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ON TRANSFINITE SEQUENCES OF MAPPINGS

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The notion of transfinite sequence of numbers and transfinite sequence of real functions was introduced by W. SIERPIŃSKI [8] and generalized by P. KOSTYRKO and T. ŠALÁT [2] and [7] for the case of mappings of sets into a metric space.

Definition 1. [7] Let $\{a_\xi\}_{\xi<\Omega}$ be a transfinite sequence of elements of the metric space (Y, ϱ). Point $a \in Y$ is said to be *the limit of the sequence* $\{a_\xi\}$ if there exists for every positive number ε a transfinite number $\eta < \Omega$ such that for any transfinite number ξ the inequality $\eta < \xi$ implies $\varrho(a_\xi, a) < \varepsilon$.

The limit of the transfinite sequence $\{a_\xi\}$ will be denoted by $\lim\limits_{\xi\to\Omega} a_\xi$. A transfinite sequence is called *convergent* if a limit in the above stated sense exists.

Definition 2. [2], [7] The transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of mappings of the set E into metric space (Y, ϱ) is said to be *convergent to the mapping* f if for every point $x \in E$ the sequence $\{f_\xi(x)\}$ is convergent and $\lim\limits_{\xi\to\Omega} f_\xi(x) = f(x)$. We write $\lim\limits_{\xi\to\Omega} f_\xi = f$.

Definition 3. The transfinite sequence $\{f_\xi\}$ $[\{a_\xi\}]$ is said to be *almost constant* if there exists a transfinite number $\eta < \Omega$ such that for every transfinite number ξ the inequality $\eta < \xi$ implies $f_\xi = f_{\eta+1}$ $[a_\xi = a_{\eta+1}]$.

It can easily be seen that a transfinite sequence of elements of (Y, ϱ) is convergent if and only if it is almost constant. There is no assumption in Definition 3 for the members of the sequence to belong to a metric space. Hence this definition is more general than Definition 1. This enables us to drop the assumption of metrization in certain other formulations.

Definition 4. [2] A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of mappings of E into the metric space (Y, ϱ) is said to be *uniformely convergent to the mapping* f if there exists for every $\varepsilon > 0$ a transfinite number $\eta < \Omega$ such that for every $x \in E$ and for every transfinite number ξ the inequality $\eta < \xi$ implies $\varrho(f_\xi(x), f(x)) < \varepsilon$.

It can be easily verified that a sequence of mappings into a metric space is uniformely convergent if and only if it is almost constant [2], [4].

Definition 5. A family $\mathscr{F}$ of mappings of E into Y is said to be *closed with respect to the transfinite convergence* if for every convergent sequence of mappings $\{f_\xi\}_{\xi<\Omega}$ with members belonging to $\mathscr{F}$ the condition $\lim_{\xi>\Omega} f_\xi \in \mathscr{F}$ holds.

Definition 6. A family $\mathscr{F}$ of mappings of E into Y is said to be *strictly closed* if every convergent sequence of mappings belonging to $\mathscr{F}$ is almost constant.

Several authors have delivered proofs of closedness and strictly closedness of different families of functions. E.g. W. Sierpiński [8] has shown that the family of real functions of real variable belonging to first Baire class is closed and the family of continuous functions is strictly closed. T. Šalát [7] demonstrated that the family of continuous mappings of a metric space into a metric space is closed. Also the family of quasicontinuous functions and the family of cliquish functions are closed, as was shown by A. Neubrunnová [5].

Definition 7. Let $\mathscr{F}$ and $\mathscr{G}$ be families of mappings of E into Y. The family $\mathscr{F}$ is said to be *dense* in $\mathscr{G}$ (with respect to the transfinite convergence) if for every mapping $g \in \mathscr{G}$ there exists a transfinite sequence $\{f_\xi\}$ with members belonging to $\mathscr{F}$ such that $g = \lim_{\xi\to\Omega} f_\xi$.

E.g. the family of approximatively continuous functions is dense in the family of functions semicontinuous from above [3].

The aim of this note is the formulation of necessary and sufficient conditions for a family of mappings to be closed or strictly closed or dense in another family.

In spite of the fact that the literature concerning the problems of closedness with respect to the transfinite convergence is fairly ample, general criterion of closedness can be found only in paper [7] by T. Šalát. It is a sufficient condition for closedness based on the notion of a determining set.

Definition 8. [1] Let $\mathscr{A}$ be a family of mappings of E into F. A set $D \subset E$ is said to be *determining for the family* $\mathscr{A}$ if two members of $\mathscr{A}$ which agree on D must agree on all of E.

Remark 1. It can easily be seen that a set determining for a family $\mathscr{A}$ is also determining for every family enclosed in $\mathscr{A}$.

The criterion of closedness given by Šalát in [7] reads as follows: "If there exists among the determinings sets for a family $\mathscr{A}$ a denumerable set, then the family $\mathscr{A}$ is closed". The sufficient condition quoted in this theorem is not a necessary condition even for strict closedness. The following exemple is a proof of this fact:

Exemple 1. Let $\{a_\xi\}_{\xi<\Omega}$ be a transfinite and one-to-one sequence of real numbers. Let $\overline{\overline{E}} = \aleph_1$ and let $E = \bigcup_{\xi>\Omega} E_\xi$ where E_ξ are denumerable sets and let $\{E_\xi\}$ be the increasing sequence of sets. Let $\{x_\xi\}_{\xi<\Omega}$ be a transfinite and one-to-one sequence of elements of E such that $x_\xi \notin E_\xi$. Let $f_\xi(x) = a_\xi$ for every $x \in E$ and $g_\xi(x) = a_\xi$ for $x \neq x_\xi$ and $g_\xi(x_\xi) = a_\xi + 1$. Let $\mathscr{A}$ be the family of all functions f_ξ and g_ξ. No denumerable set is determining for $\mathscr{A}$. In fact, let $F \subset E$ be a denumerable set. Then there exists a set E_ξ such that $F \subset E_\xi$. We come to $f_\xi(x) = g_\xi(x)$ for $x \in F$, but $f_\xi(x_\xi) \neq g_\xi(x_\xi)$. Therefore the set F is not determining for $\mathscr{A}$.

The family $\mathscr{A}$ is strictly closed. In fact, let $h_\xi \in \mathscr{A}$ and let the sequence $\{h_\xi\}_{\xi<\Omega}$ be convergent. We shall show that it is almost constant. Assume that $G \subset E$ is a denumerable set. The functions $h_\xi \mid G$ form a convergent sequence. As they are defined on a denumerable set, hence, according to Theorem 1 of paper [2], the sequence of these functions is uniformely convergent. It follows herefrom that this sequence is almost constant. There exists therefore a number $\eta < \Omega$ such that if $\eta < \xi$ then $h_\xi \mid G = h_{\eta+1} \mid G$. As $h_{\eta+1} \in \mathscr{A}$ there exists a number α such that either $h_{\eta+1} = f_\alpha$ or $h_{\eta+1} = g_\alpha$. On the set G all functions h_ξ with indeces $\xi > \eta$ are equal to f_α or to g_α. According to the definition of f_α and g_α all functions h_ξ $(\xi > \eta)$ take everywhere in G, may be with the exception of x_α, the value a_α. In the family $\mathscr{A}$ there exist only two functions of this property. They are f_α and g_α. The sequence $\{h_\xi(x_\alpha)\}_{\xi<\Omega}$ is convergent and therefore almost constant. For $\eta < \xi$ depending on whether $h_\xi = f_\alpha$ or $h_\xi = g_\alpha$ either $h_\xi(x_\alpha) = a_\alpha$ or $h_\xi(x_\alpha) = a_\alpha + 1$ holds. As the sequence $\{h_\xi(x_\alpha)\}_{\alpha<\Omega}$ is convergent it must necessarily be almost constant. There exists a number $\tau < \Omega$ such that if $\tau < \xi$ then $h_\xi(x_\alpha) = h_{\tau+1}(x_\alpha)$. Without loss of generality we may assume that $\eta < \tau$. Then it follows from $h_{\tau+1}(x_\alpha) = a_\alpha$ that for $\tau < \xi$ all functions $h_\xi = f_\alpha$, and from $h_{\tau+1}(x_\alpha) = a_\alpha + 1$ it follows that they are equal to g_α. In both cases the sequence $\{h_\xi\}$ ist almost constant.

We have proved thus that the family $\mathscr{A}$ is strictly closed, although no denumerable set is determining for this family.

Lemma 1. *Let f and f_ξ be mappings of E and let $f = \lim_{\xi\to\Omega} f_\xi$. Then there exists for each denumerable set $F \subset E$ a number η such that all mappings f_ξ with indices $\xi > \eta$ are extensions of the mapping $f \mid F$.*

Proof. Evidently $f \mid F = \lim_{\xi\to\infty} f_\xi \mid F$. It follows from Theorem 1 of paper [2] that the sequence $\{f_\xi \mid F\}$ ist almost constant. Hence there exists a number $\eta < \Omega$ such that if $\eta < \xi$ then $f_\xi \mid F = f \mid F$. The mappings f_ξ $(\eta < \xi)$ are therefore extensions of the mapping $f \mid F$.

Lemma 2. *Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite and convergent sequence of mappings of E into Y. Let $\mathscr{F}$ be the family of all elements of $\{f_\xi\}$. The sequence $\{f_\xi\}$ is almost* *constant if and only if there exists among the determining sets for $\mathscr{F}$ a set at most denumerable.*

Proof. Let F be an at most denumerable determining set for $\mathscr{F}$. It follows from Theorem 1 of [2] that the sequence $\{f_\xi \mid F\}$ is almost constant. Hence there exists a number η such that if $\eta < \xi$ then $f_\xi \mid F = f_{\eta+1} \mid F$. We have therefore $f_\xi(x) = f_{\eta+1}(x)$ for $x \in F$ and $\eta < \xi$. As F is a determining set therefore $f_\xi(x) = f_{\eta+1}(x)$ for every x. The sequence $\{f_\xi\}$ proves to be almost constant.

Assume now that the sequence $\{f_\xi\}$ is almost constant. Let $f = \lim_{\xi \to \Omega} f_\xi$. If $f = f_\xi$ for all $\xi < \Omega$ then the family $\mathscr{F}$ consists of only one mapping and every set $F \subset E$ is a determining set for $\mathscr{F}$. Evidently there are finite sets among them.

Consider now the remaining case when not all elements of the sequence are equal to its limit. Evidently the set of indices ξ for which $f_\xi \neq f$ is at most denumerable. For each coupel (ξ, ζ) such that $f_\xi \neq f$, $f_\zeta \neq f$ and $f_\xi \neq f_\zeta$ there exists a point $x_{\xi,\zeta} \in E$ such that $f_\xi(x_{\xi,\zeta}) \neq f_\zeta(x_{\xi,\zeta})$. For each mapping $f_\xi \neq f$ there exists a point x_ξ such that $f_\xi(x_\xi) \neq f(x_\xi)$. The set of all points $x_{\xi,\zeta}$ and x_ξ is non-empty and at most denumerable. Denote this set by D. We shall demonstrate that it is a determining set for $\mathscr{F}$. Indeed, if two mappings f_ξ and f_ζ belonging to $\mathscr{F}$ are not equal, then they are not equal in at least one point $x \in D$. Therefore if $f_\xi(x) = f_\zeta(x)$ for all $x \in D$ then necessarily $f_\xi = f_\zeta$.

Theorem 1. *A family $\mathscr{R}$ of mappings of E into Y is strictly closed if and only if for every transfinite and convergent sequence of mappings belonging to $\mathscr{R}$ there exists an at most denumerable determining set for the family of all terms of this sequence.*

Proof. The assertion follows easily from Lemma 2 and Definition 6.

Corollary. *Let $\mathscr{R}$ be a family of mappings. If there exists an at most denumerable determining set for $\mathscr{R}$, then the family $\mathscr{R}$ is strictly closed.*

Exemple 2. The family of all Riemann-integrable derivatives defined on the interval $\langle a, b\rangle$ is strictly closed. In fact, any set dense in $\langle a, b\rangle$ is determining for this family [1], and there exist also denumerable sets among the dense ones.

Theorem 2. *Let $\overline{\overline{E}} = \aleph_1$. A family $\mathscr{F}$ of mappings of E into Y is closed if and only if there exists for every mapping $g \notin \mathscr{F}$ a denumerable set $F \subset E$ such that no mapping $f \in \mathscr{F}$ is an extension of $g \mid F$.*

Proof. Assume that the assertion does not hold. Then there exists a mapping $g \notin \mathscr{F}$ such that for every denumerable set $A \subset E$ there exists a mapping $f \in \mathscr{F}$ such that $f \mid A = g \mid A$. Let $E = \bigcup_{\xi < \Omega} A_\xi$ where A_ξ are denumerable sets and $\xi < \zeta$ implies $A_\xi \subset A_\zeta$. There exists for every A_ξ a mapping $f_\xi \in \mathscr{F}$ such that $f_\xi \mid A_\xi =$

$= g \mid A_\xi$. There exists for any point $x \in E$ a number $\eta \in \Omega$ such that $\eta < \xi$ implies $x \in A_\xi$. Hence we have $f_\xi(x) = g(x)$ for $\eta < \xi$. Hence $g = \lim_{\xi \to \Omega} f_\xi$. Thus the family $\mathscr{F}$ is not closed and the condition given in the theorem proves to be necessary for the closedness of $\mathscr{F}$.

Assume now that $\mathscr{F}$ is not closed. Then there exist mappings $g \notin \mathscr{F}$ and $f_\xi \in \mathscr{F}$ such that $g = \lim_{\xi \to \infty} f_\xi$. By Lemma 1 there exists for any denumerable set $A \subset E$ a number η such that for $\eta < \xi$ the mappings f_ξ are extensions of the mapping $g \mid A$. The condition mentioned in the theorem is therefore not satisfied. Hence it follows that this condition is sufficient for the closedness of the family $\mathscr{F}$.

Corollary 2. *It follows from the continuum hypothesis and Theorem 2 that followings families of real functions defined on $\langle a, b\rangle$ are closed with respect to the convergence of transfinite sequences*:

a) *the family of all bounded [bounded from above] functions,*
b) *the family of all increasing [non-decreasing] functions,*
c) *the family of all functions of bounded variation,*
d) *the family of all differentiable functions,*
e) *the family of all functions satisfying the Lipschitz condition,*
f) *the family of all Riemann-integrable functions.*

Take for exemple case f). Let $\mathscr{F}$ be the family of all Riemann-integrable functions defined on the interval $\langle a, b\rangle$. Suppose that $g \notin \mathscr{F}$. In this case either g is not bounded or the set of the discontinuity points of g has positive measure. In the first case there exists a denumerable set $A \subset \langle a, b\rangle$ such that the function $g \mid A$ is not bounded. The functions $f \in \mathscr{F}$ are bounded and therefore cannot be extensions of the function $g \mid A$.

In the second case we choos for A a denumerable set such that the graph of $g \mid A$ is dense in the graph of g. The existence of such a set follows from the separability of R^2. Let $\omega(x, f)$ denote the oscillation of the function f in the point x. For each x also the condition $\omega(x, g) = \omega(x, g \mid A)$ is satisfied. For each function f being an extension of the function $g \mid A$ we have $\omega(x, f) \geqq \omega(x, g \mid A)$. The set of discontinuity points of f contains the set of discontinuity points of g. This set necessarily has positive measure. None of the functions $f \in \mathscr{F}$ can be an extension of the function $g \mid A$. Therefore by Theorem 2 the family $\mathscr{F}$ is closed.

Theorem 3. *Let $\mathscr{F}$ and $\mathscr{G}$ be families of mappings of E into Y. For the family $\mathscr{F}$ to be dense in $\mathscr{G}$ it is necessary, and if $\overline{\overline{E}} = \aleph_1$ also sufficient, that for every mapping $g \in \mathscr{G}$ and every at most denumerable set $A \subset E$ exists a mapping $f \in \mathscr{F}$ being an extension of $g \mid A$.*

Proof. Assume that the family $\mathscr{F}$ is dense in $\mathscr{G}$. For any mapping $g \in \mathscr{G}$ there exists a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of mappings $f_\xi \in \mathscr{F}$ such that $g = \lim_{\xi \to \Omega} f_\xi$. By Lemma 1 there exist among them such mappings f_ξ which are extensions of $g \mid A$. Hence the condition is necessary.

Assume now that the condition is satisfied. Assume, as in the proof of Theorem 2, that $E = \bigcup_{\xi \to \Omega} A_\xi$ where A_ξ are denumerable sets and $\xi < \zeta$ implies $A_\xi \subset A_\zeta$. Let $g \in \mathscr{G}$. According the assumption there exists for every set A_ξ a mapping $f_\xi \in \mathscr{F}$ such that $f_\xi \mid A_\xi = g \mid A_\xi$. Hence follows as was the case with Theorem 2 that $g = \lim_{\xi<\Omega} f_\xi$. The family $\mathscr{F}$ is therefore dense in $\mathscr{G}$. The condition proves to be sufficient.

Exemple 3. Let $\mathscr{A}$ be the family of all functions approximatively continuous defined on R. Let $\mathscr{B}_1$ be the family of all Baire class 1 functions. It follows from the continuum hypothesis that $\mathscr{A}$ is dense in $\mathscr{B}_1$. In fact, G. Petruska and M. Laczkovich have demonstrated in [6] that for any function $g \in \mathscr{B}_1$ and for every set A of measure zero (and therefore also for any denumerable set) there exists a function $f \in \mathscr{A}$ such that $f \mid A = g \mid A$. This implies by Theorem 3 the denseness of $\mathscr{A}$ in $\mathscr{B}_1$.

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