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ON POWERS OF NON-NEGATIVE MATRICES

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1. INTRODUCTION

Denote by $p(A)$ the number of positive elements of a matrix A. Let A be square non-negative. Then, obviously, the behaviour of the sequence $\{p(A^r)\}$ is fully determined by the combinatorial structure of the positive elements of A. In the paper [1], Z. ŠIDÁK has noticed that this sequence is not necessarily non-decreasing even when A is primitive. Further, the following theorem was deduced there:

Let A be an irreducible non-negative matrix containing at most one zero element in its main diagonal. Then $p(B) \leqq p(AB)$ for each non-negative matrix B of the same size as A and, consequently, the sequence $\{p(A^r)\}$ is non-decreasing.

It is the purpose of this note to strengthen the quoted results.

2. PRELIMINARIES

Let $A = (a_{ik})$, $B = (b_{ik})$ be matrices of the same size. Write $A \subseteqq B$ if for each pair of indices $b_{ik} = 0$ implies $a_{ik} = 0$. Let A be square non-negative. If $A^r \subseteqq A^{r+1}$ for each positive integer r then the sequence of matrices $\{A^r\}$ is said to be non-decreasing, the sequence of integers $\{p(A^r)\}$ being obviously non-decreasing.

Let $A = (a_{ij})$ be an $n \times n$ matrix. For each permutation $\{p_1, p_2, \ldots, p_n\}$ of $N = \{1, 2, \ldots, n\}$ the product $\prod_{i=1}^{n} a_{ip_i}$ is called a diagonal product of A. The well known Frobenius-König theorem states that all diagonal products of A are zero if and only if A contains an $p \times q$ zero submatrix such that $p + q > n$ (v. [2]).

Given an $n \times n$ matrix $A = (a_{ij})$, denote by $G(A)$ the directed graph consisting of vertices $\{1, 2, \ldots, n\}$ and edges $\{i, k\}$ for each $a_{ik} \neq 0$. This graph is frequently used to describe combinatorial properties of A. A sequence $\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_{l-1}, w\}$ of edges of $G(A)$ is called a connection from v to w of the length l. Denote $A^r = (a_{ik}^{(r)})$. Notice that if A is non-negative then there exists a connection from v to w of the length l in $G(A)$ if and only if $a_{vw}^{(l)} > 0$.

3.

Let A be a non-negative square matrix. If A contains at most one zero element in the main diagonal then the sequence $\{A^r\}$ is non-decreasing.

Proof. Denote by n the order of A. The case $n = 1$ being obvious, suppose $n > 1$. Let r be a positive integer. $AA^r = A^rA$ implies

$$a_{ik}^{(r+1)} = a_{ii}a_{ik}^{(r)} + \sum_{j \neq i} a_{ij}a_{jk}^{(r)} = a_{ik}^{(r)}a_{kk} + \sum_{j \neq k} a_{ij}^{(r)}a_{jk}$$

for each $i, k \in N$.

Suppose first either $i \neq k$ or $a_{ii} > 0$. Then the above equation yields that $a_{ik}^{(r)} > 0$ implies $a_{ik}^{(r+1)} > 0$.

Suppose now $a_{ii} = 0$, $a_{ii}^{(r)} > 0$. Then there is a connection c from i to i of length r in $G(A)$. $G(A)$ does not contain an edge $\{i, i\}$ and so in c there is a vertex $j \neq i$. According to the assumption, $\{j, j\}$ is in $G(A)$. Hence, there is a connection from i to i of length $r + 1$, thus $a_{ii}^{(r+1)} > 0$ which completes the proof.

Let A be a non-negative square matrix. Then $p(B) \leqq p(AB)$ for each non-negative matrix B of the same size as A if and only if A possesses a non-zero diagonal product.

Proof. Denote by n the order of A. Suppose $\prod_{i=1}^{n} a_{ip_i} > 0$. Then, obviously, the i-th row of AB contains at least as many positive elements as the p_i-th row of B does, for each $i \in N$.

Suppose that all the diagonal products of A are zero. According to the Frobenius-König theorem, there exist permutation matrices R, S such that RAS contains a $p \times q$ zero submatrix in the lower left corner and $p + q > n$. Choose an integer t, $1 \leqq t \leqq n$ and an $n \times n$ matrix C the elements of which are positive except the $(n - q) \times t$ zero submatrix in the lower left corner. Put $B = SC$. It holds $p(B) = p(C) = n^2 - (n - q)\, t$ and $p(AB) = p(RAB) \leqq n^2 - pt$, as $RAB = RASS^{-1}B = RASC$ contains the $p \times t$ zero submatrix in the left down corner. Accordingly, $p(B) - p(AB) \geqq t(p + q - n) > 0$ which completes the proof.

As an immediate consequence the following corollary is obtained.

Let A be a square non-negative matrix possessing a non-zero diagonal product. Then the sequence $\{p(A^r)\}$ is non-decreasing.

References

[1] *Z. Šidák:* O počtu kladných prvků v mocninách nezáporné matice. Čas. pěst. mat. *89* (1964), 28—30.

[2] *A. Vrba:* An application of Halls' theorems to matrices. Čas. pěst. mat. *98* (1973), 288—291.

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