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ON ASYMPTOTIC BEHAVIOUR OF CENTRAL DISPERSIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1.1. Consider a differential equation

$$\text{(q)} \qquad y'' = q(t)\,y\,, \quad q \in C^0[a, b)\,, \quad b \leqq \infty$$

where $C^n[a, b)$ (n being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order n on $[a, b)$. In all the paper we will deal only with oscillatory ($t \to b_-$) differential equations (i.e. every non-trivial solution has infinitely many zeros on every interval of the form $[t_0, b)$, $t_0 \in [a, b)$).

Let y be a non-trivial solution of (q) vanishing at $t \in [a, b)$. If $\varphi(t)$ is the first zero of y lying to the right from t, then φ is called the basic central dispersion of the 1-st kind of (q) (briefly, dispersion of (q)). The function φ has the following properties (see [2] § 13):

(1)

1) $\varphi \in C^3[a, b)$,

2) $\varphi'(t) > 0$ on $[a, b)$,

3) $\varphi(t) > t$ on $[a, b)$,

4) $\lim_{t \to b_-} \varphi(t) = b$,

5) $-\dfrac{1}{2}\dfrac{\varphi'''}{\varphi'} + \dfrac{3}{4}\left(\dfrac{\varphi''}{\varphi'}\right)^2 + q(\varphi)\,\varphi'^2 = q(t),\ t \in [a, b)\,,$

6) $\varphi'(t) = \dfrac{q(t_1)}{q(t_2)},\ t < t_1 < t_2 < \varphi(t)\,.$

1.2. In our later considerations we will generalize some of the following results which have been proved by the author of [3] (see also [1]):

Theorem 1. *Let* $q \in C^n[a, \infty)$, $p \in C^n[a, \infty)$ $(n \geqq 0$ *being an integer) and let* $q \in C^1[a, \infty)$ *if* $n = 0$. *Let* $\limsup_{t\to\infty} q(t) < 0$, $\liminf_{t\to\infty} q(t) > -\infty$, $\lim_{t\to\infty} (q(t) - p(t)) = 0$, $\lim_{t\to\infty} q'(t) = 0$ *and if* $n > 0$ *let* $\lim_{t\to\infty} q^{(k)}(t) = 0$, $\lim_{t\to\infty} p^{(k)}(t) = 0$ *for* $k = 1, 2, \ldots, n$. *If* φ, $\bar{\varphi}$ *are the dispersions of* (q) *and* $y'' = p(t)\, y$, *respectively, then*

$$\lim_{t\to\infty} (\varphi(t) - \bar{\varphi}(t)^{(k)} = 0\,, \quad k = 1, 2, \ldots, n + 3\,.$$

Lemma 1. *Let* $q \in C^0[a, b)$, $\limsup_{t\to b-} q(t) < 0$. *Let* φ *be the dispersion of the differential equation* (q). *Then there exists a number* $k > 0$ *such that*

$$\varphi(t) - t \leqq k\,, \quad t \in [a, b)\,.$$

2. Lemma 2. *Let* φ *be the dispersion of an oscillatory* $(t \to b_-)$ *differential equation* (q), $q \in C^1[a, b)$, $\limsup_{t\to b-} q(t) < 0$. *Let*

$$\lim_{t\to b-} \max_{x\in[t,\varphi(t)]} |q'(x)|\, (\varphi(t) - t) = 0\,.$$

Then

$$\lim_{t\to b-} \varphi'(t) = 1\,,$$

$$\lim_{t\to b-} \varphi''(t) = \lim_{t\to b-} \varphi'''(t) = 0\,.$$

Proof. It follows from the assumption $\limsup_{t\to b_-} q(t) = c < 0$ and from Lemma 1 that there exist numbers $t_0 \in [a, b)$, $K > 0$ such that we have

$$\varphi(t) - t \leqq K\,, \quad t \in [a, b)\,,$$

$$q(t) \leqq \frac{c}{2}\,, \quad t \in [t_0, b)\,.$$

Then according to (1) 6, we obtain for $t \geqq t_0$ that

$$|q(\varphi)\,(1 - \varphi')| = |q(\varphi)| \cdot \left|1 - \frac{q(t_1)}{q(t_2)}\right| = \left|\frac{q(\varphi)}{q(t_2)}\right| \cdot |q(t_2) - q(t_1)| \leqq$$

$$\leqq \frac{|q(t_2)| + M_1(t)}{|q(t_2)|} \cdot |q'(\xi)| \cdot (t_2 - t_1) \leqq \left(1 + \frac{2}{c} M_2\right) \max_{x\in[t,\varphi(t)]} |q'(x)|\, (\varphi(t) - t)$$

holds where $t < t_1 < t_2 < \varphi(t)$, $\xi \in (t_1, t_2)$, $\eta \in (t_2, \varphi(t))$,

$$M_1(t) = |q'(\eta)|\, (\varphi(t) - t_2) \leqq \max_{x\in[t,\varphi(t)]} |q'(x)|\, (\varphi - t) \xrightarrow[t\to b_-]{} 0\,,$$

$$M_2 = \max_{t\in[a,b)} M_1(t)\,.$$

Hence it follows

$$\text{(2)} \qquad \lim_{t \to b_-} |q(\varphi)(1 - \varphi')| = 0$$

and thus $\lim_{t \to b_-} \varphi'(t) = 1$. So the first part of the statement is proved.

According to (2) and (1) 5, we have

$$\lim_{t \to b_-} |-\tfrac{1}{2}\varphi'''\varphi' + \tfrac{3}{4}\varphi''^2| = \lim_{t \to b_-} |q(t) - q(\varphi)\varphi'^2| =$$

$$= \lim_{t \to b_-} |(q(t) - q(\varphi)) + q(\varphi)(1 - \varphi'^2)| = \lim_{t \to b_-} |q(t) - q(\varphi)| =$$

$$= \lim_{t \to b_-} |q'(\xi)(t - \varphi)| \leqq \lim_{t \to b_-} \max_{x \in [t, \varphi(t)]} |q'(x)|(\varphi - t) = 0$$

where $\xi \in (t, \varphi)$. So

$$\text{(3)} \qquad \lim_{t \to b_-} |-\tfrac{1}{2}\varphi'''\varphi' + \tfrac{3}{4}\varphi''^2| = 0\,.$$

Suppose $\lim_{t \to b_-} \varphi''^2 = c > 0$. Then $\lim_{t \to b_-} \varphi'(t) = \pm\infty$ but this is in contradiction with the proved part of the lemma. Assume that $\lim_{t \to b_-} \varphi''^2$ does not exist. Let $M = \{t \in [a, b), \varphi'''(t) = 0\}$. Then the set M contains every local maximum of the function φ''^2 and the point $t = b$ is an accumulation point of M. According to (3),

$$\lim_{\substack{t \to b_- \\ t \in M}} \varphi''^2(t) = 0$$

holds and hence we have $\lim_{t \to b_-} \varphi''^2(t) = 0$. But this is in contradiction with our assumption.

Thus $\lim_{t \to b_-} \varphi''(t) = 0$ and the rest of the statement follows from (3).

Lemma 3. *Let* (q), ($\bar{\text{q}}$) *be oscillatory* $(t \to b_-)$ *differential equations such that* $q \in C^0[a, b)$, $\bar{q} \in C^1[a, b)$, $\limsup_{t \to b_-} \bar{q}(t) < 0$ *and* $\lim_{t \to b_-} (q(t) - \bar{q}(t)) = 0$. *Let* $\varphi(\bar{\varphi})$ *be the dispersion of* (q) (($\bar{\text{q}}$)) *and let*

$$\lim_{t \to b_-} \max_{x \in [t, \varphi(t)]} |\bar{q}'(x)| (\bar{\bar{\varphi}}(t) - t) = 0$$

where $\bar{\bar{\varphi}}(t) = \max(\varphi(t), \bar{\varphi}(t))$. *Then*

$$\lim_{t \to b_-} \varphi'(t) = 1\,,$$

$$\lim_{t \to b_-} \varphi''(t) = \lim_{t \to b_-} \varphi'''(t) = 0\,.$$

Proof. By virtue of (1) 6, we have:

$$\bar{q}(\bar{\varphi}(t))\,(\varphi'(t) - \bar{\varphi}'(t)) = \bar{q}(\bar{\varphi}(t)) \cdot \left(\frac{q(t_1)}{q(t_2)} - \frac{\bar{q}(t_3)}{\bar{q}(t_4)}\right) =$$

$$= \frac{\bar{q}(\bar{\varphi}(t))}{q(t_2)\,\bar{q}(t_4)}\,(q(t_1)\,\bar{q}(t_4) - \bar{q}(t_3)\,q(t_2)) =$$

$$= \frac{\bar{q}(\varphi(t))}{q(t_2)\,\bar{q}(t_4)}\,[\bar{q}(t_4)\,(q(t_1) - \bar{q}(t_1)) + \bar{q}(t_3)\,(\bar{q}(t_2) - q(t_2)) +$$

$$+ \bar{q}(t_1)\,(\bar{q}(t_4) - \bar{q}(t_2)) - \bar{q}(t_2)\,(\bar{q}(t_3) - \bar{q}(t_1))]\,,$$

where

$$t < t_1 < t_2 < \varphi(t)\,, \quad t < t_3 < t_4 < \bar{\varphi}(t)\,.$$

It follows from the relations

$$\limsup_{t \to b_-} \bar{q}(t) = c < 0\,, \quad \lim_{t \to b_-} (q(t) - \bar{q}(t)) = 0$$

that there exists $t_0 \in [a, b)$ such that

$$|q(t)| \geqq \frac{1}{2}\,|\bar{q}(t)| \geqq \frac{|c|}{4}\,, \quad t \in [t_0, b)\,.$$

Then the following inequalities are valid for $t \in [t_0, b)$ and $t_5 \in [t, \bar{\bar{\varphi}}(t))$ (by the Taylor Theorem):

$$\left|\frac{\bar{q}(\bar{\varphi}(t)) \cdot \bar{q}(t_5)}{q(t_2) \cdot \bar{q}(t_4)}\right| \leqq 2 \left|\frac{\bar{q}(\bar{\varphi}(t))}{\bar{q}(t_2)} \cdot \frac{\bar{q}(t_5)}{\bar{q}(t_4)}\right| \leqq 2 \cdot \frac{|\bar{q}(t_2)| + M_1(t)}{|\bar{q}(t_2)|} \cdot \frac{|\bar{q}(t_4)| + M_1(t)}{|\bar{q}(t_4)|} \leqq$$

$$\leqq 2 \cdot \left(1 + \frac{2 \cdot M_1(t)}{|c|}\right)\left(1 + \frac{2 \cdot M_1(t)}{|c|}\right) \leqq 2 \cdot \left(1 + \frac{2 \cdot M_2}{|c|}\right)^2 = M_3 < \infty\,.$$

Here

$$M_1(t) = \max_{x \in [t, \bar{\varphi}(t)]} |\bar{q}'(x)|\,(\bar{\bar{\varphi}}(t) - t) \xrightarrow[t \to b_-]{} 0\,, \quad M_2 = \max_{t \in [t_0, b)} M_1(t)\,.$$

Hence

$$|\bar{q}(\bar{\varphi}(t))\,(\varphi'(t) - \bar{\varphi}'(t))| \leqq M_3 \cdot [|q(t_1) - \bar{q}(t_1)| + |\bar{q}(t_2) - q(t_2)| +$$

$$+ |\bar{q}'(\xi_1)|\,(\bar{\bar{\varphi}} - t) + |\bar{q}'(\xi_2)|\,(\bar{\bar{\varphi}} - t)] \xrightarrow[t \to b_-]{} 0$$

for $\xi_1 \in (t_2, t_4)$, $\xi_2 \in (t_1, t_3)$ and thus

(4) $$\lim_{t \to b_-} |\bar{q}(\bar{\varphi}(t))\,(\varphi'(t) - \bar{\varphi}'(t))| = 0\,.$$

This implies $\lim_{t\to b_-}(\varphi'(t) - \bar{\varphi}'(t)) = 0$ and by virtue of Lemma 2 we can see that $\lim_{t\to b_-} \varphi'(t) = 1$ which proves the first part of the lemma.

The dispersions φ and $\bar{\varphi}$ fulfil the non-linear differential equation (1) 5:

$$-\frac{1}{2}\frac{\varphi'''}{\varphi'} + \frac{3}{4}\frac{\varphi''^2}{\varphi'^2} = -q(\varphi)\,\varphi'^2 + q(t)\,,$$

$$-\frac{1}{2}\frac{\bar{\varphi}'''}{\bar{\varphi}'} + \frac{3}{4}\frac{\bar{\varphi}''^2}{\bar{\varphi}'^2} = -\bar{q}(\bar{\varphi})\,\bar{\varphi}'^2 + \bar{q}(t)\,.$$

Subtracting and modifying these equations we get (by (4) and the proved part of Lemma 3):

$$\begin{aligned} A = \left|-\frac{1}{2}\left(\frac{\varphi'''}{\varphi'} - \frac{\bar{\varphi}'''}{\bar{\varphi}'}\right) + \frac{3}{4}\left(\frac{\varphi''^2}{\varphi'^2} - \frac{\bar{\varphi}''^2}{\bar{\varphi}'^2}\right)\right| = \\ = |q(t) - \bar{q}(t) - q(\varphi)\,\varphi'^2 + \bar{q}(\bar{\varphi})\,\bar{\varphi}'^2| = \\ = |(q(t) - \bar{q}(t)) - \bar{q}(\bar{\varphi})\,(\varphi' - \bar{\varphi}')\,(\varphi' + \bar{\varphi}') - \\ - \varphi'^2[(q(\varphi) - \bar{q}(\varphi)) + (\bar{q}(\varphi) - \bar{q}(\bar{\varphi}))]| \leqq |q(t) - \bar{q}(t)| + \\ + |\bar{q}(\bar{\varphi})\,(\varphi' - \bar{\varphi}')\,(\varphi' + \bar{\varphi}')| + \varphi'^2|q(\varphi) - \bar{q}(\varphi)| + \\ + \varphi'^2 \max_{x\in[t,\bar{\bar{\varphi}}(t)]} |\bar{q}'(x)|\,(\bar{\bar{\varphi}}(t) - t) \xrightarrow[t\to b_-]{} 0\,. \end{aligned}$$

Taking into account Lemma 2 (for $q \equiv \bar{q}$ we have $\lim_{t\to b_-} \bar{\varphi}''(t) = \lim_{t\to b_-} \bar{\varphi}'''(t) = 0$) we can see from this that

$$\lim_{t\to b_-} A\varphi'^2 = \lim_{t\to b_-} \left|-\tfrac{1}{2}\varphi''' \,.\, \varphi' + \tfrac{3}{4}\varphi''^2\right| = 0 \tag{5}$$

holds. The relation (5) is the same as the relation (3) and therefore we can prove in the same way as in Lemma 1 that

$$\lim_{t\to b_-} \varphi''(t) = \lim_{t\to b_-} \varphi'''(t) = 0\,.$$

So the statement of the lemma is proved.

Theorem 2. *Let* (q), ($\bar{\text{q}}$) *be oscillatory* ($t \to \infty$) *differential equations such that* $q \in C^0[a, \infty)$, $\bar{q} \in C^1[a, \infty)$, $\limsup_{t\to\infty} \bar{q}(t) < 0$, $\lim_{t\to\infty}(q(t) - \bar{q}(t)) = 0$, $\lim_{t\to\infty} \bar{q}'(t) = 0$. *Let* φ *be the dispersion of* (q). *Then*

$$\lim_{t\to\infty} \varphi(t) = 1\,,\quad \lim_{t\to\infty} \varphi''(t) = \lim_{t\to\infty} \varphi'''(t) = 0\,.$$

Proof. Let $\bar{\varphi}$ be the dispersion of ($\bar{\text{q}}$). It follows from Lemma 1 that there exists a constant $M > 0$ such that $\varphi(t) - t \leqq M$, $\bar{\varphi}(t) - t \leqq M$, $t \in [a, \infty)$.

Thus

$$\lim_{t \to \infty} \max_{x \in [t, \bar{\bar{\varphi}}(t)]} |\bar{q}'(x)| (\bar{\bar{\varphi}}(t) - t) = 0$$

where $\bar{\bar{\varphi}}(t) = \max(\varphi(t), \bar{\varphi}(t))$. This together with Lemma 3 implies the statement of the theorem.

Theorem 3. *Let* (q), ($\bar{\text{q}}$) *be oscillatory* $(t \to \infty)$ *differential equations such that* $q \in C^0[a, \infty)$, $q \in C^1[a, \infty)$, $\lim_{t \to \infty} (q(t) - \bar{q}(t)) = 0$, $\lim_{t \to \infty} q(t) = -\infty$, $|\bar{q}'(t)| \leqq \text{const.}$ *for* $t \in [a, \infty)$. *Let* φ *be the dispersion of* (q). *Then*

$$\lim_{t \to \infty} (\varphi(t) - t) = 0\,, \quad \lim_{t \to \infty} \varphi'(t) = 1\,, \quad \lim_{t \to \infty} \varphi''(t) = \lim_{t \to \infty} \varphi'''(t) = 0\,.$$

Proof. Let $C < 0$ be an arbitrary number. As $\lim_{t \to \infty} \bar{q}(t) = -\infty$, there exists a number t_1, $t_1 \in [a, \infty)$ such that $q(t) < C$, $t \in [t_1, \infty)$. From the Sturm Comparison Theorem for the equations (q) and $y'' = C \,.\, y$ we obtain

$$0 < \varphi(t) - t \leqq \frac{\pi}{\sqrt{-C}}\,, \quad t \in [t_1, \infty)\,.$$

Hence $\lim_{t \to \infty} (\varphi(t) - t) = 0$. We can prove similarly that $\lim_{t \to \infty} (\bar{\varphi}(t) - t) = 0$ where $\bar{\varphi}$ is the dispersion of ($\bar{\text{q}}$). Thus

$$\lim_{t \to \infty} \max_{x \in [t, \bar{\bar{\varphi}}(t)]} |\bar{q}'(x)| (\bar{\bar{\varphi}}(t) - t) = 0$$

where $\bar{\bar{\varphi}}(t) = \max(\varphi(t), \bar{\varphi}(t))$ and the statement of the theorem follows from Lemma 3.

References

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