Časopis pro pěstování matematiky

Gary Chartrand; Raymond E. Pippert
Locally connected graphs

*Časopis pro pěstování matematiky*, Vol. 99 (1974), No. 2, 158--163

Persistent URL: `http://dml.cz/dmlcz/117838`

Časopis pro pěstování matematiky, roč. 99 (1974), Praha

# LOCALLY CONNECTED GRAPHS

GARY CHARTRAND, Kalamazoo and RAYMOND E. PIPPERT, Fort Wayne

(Received December 4, 1972)

## INTRODUCTION

One of the most elementary yet most important properties that a graph can possess is that of being connected. This is a global property in the sense that it is defined in terms of all pairs of vertices in the graph. It is the object of this paper to present results dealing with graphs which are connected in a localized sense.

## PRELIMINARY DEFINITIONS

In this section we define several terms which will occur throughout the paper.

If $W$ is a nonempty subset of the vertex set of a graph $G$, then the *subgraph induced by* $W$ is that graph with vertex set $W$ and whose edges are those edges in $G$ joining two vertices of $W$.

The *neighboring vertices* of a vertex $v$ in a graph $G$ are those vertices in $G$ adjacent with $v$. The *neighborhood* $N(v)$ of $v$ is the subgraph induced by the neighboring vertices of $v$. The graph $G$ is *locally connected* if the neighborhood of every vertex of $G$ is connected.

The *complete graph* $K_p$ is that graph with $p$ vertices every two of which are adjacent. The *complete n-partite graph* $K(p_1, p_2, \ldots, p_n)$, $n \geqq 2$, is the graph whose vertex set can be partitioned as $V_1 \cup V_2 \cup \ldots \cup V_n$, where $|V_i| = p_i$, $i = 1, 2, \ldots, n$, such that $uw$ is an edge if and only if $u \in V_i$ and $w \in V_j$, $i \neq j$.

## ELEMENTARY PROPERTIES OF LOCALLY CONNECTED GRAPHS

First, it should be noted that neither the property of being connected nor the property of being locally connected implies the other. For example, every cycle of length $n \geqq 4$ is connected, but the neighborhood of every vertex in this graph consists of two isolated vertices. Thus the graph is not locally connected. Conversely, let $G$ be the disconnected graph with two components, each of which is a triangle. The neighboorhood of every vertex of $G$ is the connected graph $K_2$; therefore, $G$ is locally connected.

Since every connected graph $H$ contains a spanning tree (a tree containing every vertex of $H$) and every tree contains one less edge than vertex, we obtain the following. (The degree of a vertex $v$ in a graph is denoted by deg $v$.)

**Proposition 1.** *If $v$ is a vertex in a locally connected graph $G$, then $v$ belongs to at least* deg $v - 1$ *triangles.*

The preceding proposition implies that every locally connected graph of sufficiently large order contains a relatively large number of triangles. Indeed, if a graph $G$ contains no triangles, then the neighborhood of each vertex of $G$ contains no edges.

The simplest neighborhood in any locally connected graph is a tree. Another observation can now be made. A *wheel* is a cycle together with an aditional vertex adjacent with every vertex of the cycle.

**Proposition 2.** *Every neighborhood in a graph $G$ is a forest if and only if $G$ has no wheels.*

**Corollary.** *Every neighborhood in a graph $G$ is a tree if and only if $G$ has no wheels and is locally connected.*

While the minimum number of edges in a connected graph $G$ occurs when $G$ is a tree, the maximum number of edges occurs when $G$ is complete. In this connection, we make the following observation.

**Proposition 3.** *Every neighborhood in a graph $G$ is complete if and only if every component of $G$ is complete.*

Proof. If every component of $G$ is complete, then it is clear that every neighborhood in $G$ is complete. Conversely, suppose $H$ is a graph in which every neighborhood is complete. Assume $H$ contains a component $H_1$ which is not complete. Then in $H_1$, there are two vertices $u$ and $w$ which are not adjacent. Since $u$ and $w$ belong to the same component, there exists a $u - w$ path, say $u = v_0, v_1, \ldots, v_n = w$. Let $k$ be the least $i$ such that $v_i v_n$ is an edge of $H_1$. Thus, $v_k$ is adjacent to $v_{k-1}$ and $v_n$, but $v_{k-1}$ and $v_n$ are not adjacent to each other. However, then, the neighborhood of $v_k$ is not complete, which is a contradiction.

**Corollary.** *Every neighborhood in a connected graph $G$ is complete if and only if $G$ is complete.*

A graph $G$ is *n-connected* if the removal of fewer than $n$ vertices results in neither a disconnected graph nor the trivial graph consisting of a single vertex. A graph $G$ is *locally n-connected* if the neighborhood of every vertex of $G$ is $n$-connected. We now investigate the relationship between connectedness and local connectedness.

**Proposition 4.** *If a graph $G$ is locally n-connected, $n \geqq 1$, then every component of $G$ is $(n + 1)$-connected.*

Proof. Suppose there exists a component $G_1$ of $G$ which is not $(n + 1)$-connected. Then there exists a set $T$ of $k(\leqq n)$ vertices of $G_1$ such that $G_1 - T$ is disconnected or consists of a single vertex. If $G_1 - T$ consists of a single vertex, then $G_1$ has $k + 1$ vertices, implying that the neighborhood of a vertex of $G_1$ has at most $n$ vertices and that, consequently, no neighborhood is $n$-connected. Thus, $G_1 - T$ is disconnected. Let $v \in T$, and suppose $u$ and $w$ are neighboring vertices of $v$ in different components of $G_1 - T$. Therefore, the minimum number of vertices separating $u$ and $w$ in $G_1 - v$ is at most $k - 1$ $(\leqq n - 1)$. This implies that the minimum number of vertices separating $u$ and $w$ in $N(v)$ does not exceed $n - 1$. However, $N(v)$ is $n$-connected, and this is a contradiction, which completes the proof.

It is well-known that every 6-connected graph is nonplanar, while there are 5-connected graphs which are planar (such as the graph of the icosahedron). For local connectedness, however, the situation is quite different.

**Proposition 5.** *Every locally 3-connected graph is nonplanar.*

Proof. Let $G$ be a locally 3-connected graph, and let $v$ be a vertex of $G$. If $N(v)$ is complete, then since $N(v)$ is 3-connected, $N(v)$ contains the complete graph $K_4$ as a subgraph. In $G$, the vertex $v$ is adjacent to all vertices of $K_4$ so that $G$ contains $K_5$ as a subgraph, and by Kuratowski's theorem, $G$ is nonplanar.

If $N(v)$ is not complete, then there exist two nonadjacent vertices $u$ and $w$. By Whitney's theorem, there exist three disjoint $u - w$ paths, each of which has length at least two. In $G$, the vertex $v$ is adjacent to the interior vertices of these three paths; thus, $G$ contains a subgraph homeomorphic with $K(3, 3)$. Again, by Kuratowski's theorem, $G$ is nonplanar.

We note that the graph $K_4$ is planar and locally 2-connected so that, in a certain sense, the preceding proposition is best possible; however, from the proof of this proposition, a somewhat stronger version holds.

**Corollary 5a.** *If a graph $G$ contains a vertex whose neighborhood is 3-connected, then $G$ is nonplanar.*

Examples can be given to show that among the connected, locally connected graphs, there are many hamiltonian graphs (graphs with a cycle containing every vertex). Along this line, we present the next result. (The maximum degree among the vertices of a graph $G$ is denoted $\Delta(G)$.)

**Proposition 6.** *Let $G$ be a connected, locally connected graph with at least three vertices. If $\Delta(G) \leqq 4$, then either $G$ is hamiltonian or $G = K(1, 1, 3)$.*

Proof. Since $G$ is connected and locally connected, it follows by Proposition 4 that $G$ is 2-connected. Assume that $G$ is not hamiltonian. Then a cycle $C$ of maximum length in $G$ does not contain all vertices of $G$. Since $G$ is connected, there exists a vertex $w$ not on $C$ adjacent to a vertex $u$ on $C$. Let $u_1$ and $u_2$ be the vertices consecutive to $u$ on $C$. The vertex $w$ is adjacent to neither $u_1$ nor $u_2$; for otherwise a cycle exists having length exceeding that of $C$.

Since the neighborhood of $u$ is connected, there exists a vertex $v$ (different from $w, u_1, u_2$) which is adjacent to $u$ such that $v$ is adjacent to at least one of $u_1$ and $u_2$, say $u_1$. Necessarily, $v$ lies on $C$; for otherwise there exists a cycle whose length exceeds that of $C$. Now, $v$ is adjacent to $w$, since the neighborhood of $u$ is connected and the neighborhood of $u$ cannot contain more than four vertices. The vertices $u_1$ and $v$ are consecutive on $C$, for otherwise $v$ has degree at least 5. If $u_1$ and $u_2$ are adjacent, then $G$ contains a cycle whose length exceeds that of $C$. Therefore, $v$ is adjacent to $u_2$, implying that $u_2$ and $v$ are consecutive on $C$. Hence, $G$ contains $K(1, 1, 3)$ as an induced subgraph.

We claim that $G = K(1, 1, 3)$; for otherwise, there exists another vertex $x$ adjacent with one of $u_1, u_2, w$, say $u_1$. Hence $u$, $v$, and $x$ are vertices in $N(u_1)$, but $w$ and $u_2$ do not belong to $N(u_1)$. Since $N(u_1)$ is connected, there exists an $x - u$ path in $N(u_1)$. This implies that at least one of $u$ and $v$ has degree exceeding four, which is a contradiction.

## SUFFICIENT CONDITIONS FOR LOCALLY CONNECTED GRAPHS

We now turn our attention from properties of locally connected graphs to conditions which are sufficient for a graph to be locally connected. We present one dealing with the degrees of the vertices.

**Proposition 7.** *Let $G$ be a graph of order $p$ such that for every pair $x$, $y$ of vertices,* $\deg x + \deg y > \frac{4}{3}(p - 1)$. *Then $G$ is locally connected.*

Proof. Suppose $G$ satisfies the hypothesis of the proposition but is not locally connected. Thus, there exists a vertex $v$ of $G$ such that $N(v)$ is not connected. Let $u$ be a vertex in a smallest component of $N(v)$, and assume this component has order $m_1$. Let $w$ be a vertex in one of the other components of $N(v)$, where the union of the components of $N(v)$ not containing $u$ has order $m_2$. Let $k$ denote the number of vertices different from $v$ which are not in $N(v)$.

By assumption, $\deg v = m_1 + m_2$. Since $u$ is adjacent to no vertex of $N(v)$ in a component not containing $u$, it follows that $\deg u \leqq p - m_2 - 1$. By hypothesis,

$$\deg u > \tfrac{4}{3}(p - 1) - \deg v = \tfrac{4}{3}(p - 1) - m_1 - m_2 .$$

Thus,

$$p - m_2 - 1 > \tfrac{4}{3}(p - 1) - m_1 - m_2 ,$$

so that

$$m_1 > \tfrac{1}{3}(p - 1) .$$

However, $m_2 \geqq m_1$, so that

$$m_2 > \tfrac{1}{3}(p - 1) .$$

Therefore,

$$k < \tfrac{1}{3}(p - 1) .$$

Now,

$$\deg u + \deg w \leqq (m_1 + k) + (m_2 + k) = (p - 1) + k < \tfrac{4}{3}(p - 1) ,$$

but this is a contradiction.

We obtain an immediate corollary. (The minimum degree among the vertices of a graph $G$ is denoted by $\delta(G)$.)

**Corollary 7a.** If $G$ is a graph of order $p$ for which $\delta(G) > \tfrac{2}{3}(p - 1)$, then $G$ is locally connected.

The previous proposition can be extended to locally $n$-connected graphs.

**Proposition 8.** *Let $G$ be a graph of order $p$ such that for every pair $x$, $y$ of vertices*

$$\deg x + \deg y > \tfrac{4}{3}[p + \tfrac{1}{2}(n - 3)] ,$$

*where $1 \leqq n \leqq p - 2$. Then $G$ is locally $n$-connected.*

Proof. Assume that there exists a graph $G$ such that for $1 \leqq n \leqq p - 2$,

$$\deg x + \deg y > \tfrac{4}{3}[p + \tfrac{1}{2}(n - 3)]$$

for every pair $x$, $y$ of vertices of $G$ but such that $G$ is not locally $n$-connected. Hence there exists a vertex $v$ such that $N(v)$ is not $n$-connected. We consider two cases.

**Case 1.** *Assume $N(v) = K_j$, for some $j \leqq n$.* Suppose that there exists a vertex $u \neq v$, such that $u$ is not adjacent with $v$. Then $\deg v = j$ and $\deg u \leqq p - 2$ so that $\deg u + \deg v \leqq p - 2 + j \leqq p - 2 + n$. By hypothesis,

$$p - 2 + n > \tfrac{4}{3}[p + \tfrac{1}{2}(n - 3)] ,$$

which implies that $n > p$. However, this is impossible. If there is no such vertex $u$, then $N(v) = K_{p-1}$ and $G = K_p$. Here $G$ is locally $n$-connected for every $n$, $1 \leqq n \leqq \leqq p - 2$, and this is a contradiction.

**Case 2.** *Assume $N(v)$ contains a set $S$ of $s(<n)$ vertices whose removal from $N(v)$ disconnects $N(v)$.* Let $u$ be a vertex in a component of $N(v) - S$ of minimum order $m_1$, and let $w$ be a vertex in one of the other components of $N(v) - S$, where the union of the other components of $N(v) - S$ has order $m_2$. Now $\deg v = m_1 + m_2 + s$ and $\deg u \leqq p - m_2 - 1$. By hypothesis, $\deg u + \deg v > \alpha$, where

$$\alpha = \tfrac{4}{3}[p + \tfrac{1}{2}(n - 3)] .$$

Thus, $\deg u > \alpha - \deg v$ so that $p - m_2 - 1 > \alpha - m_1 - m_2 - s$. Hence $m_1 > > \alpha - p - s + 1$, and since $m_2 \geqq m_1$, it follows also that $m_2 > \alpha - p - s + 1$. Let $k = p - m_1 - m_2 - s - 1$. Then $k < p - 2\alpha + 2p + 2s - 2 - s - 1 = = 3p - 2\alpha + s - 3$. Now $\deg u + \deg w \leqq (m_1 + s + k) + (m_2 + s + k) = = p + s + k - 1 < p + s - 1 + (3p - 2\alpha + s - 3) = 4p + 2s - 2\alpha - 4 \leqq 4p + + 2n - 2\alpha - 6 = \alpha$. This is a contradiction, and the desired result follows.

We have a corollary in this case also.

**Corollary 8a.** *If $G$ is a graph of order $p$ for which $\delta(G) > \tfrac{2}{3}[p + \tfrac{1}{2}(n - 3)]$, where $1 \leqq n \leqq p - 2$, then $G$ is locally $n$-connected.*

Both of the preceding results are best possible as we shall now illustrate. Let $n$ and $p$ be positive integers, where $p \geqq n + 2$ and $p \equiv n \pmod 3$. Let $G'$ be a complete graph of order $p$. Denote a vertex of $G'$ by $v$ and some other set of $n - 1$ vertices of $G'$ by $S$. The remaining $p - n$ vertices can be divided into 3 sets of $\tfrac{1}{3}(p - n)$ vertices each. Denote these sets by $S_1$, $S_2$, and $S_3$. Delete all edges joining $v$ with elements of $S_3$ as well as all edges joining elements of $S_1$ and elements of $S_2$, calling the resulting graph $G$. For all vertices $x$ and $y$ of $G$, we have $\deg x + \deg y \geqq \geqq \tfrac{4}{3}[p + \tfrac{1}{2}(n - 3)]$, and $\delta(G) = \tfrac{2}{3}[p + \tfrac{1}{2}(n - 3)]$. However, the neighborhood of $v$ is not $n$-connected; therefore $G$ is not locally $n$-connected.

*Author's addresses:* Gary Chartrand, Western Michigan University, Kalamazoo, Michigan 49001, U.S.A.; Raymond E. Pipert, Purdue University, Fort Wayne, Indiana, U.S.A.