [\varphi] \subset \overline{K_2((Z', \varphi(Z')); L)}$$

where $\bar{K}$ is the closure of K in the norm of R^{N+1} (see Fig. 1).

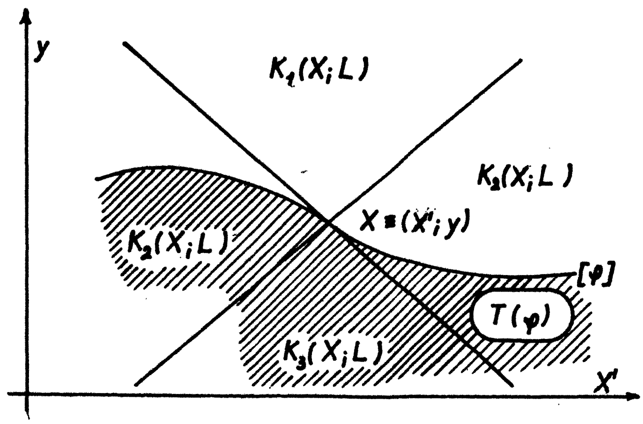

Fig. 1.

Proof. Rewriting (2,2) in terms of the inequalities which define K_2 we obtain exactly the Lipschitz property of φ.

In the following, an alternative characterisation of lipschitzian functions will be useful; cf. also [2]:

Lemma 2,2. *Let φ be a function defined on R^N, and let us denote by $T(\varphi)$ the set of points "under the graph of φ":*

$$T(\varphi) = \{X = (X', y);\ y < \varphi(X')\}\,. \tag{2,3}$$

Then φ has the Lipschitz property with a constant L iff the following inclusions hold for arbitrary $X' \in R^N$ (see Fig. 1):

$$\text{(2,4)}\quad \text{(i)}\qquad T(\varphi) \supset K_3((X', \varphi(X')); L)\,,$$

$$\text{(ii)}\qquad R^{N+1} - \overline{T(\varphi)} \supset K_1((X', \varphi(X')); L)\,.$$

Remark: For a function φ defined on an open set $A \subset R^N$ we obtain completely analogous characterisations locally, i.e., by considering intersections of $T(\varphi)$, $K_1(X; L)$ etc. with a sufficiently small ball; cf. once more [2]. Such localization will be used in the following.

Now we shall consider geometric properties of the derivative at a given point.

Definition 2,2. Let $X_0 = (X'_0, y_0) \in R^{N+1}$ be a given point and let $H = (h_1, h_2, \ldots$ $\ldots, h_N)$. Let us denote by $p(X_0; H)$ the hyperplane

$$p(X_0; H) = \{(X', y);\ y = y_0 + (H, X' - X'_0)\} \tag{2,5}$$

and by $P(X_0; H)$ the set of points under this hyperplane:

$$P(X_0; H) = \{(X', y);\ y < y_0 + (H, X' - X'_0)\} \tag{2,6}$$

(see Fig. 2). Given $\varepsilon > 0$, let us denote by $A(X_0; H, \varepsilon)$ the cone

$$\text{(2,7)}\qquad A(X_0; H, \varepsilon) =$$

$$= \{Z = (Z', z);\ z = y_0 + (H, Z' - X'_0) + t|Z' - X'_0|,\ t \in (-\varepsilon, \varepsilon)\}$$

(see Fig. 3).

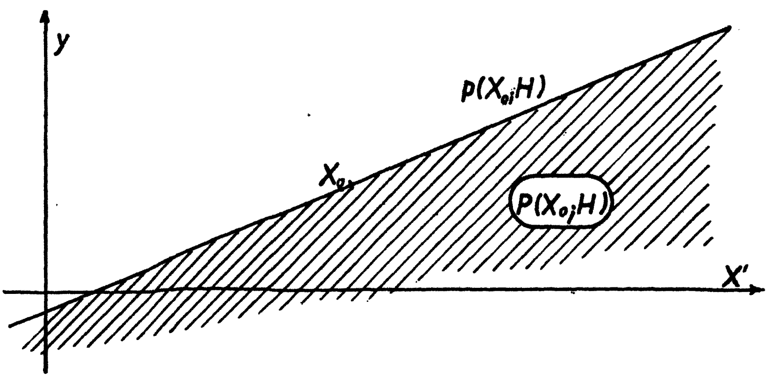

Fig. 2.

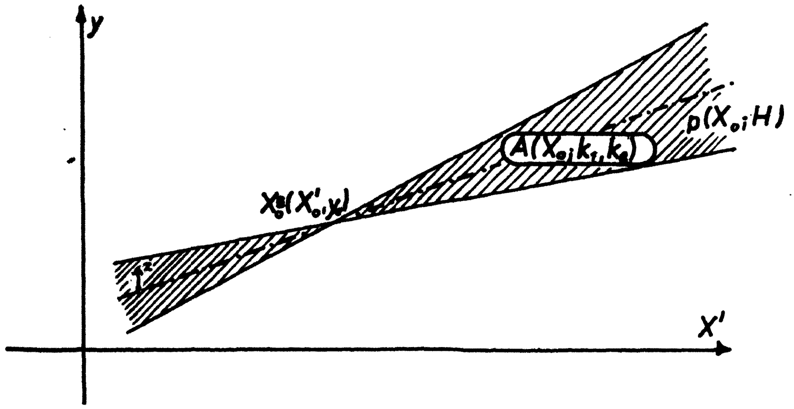

Fig. 3.
[Correction: instead of $A(X_0; k_1, k_2)$ read $A(X_0; H, \varepsilon)$]

Lemma 2,3. *Let φ be a function defined on R^{N+1}, and let $X'_0 \in R^N$ be a given point, $X_0 = (X'_0, \varphi(X'_0))$; let $H = (h_1, h_2, \ldots, h_N)$.*

Then φ has at the point X'_0 the gradient equal to H iff for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[\varphi] \cap \{(Z', z);\ |Z'| < \delta\} \subset \overline{A(X_0; H, \varepsilon)}\,.$$

Proof. The assertion is exactly a geometric transliteration of the definition of the gradient.

In the following lemma, we give an assertion regarding the Dini derivatives. For the sake of convenience, let us suppose $N = 1$.

Lemma 2,4. *Let φ be a function defined on R^1, and let $x_0 \in R^1$ be a fixed point, $X_0 = (x_0, \varphi(x_0))$. Let k be a real number, and let*

$$D^+ \varphi(x_0) = \limsup_{x \to x_0+} \frac{\varphi(x) - \varphi(x_0)}{x - x_0}$$

be the right upper Dini derivative of φ. Let the inclusion

$$\{(x, y) \in [\varphi];\ x_0 < x < x_0 + \delta\} \subset P(X_0, k)$$

holds for some positive δ.

Then $D^+ \varphi(x_0) \leqq k$.

Analogously, let $D_- \varphi(x_0)$ be the left lower Dini derivative, and let

$$\{(x, y) \in [\varphi];\ x_0 - \delta < x < x_0\} \subset P(X_0; k)\,.$$

Then $D_- \varphi(x_0) \leqq k$.

Proof. The assertions follow immediately from the definitions of D^+, D_-, $P(X_0; k)$.

The arithmetic formulation of these assertions is rather complicated but their geometrical meaning is quite clear.

Let us now define some geometrical notions which shall be used in the sequel.

Definition 2,3. Let $X \in R^{N+1}$ be a given point, and let d be a positive number. By $B(X; d)$ and $C(X; d)$ we denote the ball and sphere, respectively, with a centre X and radius d:

$$B(X; d) = \{Z \in R^{N+1};\ |X - Z| < d\},$$

$$C(X; d) = \{Z \in R^{N+1};\ |X - Z| = d\}.$$

Let K be a subset of R^{N+1}; by dist (X, K) we denote the distance between X and K:

$$\operatorname{dist}(X, K) = \inf\{|X - Z|;\ Z \in K\}.$$

3. LEMMAS CONCERNING LIPSCHITZIAN FUNCTIONS

Lemma 3,1. *Let φ be a lipschitzian function defined on R^N, with a Lipschitz constant L, and let $d > 0$. Let $T_d(\varphi)$ be the set*

$$T_d(\varphi) = \{X \in R^{N+1};\ \operatorname{dist}(X, T(\varphi)) < d\}. \tag{3,1}$$

Then there exists a function φ_d defined on R^N, such that $T_d(\varphi)$ lies under $[\varphi_d]$:

$$T_d(\varphi) = T(\varphi_d); \tag{3,2}$$

moreover, this function has the Lipschitz property with the same constant L.

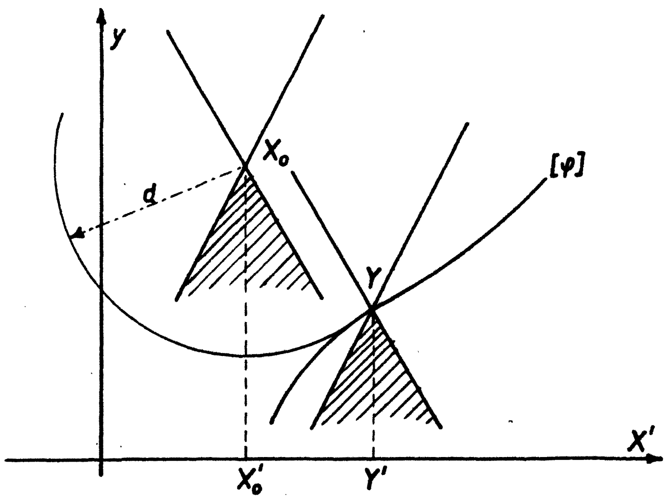

Fig. 4.

Proof. Let $X_0' \in R^N$ be a given point. Because of the Lipschitz property of φ, $T(\varphi) \subset R^{N+1} - K_1((X_0', \varphi(X_0')), L) \equiv Q$ (Lemma 2,2) and $\operatorname{dist}((X_0', \xi), T(\varphi)) \geqq \operatorname{dist}((X_0', \xi), Q) \to \infty$ for $\xi \to \infty$; so there exists a point $X = (X_0', \xi)$ such that $\operatorname{dist}(X, T(\varphi)) = d$. Let $Y = (Y', \eta)$ be a point at which this distance is attained: $|X - Y| = d$. Obviously such a point exists and $\eta = \varphi(Y')$.

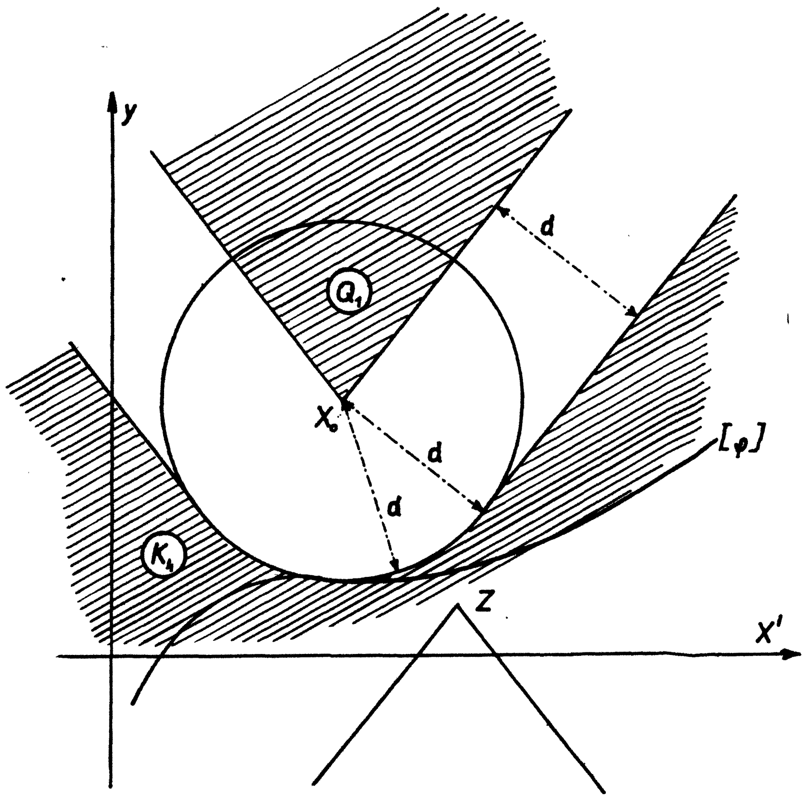

Fig. 5.

Once more because of the Lipschitz property, $K_3(Y; L) \subset T(\varphi)$ and hence $K_3(X; L) \subset T_d(\varphi)$ (see Fig. 4). On the other hand, $T(\varphi) \cap B(X; d) = \emptyset$. But $Z \in T(\varphi)$ implies $K_3(Z; L) \subset T(\varphi)$ and hence $T(\varphi) \subset K_4 = \bigcup K_3(Z; L)$, where the union is taken over all Z such that $K_3(Z; L) \cap B(X; d) = \emptyset$ (see Fig. 5).

Let us consider the set $Q_1 = \{Z \in R^{N+1}; \operatorname{dist}(Z, K_4) > d\}$. Obviously $Q_1 = K_1(X; L)$ and $Q_1 \cap T_d(\varphi) = \emptyset$. In particular we obtain $\bar{X} = (X_0', \xi) \in T_d(\varphi)$ for $\xi < \xi_0$ and $\bar{X} \notin T_d(\varphi)$ for $\xi > \xi_0$ and hence we can put $\varphi_d(X_0') = \xi_0$. The Lipschitz property of φ_d is now a consequence of Lemma 2,2.

Lemma 3,2. *Let φ be a lipschitzian function defined on R^N, and let φ_d $(d > 0)$ be a function defined in Lemma 3,1. Let $X_0' \in R^N$ be such a point that there exists*

$$\operatorname{grad} \varphi(X'_0) = (h_1, \ldots, h_N) ;$$

then

(3,3) (i) $$\limsup_{d \to 0^+} D_i^+ \varphi_d(X'_0) \leqq h_i ,$$

(ii) $$\liminf_{d \to 0^+} D_{i,-} \varphi_d(X'_0) \geqq h_i ,$$

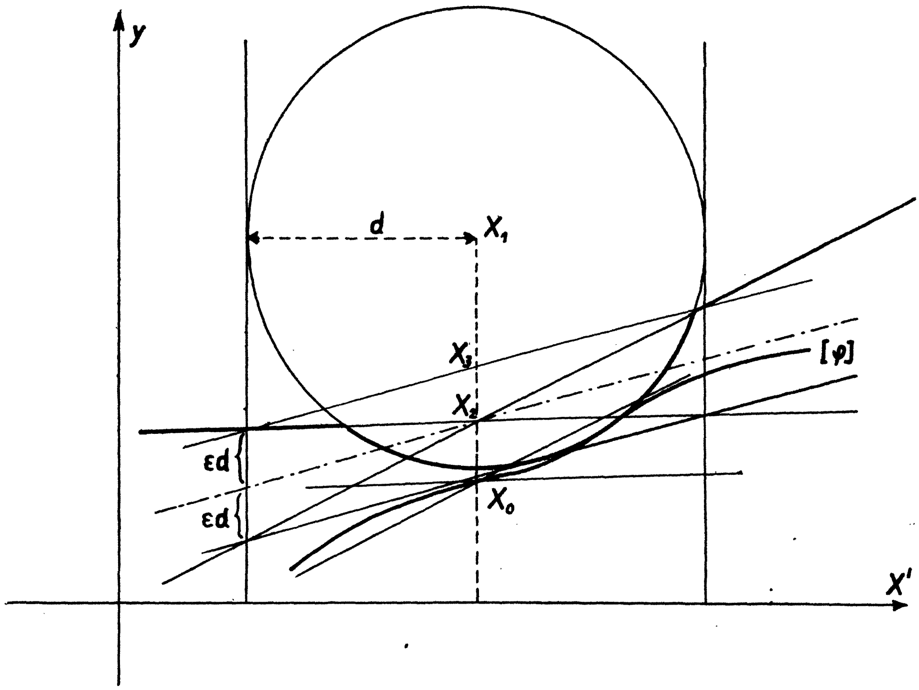

Fig. 6.

where D_i^+, $D_{i,-}$ stands for the Dini derivatives with respect to the i-th variable.

Proof. Let us supposed $H \neq 0$ (the case $H = 0$ is even easier). Without loss of generality, we can suppose $h_1 > 0$, $h_2 = h_3 = \ldots = h_N = 0$ (if this is not the case we can use a rotation of the co-ordinate system); further, we can suppose $X'_0 = (x_{0,1}, 0, \ldots, 0)$.

Let $\varepsilon > 0$, and let $X_0 = (X'_0, \varphi(X'_0))$. According to Lemma 2,3, $[\varphi]$ locally lies in a set $A(X_0; H, \varepsilon)$. Considering d small enough we can suppose that all the graph lies in this set. Let $y_1 = \varphi_d(X'_0)$ and $X_1 = (X'_0, y_1)$; we have $X_1 \notin A(X_0; H, \varepsilon)$. It is $T(\varphi) \cap B(X_1; d) = \emptyset$ and hence

$$T(\varphi) \subset [A(X_0; H, \varepsilon) \cup P(X_0; H)] - B(X_0; d) \subset$$
$$\subset [A(X_2; H, \varepsilon) \cup P(X_2; H)] - B(X_0; d) \equiv Q_2$$

where $X_2 = (X'_0, y_2)$, $y_2 > \varphi(X'_0)$ (see Fig. 6).

On the other hand, $T(\varphi) \cap B(X_1; d) = \emptyset$ yields $y_2 > \varphi(X'_0)$ for y_2 such that $p((X'_0; y_2 - d\varepsilon); H)$ is a tangent hyperplane to $B(X_1; d)$ (see Fig. 6). Let $X_3 = (X'_0, y_2 + d\varepsilon)$.

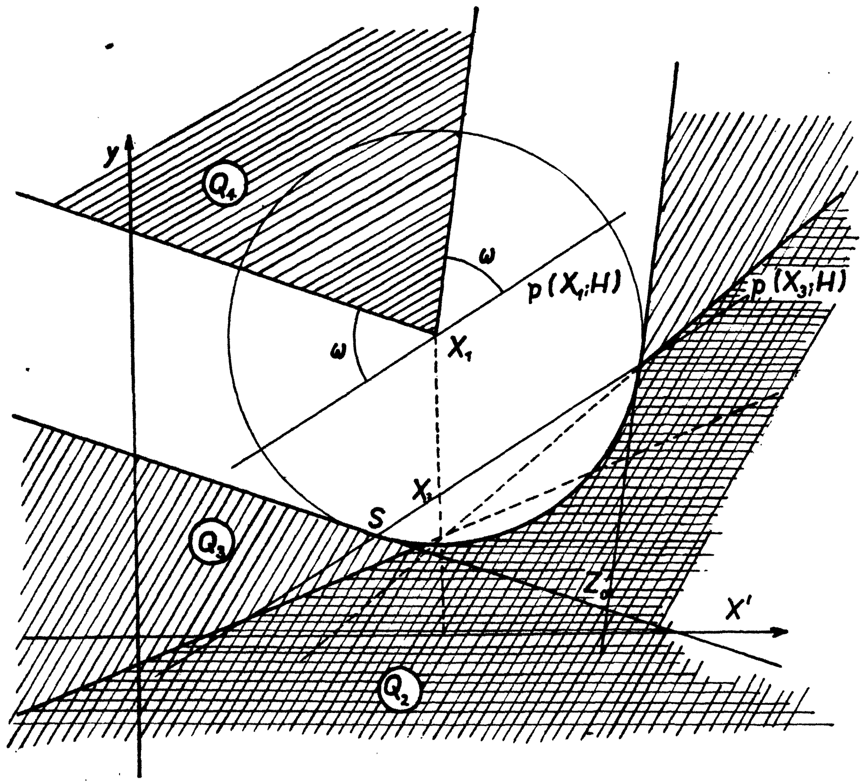

Fig. 7.

Constructing at all points $S \in C(X_1; d) \cap p(X_3; H)$ the tangent to $C(X_1; d)$ which lies in the (two-dimensional) plane containing X_1, S and a vector V perpendicular to $p(X_3; H)$ we obtain a set Q_3 – see Fig. 7 – which is, in fact, congruent with the set K_4 from the previous lemma (with a suitable L). This set contains Q_2. To see it, let us note that it is sufficient to consider the problem in the space R^3 which contains the axes y, x_1 and an arbitrary point $(Z', 0) \in R^{N+1}$. In this case, it is sufficient to show that $Q_2 \cap p \subset Q_3 \cap p$ for an arbitrary plane p containing a straight line $X_1 + tV$, $t \in R^1$. Such a p intersects the boundary of $A(X_0; H, \varepsilon)$ in a certain hyperbola for which the inclusion is obvious; see Fig. 8.

Now, as in the previous lemma, we shall construct the domain

$$Q_4 = \{Z \in R^{N+1}; \operatorname{dist}(Z; Q_3) > d\}$$

(see Fig. 7). This domain is a cone such that the angless between its surface-lines and the hyperplane $p(X_1, H)$ are constant, say ω. Elementary calculation yields that $\omega = \omega(H; \varepsilon)$ where $\lim_{\varepsilon \to 0^+} \omega(H; \varepsilon) = 0$. The assertion of Lemma 3,2 is now a consequence of Lemma 2,4.

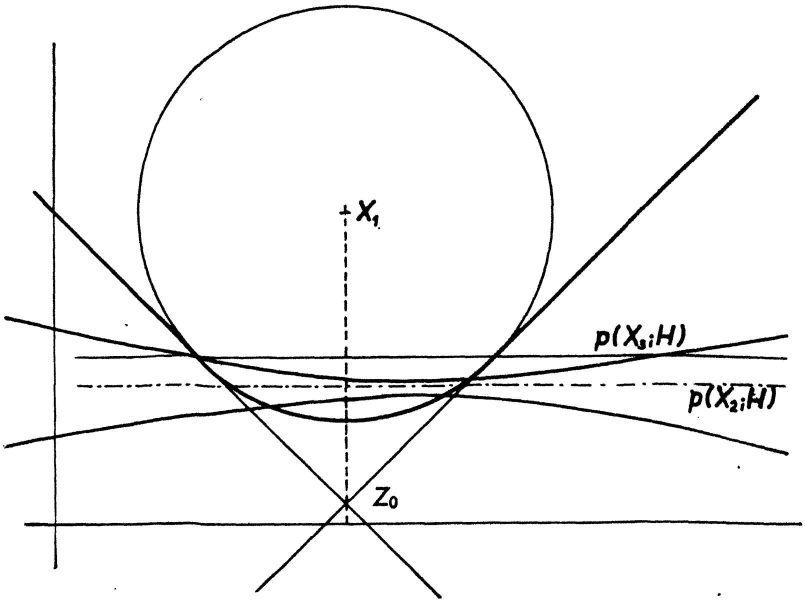

Fig. 8.

Lemma 3,3. *Let φ be a lipschitzian function defined on R^N, and let φ_d $(d > 0)$ be the function defined in Lemma 3,1. Then the functions φ, φ_d have all partial derivatives of the first order almost everywhere, and for arbitrary $p \geqq 1$*

$$\lim_{d \to 0^+} \int_K \left| \frac{\partial \varphi_d}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^p \mathrm{d}X' = 0$$

for every compact subset $K \subset R^N$ $(i = 1, 2, \ldots, N)$.

Proof. The function φ being lipschitzian it has the first partial derivatives on R^N except for a set of measure zero. Let us now consider an arbitrary sequence $d(n)$, $d(n) \to 0^+$ for $n \to \infty$. Then, by the same argument, the functions $\varphi_{d(n)}$, φ have derivatives on the set $R^N - M$, mes $M = 0$, and on this set we have

$$\frac{\partial \varphi_{d(n)}}{\partial x_i} = D_i^+ \varphi_{d(n)} = D_{i,-} \varphi_{d(n)} \,.$$

Lemma 3,2 then yields

$$\frac{\partial \varphi_{d(n)}(X')}{\partial x_i} \to \frac{\partial \varphi(X')}{\partial x_i}$$

on $R^N - M$, and we have

$$\left|\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi_{d(n)}}{\partial x_i}\right|^p < (2L)^p$$

(Lemma 3,1). The assertion then follows from the Lebesgue dominated convergence theorem.

4. APPROXIMATION OF LIPSCHITZIAN DOMAIN

We shall consider a bounded domain $\Omega \subset R^{N+1}$ with a lipschitzian boundary, i.e. a domain whose boundary is locally representable as a graph of a lipschitzian function in a convenient co-ordinate system. Thanks to the compactness of the bounded closed set $\partial\Omega$, we can describe $\partial\Omega$ with the help of a finite number of co-ordinate systems, and so we can define:

Definition 4,1. Let $\Omega \subset R^{N+1}$ be a bounded domain. We say that Ω has a lipschitzian boundary if there exists a finite number of ("local") co-ordinate systems $(X'_r, y_r) = (x_{1,r}, \ldots, x_{N,r}, y_r)$ $(r = 1, 2, \ldots, m)$ and a finite number of lipschitzian functions a_r, $r = 1, 2, \ldots, m$ (with a Lipschitz constant L), defined on the neighbourhood of zero

$$\Delta = \{X'_r \in R^N;\ |x_{i,r}| < \beta,\ i = 1, 2, \ldots, N\} \quad (\beta > 0)$$

such that

(4,1) (i) $X = (X'_r, y_r) \in \Omega$ for $X'_r \in \Delta$, $a_r(X'_r) - \beta < y_r < a_r(X'_r)$,

(ii) $X \notin \bar{\Omega}$ for $X'_r \in \Delta$, $a_r(X'_r) < y_r < a_r(X'_r) + \beta$,

(iii) for every $X \in \partial\Omega$ there exists r $(r = 1, 2, \ldots, m)$ and $X'_r \in \Delta$ such that $X = (X'_r, a_r(X'_r))$ in the corresponding co-ordinate system.

(See [3]; cf. also [2].)

Theorem 4,1. *Let $\Omega \subset R^{N+1}$ be a bounded domain with a lipschitzian boundary, and let us denote by Ω_d $(d > 0)$ the domain*

$$\Omega_d = \{X \in R^{N+1};\ \operatorname{dist}(X, \Omega) < d\}. \tag{4,2}$$

Then for d small enough, Ω_d has also a lipschitzian boundary, and

$$\bar{\Omega} = \lim_{d \to 0^+} \Omega_d \ \Big(= \bigcap_{d > 0} \Omega_d\Big). \tag{4,3}$$

Moreover, the boundary of Ω_d is described with the help of the same co-ordinate systems as $\partial\Omega$, and the corresponding functions $a_{r,d}$ approximate a_r in the following sense:

(4,4) (i) $a_{r,d} \rightrightarrows a_r$ on Δ (*i.e. the convergence is uniform*);

(ii) $a_{r,d}$ *are lipschitzian functions with the same constant L as* a_r;

(iii) $\dfrac{\partial a_{r,d}}{\partial x_{i,r}}$ $(i = 1, 2, \ldots, N)$

exists almost everywhere in Δ *and, for arbitrary* $p \geqq 1$,

$$\lim_{d \to 0^+} \int_\Delta \left| \frac{\partial a_{r,d}}{\partial x_{i,r}} - \frac{\partial a_r}{\partial x_{i,r}} \right|^p \mathrm{d}X'_r = 0 .$$

Proof. The assertions follow from lemmas of previous section with the help of the technique of localization; cf. the remark following Lemma 2,1. To this end, we use the fact that there obviously exists $\beta' < \beta$ such that replacing β in Definition 4,1 by this β' we obtain again a description of the boundary.

5. SMOOTH APPROXIMATION OF LIPSCHITZIAN DOMAINS

In this section, we shall consider an approximation of lipschitzian domains with the help of C^∞-domains. The possibility of such approximation was proved by Nečas in [3] but his proof is rather complicated. Massari and Pepe obtained a similar result in [2]; the proof given there is simpler but the result obtained is weaker than that of Nečas. With the help of Theorem 4,1 of the present paper we can obtain an assertion which is more or less equivalent to the Nečas theorem. Our proof, though not very simple either, is by our opinion more transparent than that in [3].

As in the previous sections, we shall study first the approximation of lipschitzian functions; our construction will not depend on co-ordinate system. To this end, we shall use mollifiers.

Definition 5,1. Let us define the mollifier $\omega = \omega(X)$, $X \in R^{N+1}$, $X = (X', y)$ by

$$\omega(X) = \begin{cases} \varkappa \exp\left(|X|^2 - 1\right)^{-1}, & |X| < 1, \\ 0, & |X| \geqq 1, \end{cases} \tag{5,1}$$

where $\varkappa$ is such that

$$\int_{B(0;1)} \omega(X)\,\mathrm{d}X = 1 . \tag{5,2}$$

For $h > 0$, let us denote

$$\omega_h(X) = h^{-N-1}\,\omega\left(\frac{X}{h}\right). \tag{5,3}$$

Now, let φ be a lipschitzian function defined on R^N and let Φ be the characteristic function of $T(\varphi)$:

$$(5,4)\qquad \Phi(X', y) = \begin{cases} 1, & y < \varphi(X'), \\ 0, & y \geqq \varphi(X'). \end{cases}$$

For $h > 0$, let us define Φ_h as

$$(5,5)\qquad \Phi_h(X) = (\omega_h * \Phi)(X) = \int_{B(0;h)} \Phi(X - Y)\, \omega_h(Y)\, \mathrm{d}Y.$$

The properties of a mollifier (see e.g. [3]) imply that the function Φ_h has continuous derivatives of all orders and

$$(5,6)\qquad \frac{\partial \Phi_h}{\partial x_i} = \left(\frac{\partial \omega_h}{\partial x_i} * \Phi\right), \quad \frac{\partial \Phi_h}{\partial y} = \left(\frac{\partial \omega_h}{\partial y} * \Phi\right).$$

Further, substituting $Y = -hZ$, $Z = (Z', z)$ into (5,5) and (5,6) we obtain from (5,4)

$$(5,7)\quad \Phi_h(X) = \int_{\Omega(X;h)} \omega(Z)\, \mathrm{d}Z, \quad \frac{\partial \Phi_h}{\partial x_i} = \int_{\Omega(X;h)} \frac{\partial \omega(Z)}{\partial x_i} \mathrm{d}Z, \quad \frac{\partial \Phi_h}{\partial y} = \int_{\Omega(X;h)} \frac{\partial \omega(Z)}{\partial y} \mathrm{d}Z,$$

$$\Omega(X; h) = \{Z \in B(0; 1);\ Z = (Z', z),\ z < h^{-1}\varphi(X' + hZ') - h^{-1}y\}$$

(see Fig. 9). Let us now consider Φ_h as a function of y, for fixed $X' \in R^N$.

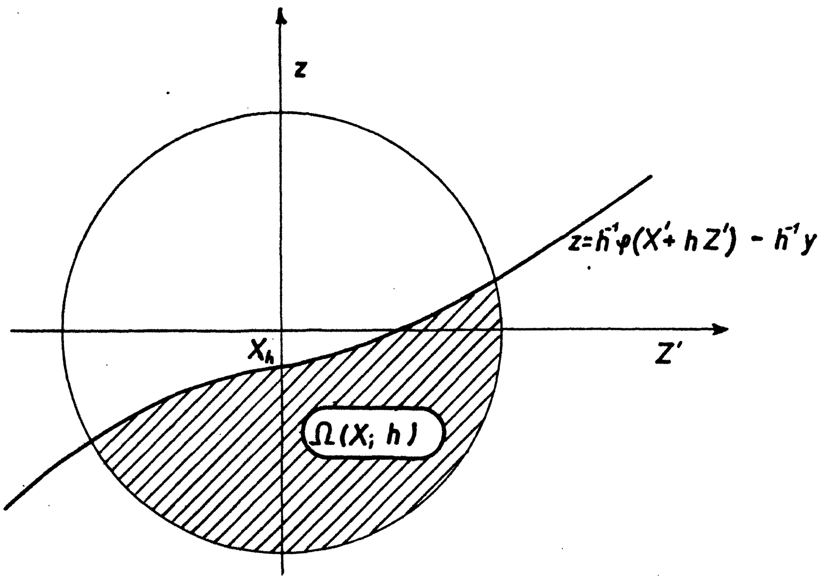

Fig. 9.

Lemma 5,1. *Given $X' \in R^N$, then:*
(i) *For every $y \in R^1$ and for every $h > 0$ we have*

$$0 \leqq \Phi_h(X) \leqq 1 \quad (X = (X', y))$$

and there exists $K_h \in R^1$ such that

$$\Phi_h(X) = 0 \quad \text{for} \quad y > \varphi(X') + K_h\,,$$

$$\Phi_h(X) = 1 \quad \text{for} \quad y < -K_h + \varphi(X')\,, \quad \lim_{h \to 0^+} K_h = 0\,.$$

(ii) *The function $\Phi_h(X)$ is a nonincreasing function of y, and it is even (strictly) decreasing provided $0 \neq \Phi_h(X) \neq 1$.*

Proof. The assertions follow easily from (5,7). Obviously, for $y_1 < y_2$ and $X_1 = (X', y_1)$, $X_2 = (X', y_2)$ we have

$$B(0; 1) \supset \Omega(X_1; h) \supset \Omega(X_2; h) \supset \emptyset$$

and hence

$$1 \geqq \Phi_h(X_1) \geqq \Phi_h(X_2) \geqq 0\,.$$

Moreover, $\operatorname{dist}(X, T(\varphi)) > h$ implies $\Omega(X; h) = \emptyset$; analogously $\operatorname{dist}(X, R^{N+1} - T(\varphi)) > h$ implies $\Omega(X; h) = B(0,1)$, which yields (i). To complete the proof of (ii), let us observe that in the case $0 \neq \Phi_h(X_2) \neq 1$ we have $\emptyset \neq \Omega(X_2; h) \neq B(0, 1)$ and hence

$$\emptyset \neq \Omega = \Omega(X_1; h) - \Omega(X_2; h)\,, \quad 0 < \Phi_h(X_1) - \Phi_h(X_2) = \int_\Omega \omega(Z)\, dZ$$

(see Fig. 10).

Lemma 5,1 guarantees that for every $t \in (0, 1)$, $X' \in R^N$ there exists exactly one $y \in R^1$ such that

$$\Phi_h(X', y) = t$$

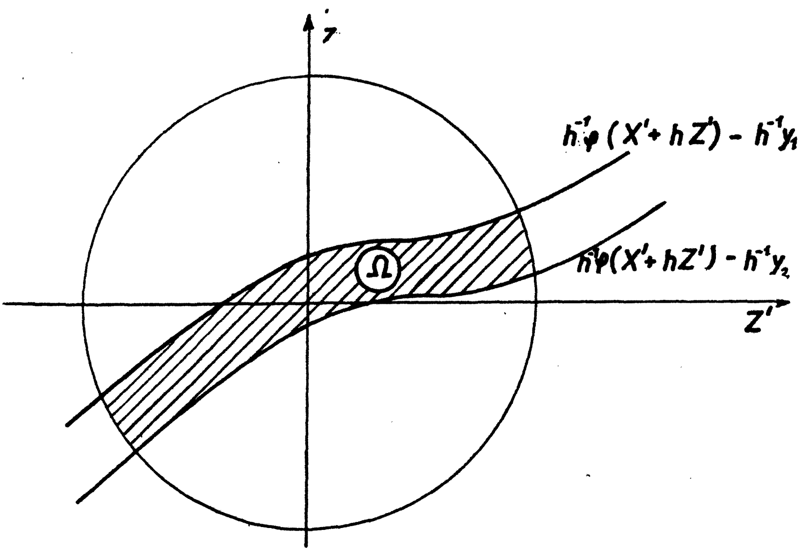

Fig. 10.

and hence we can define a function Ψ_h:

Definition 5,2. Let $h > 0$. By Ψ_h we denote a function defined on R^N satisfying for every $X' \in R^N$ the equation

$$\Phi_h(X', \Psi_h(X')) = \tfrac{1}{2}\,. \tag{5,8}$$

Obviously, $\lim_{h \to 0^+} \Psi_h(X') = \varphi(X')$ uniformly with respect to $X' \in R^N$. We show in the following that Ψ_h are infinitely continuously differentiable and approximate φ in the same sense as the functions φ_d from Section 3 (cf. Lemma 3,2, Lemma 3,3).

Lemma 5,2. *The function Ψ_h has continuous derivatives of all orders, and*

$$\left|\frac{\partial \Psi_h}{\partial x_i}\right| \leqq S\,, \quad i = 1, 2, \dots, N\,, \tag{5,9}$$

where S depends only on the Lipschitz constant L of the function φ (and is independent on h).

Proof. To prove the smoothness of Ψ_h, it is sufficient to show that

$$\frac{\partial \Phi_h}{\partial y} \neq 0$$

and to apply the implicit function theorem. To prove (5,9), we show in addition that

$$\text{(i)} \quad \left|\frac{\partial \Phi_h(X)}{\partial x_i}\right| \leqq M \tag{5,10}$$

$$\text{(ii)} \quad \left|\frac{\partial \Phi_h(X)}{\partial y}\right| \geqq m > 0$$

where M, m depend only on L.

It follows immediately from (5,7) that

$$\left|\frac{\partial \Phi_h}{\partial x_i}\right| \leqq \int_{B(0;1)} \frac{\partial \omega}{\partial x_i}(Z)\,\mathrm{d}Z\,,$$

which gives (5,10) (i). Let us now prove (ii). To this end, let us observe that if φ has the Lipschitz property with a constant L, the same holds for $\tilde{\varphi}_h$:

$$\tilde{\varphi}_h(Z') = h^{-1}\,\varphi(X' + hZ') - h^{-1}y\,.$$

Further, there exists $\varrho < 1$ such that

$$\int_{\varrho < |Z| < 1} \omega(Z)\,\mathrm{d}Z < \tfrac{1}{2} < \int_{|Z| < \varrho} \omega(Z)\,\mathrm{d}Z\,. \tag{5,11}$$

Let $X = (X', y)$ be such a point that $\Phi_h(X) = \frac{1}{2}$. Then (5,11) yields $[\tilde{\varphi}_h] \cap \cap B(0; \varrho) \neq \emptyset$. Indeed, if the opposite were true then either $B(0; \varrho) \subset \Omega(X; h)$ and hence $\Phi_h(X') > \frac{1}{2}$ or $B(0; \varrho) \cap \Omega(X; h) = \emptyset$ and $\Phi_h(X') < \frac{1}{2}$, which is a contradic-

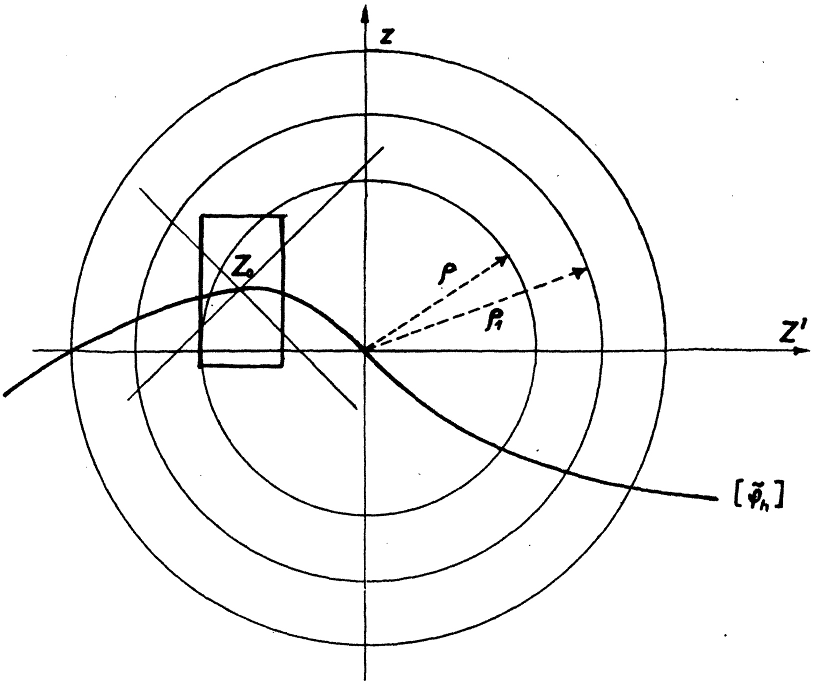

Fig. 11.

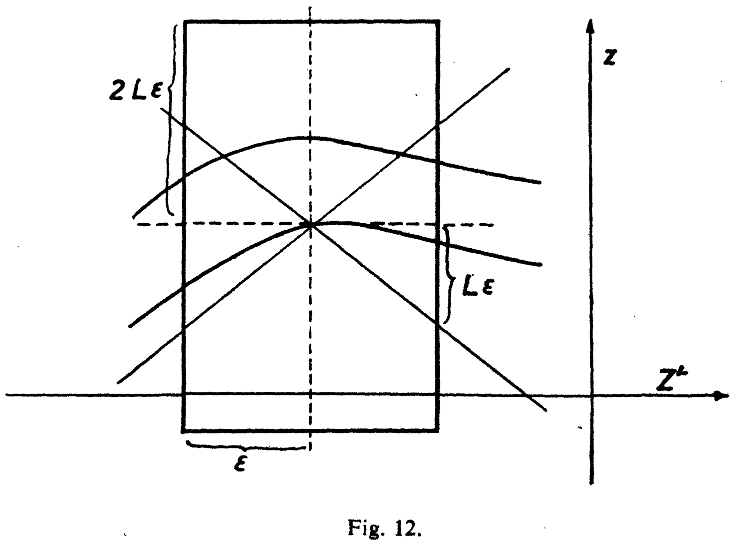

Fig. 12.

tion. Let $Z_0 = (Z'_0, z)$ be a point from this intersection. There exists $\varepsilon > 0$ depending only on ϱ such that the set

$$\mathfrak{M} = \{U = (U', u) \in R^{N+1};\ |U' - Z'_0| < \varepsilon,\ |u - z_0| < 2L\varepsilon\}$$

lies in $B(0; \varrho_1)$ where $\varrho_1 = \frac{1}{2}(\varrho + 1)$ (see Fig. 11), and hence $r = \min\{\omega(U); U \in \mathfrak{M}\} > > 0$. Because of the Lipschitz property we have $[\tilde{\varphi}_h] \subset K_2(Z_0; L)$ and hence for $|\lambda| < \varepsilon L h$ we obtain

$$\{U = (U', u);\ |U' - Z'_0| < \varepsilon,\ u = \tilde{\varphi}_h(U') + \lambda h^{-1}\} \subset \mathfrak{M} \tag{5,12}$$

(see Fig. 12). Let us suppose $\lambda < 0$ and denote

$$\Omega = \Omega((X', y + \lambda); h) - \Omega(X; h)\,, \quad \Omega_0 = \Omega \cap \mathfrak{M}\,.$$

Then using (5,12) we obtain by means of the Fubini theorem

$$\Phi_h(X', y + \lambda) - \Phi_h(X', y) = \int_\Omega \omega(Z)\, dZ \geqq \int_{\Omega_0} \omega(Z)\, dZ \geqq \tag{5,13}$$

$$\geqq \int_{\Omega_0} r\, dZ = r|\lambda| \int_{|Z' - Z'_0| < \varepsilon} dZ'\,;$$

for $\lambda > 0$ we obtain an analogous estimate. Dividing the inequality by λ and passing to the limit we obtain

$$\frac{\partial \Phi_h(X)}{\partial y} \leqq -r \int_{|Z'| < \varepsilon} dZ' < 0$$

which is (ii).

Lemma 5,3. *Let* $X'_0 \in R^N$ *be such a point that there exists* grad $\varphi(X'_0) = H$. *Then*

$$\lim_{h \to 0+} \frac{\partial \Psi_h}{\partial x_i}(X'_0) = \frac{\partial \varphi}{\partial x_i}(X'_0)\,, \quad i = 1, 2, \ldots, N\,.$$

Proof. According to the implicit function theorem we have

$$\frac{\partial \Psi_h}{\partial x_i}(X') = -\left(\frac{\partial \Phi_h(X)}{\partial x_i}\right) : \left(\frac{\partial \Phi_h(X)}{\partial y}\right)$$

where $X = (X'_0, \Psi_h(X'_0))$, and we can express the function Φ_h as well as its derivatives in the form (5,7). We denote $X_h = (0, \tilde{\varphi}_h(0))$ (see Fig. 10). Let us observe that if the function φ has at the point X the gradient equal to H, the same holds for $\tilde{\varphi}_h(Z)$ at the origin; moreover, if $[\varphi]$ lies in $A(X_0; H, \varepsilon)$ for $|X' - X'_0| < \delta$ then $[\tilde{\varphi}_h]$ lies in $A(X_h; H, \varepsilon)$ for $|Z'| < \delta/h$. Hence choosing h sufficiently small we can suppose for given ε that $[\tilde{\varphi}_h] \subset \overline{A(X_h; H, \varepsilon)}$ for all Z considered. This together with the obvious inclusion $p(X_h; H) \subset \overline{A(X_h; H, \varepsilon)}$ yields

$$\left| \int_{\Omega(X_0;h)} \omega(Z)\,\mathrm{d}Z - \int_{P(X_h;H)} \omega(Z)\,\mathrm{d}Z \right| \leqq 2 \int_{\Omega_1(X_0;h)} \omega(Z)\,\mathrm{d}Z,$$

where $\Omega_1(X_0; h) = B(0; 1) \cap \overline{A(X_h; H, \varepsilon)}$. However, the last integral tends to zero if $h \to 0$ and so

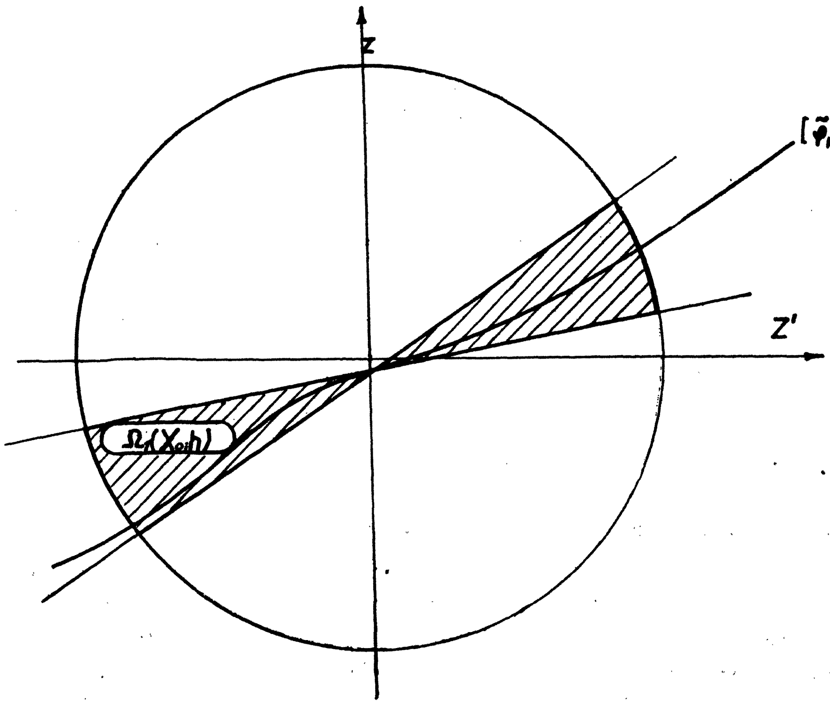

Fig. 13.

$$\int_{P(X_h;H)} \omega(Z)\,\mathrm{d}Z \to \tfrac{1}{2}, \quad X_h \to 0$$

(let us note that $\omega(Z)$ depends only on $|Z|$ and hence

$$\int_{P(0;H)} \omega(Z)\,\mathrm{d}Z = \tfrac{1}{2}$$

for arbitrary H).

By the same argument we obtain

$$\left| \int_{\Omega(X_0;h)} \frac{\partial \omega}{\partial x_i}(Z)\,\mathrm{d}Z - \int_{P(X_h;H)} \frac{\partial \omega}{\partial x_i}(Z)\,\mathrm{d}Z \right| \leqq 2 \int_{\Omega_1(X_0;h)} \left| \frac{\partial \omega}{\partial x_i}(Z) \right| \mathrm{d}Z$$

and hence

$$\lim_{h\to 0} \frac{\partial \Phi_h}{\partial x_i} = \int_{P(0;H)} \frac{\partial \omega}{\partial x_i}(Z)\,\mathrm{d}Z\,, \quad i = 1, 2, \ldots, N\,;$$

the same holds for the derivatives of Φ_h and ω with respect to y.

Applying the procedure to the function $\varphi_0(Z') = (Z', H)$ we obtain $\Psi_{0,h} = \varphi_0$ for every $h > 0$ and hence

$$h_i = \frac{\partial \Psi_{0,h}}{\partial x_i} = -\frac{\displaystyle\int_{P(0;H)} \frac{\partial \omega}{\partial x_i}(Z)\,\mathrm{d}Z}{\displaystyle\int_{P(0;H)} \frac{\partial \omega}{\partial y}(Z)\,\mathrm{d}Z}\,,$$

which proves the lemma.

Lemma 5,4. *Let φ be a lipschitzian function defined on R^N, and let Ψ_h $(h > 0)$ be the function defined by* (5,8). *Let $p \geqq 1$; then φ has partial derivatives of the first order almost everywhere and*

$$\lim_{h\to 0+} \int_K \left| \frac{\partial \Psi_h}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^p \mathrm{d}X' = 0$$

for every compact set $K \subset R^N$.

Proof is the same as that of Lemma 3,3.

Theorem 5,1. *Let $\Omega \subset R^N$ be a bounded lipschitzian domain and let d_n be an arbitrary sequence such that $0 < d_{n+1} < d_n$ and $\lim d_n = 0$. Then there exist domains Ω_n with infinitely differentiable boundaries which approximate Ω in the sense of Theorem* 4,1, *i.e. the boundaries of Ω_n are described with the help of the same co-ordinate systems as $\partial\Omega$, and the corresponding functions $a_{r,n}$ defined and infinitely differentiable on R^N approximate a_r in the following sense:*

(5,14) (i) $$a_{r,n} \rightrightarrows a_r \quad \text{on} \quad \Delta$$

(ii) *there exists a constant M depending only on the Lipschitz constant of the functions a_r such that*

$$\left| \frac{\partial a_{r,n}}{\partial x_{i,r}} \right| < M\,,$$

(iii) $$\lim_{n > \infty} \int_\Delta \left| \frac{\partial a_{r,n}}{\partial x_{i,r}} - \frac{\partial a_r}{\partial x_{i,r}} \right|^p \mathrm{d}X'_r = 0$$

for arbitrary $p \geqq 1$,

(iv) $$a_r(X'_r) + d_{n+1} < a_{r,n}(X'_r) < a_r(X'_r) + d_n\,.$$

Proof. The domains $\tilde{\Omega}_n = \{Z \in R^{N+1};\ \operatorname{dist}(Z, \Omega) < \frac{1}{2}(d_n + d_{n+1})\}$ fulfil (4,4) (i)–(iii); it is now sufficient to construct smooth approximations of these domains with the help of the previous lemmas of this section.

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