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ON SOME GRAPH-THEORETICAL PROBLEMS OF V. G. VIZING

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In [2] V. G. VIZING suggests a number of unsolved graph-theoretical problems. Here we shall solve partially two of them.

I.

The first problem we shall investigate is the following one:

Which is the maximal number of edges that a graph with n vertices and with a given Hadwiger number can have?

Here this problem is solved for Hadwiger number 3.

We say that a graph G can be contracted onto a graph H if and only if the graph H can be obtained from G by a finite number of the following operations:

(a) deleting an edge;

(b) deleting an isolated vertex;

(c) identifying two neighbouring vertices, i.e. replacing of two neighbouring vertices x and y by a new vertex neighbouring to exactly all vertices which were neighbouring to at least one of the vertices x and y.

We consider only finite undirected graphs without loops and multiple edges.

The Hadwiger number $\eta(G)$ of a graph G is the maximal number of vertices of a complete graph onto which G can be contracted.

By $\lambda_k(n)$ for any positive integer n we shall denote the maximal number of edges of a graph of Hadwiger number k with n vertices.

The graphs with Hadwiger number 3 are graphs which can be contracted onto a triangle, but not onto a complete graph with four vertices.

If G is a graph, C a circuit in G, then a diagonal arc of C in G is an arc joining two vertices of C whose internal vertices do not belong to C. Two vertex-disjoint diagonal arcs P_1 and P_2 of C will be called topologically crossing if and only if the circuit C and the arcs P_1 and P_2 cannot be drawn in the plane so that the arcs P_1 and P_2 might be drawn in the interior of the drawing of C without crossing each other. (This term is defined only for the use of this paper.)

The following lemma is evident.

Lemma 1. *A graph G has Hadwiger number not exceeding 3 if and only if no circuit C in G has two vertex-disjoint topologically crossing diagonal arcs.*

We shall prove another lemma.

Lemma 2. *For every positive integer $n \geqq 2$ any graph of Hadwiger number 3 with n vertices and maximal possible number of edges is connected and without articulations.*

Proof. Assume that such a graph G is disconnected. Then we join two vertices of different connected components of G by an edge e; the graph thus obtained will be denoted by G'. The edge e is a bridge in G', therefore it belongs neither to a circuit, nor to a diagonal arc of some circuit in G'. This means that all circuits and their diagonal arcs in G' are those of G and G' has also Hadwiger number 3, which is a contradiction with the maximality of G.

Now assume that G is connected and contains an articulation a. Let L_1, L_2 be two lobes whose common vertex is a. Let u_1 and u_2 be vertices of L_1 and L_2 respectively joined by an edge with a. Let G'' be a graph obtained from G by adjoining an edge h joining u_1 and u_2. Let C be a circuit in L_1, let P be a diagonal arc of C in G'' not contained in G and joining the vertices v_1 and v_2 of C. Then P consists of an arc P_1 from v_1 (or v_2) to a, an arc P_2 from a to u_2, the edge h and an arc P_3 from u_1 to v_2 (or v_1). Then there exists an arc P' in G joining v_1 and v_2 and consisting of the path P_1, the edge au_1 and the path P_3. This is either a diagonal arc of C, or an edge of C (if the vertices v_1, v_2 are identical with u_1, a). If some other diagonal arc P'' of c vertex-disjoint with P forms together with P a pair of topologically crossing diagonal arcs of C in G'', then P'' is in L_1, because it contains neither a nor h. This means that P'' forms a pair of topologically crossing diagonal arcs of C also with P' and this pair is also in G, which is a contradiction. Analogously we can consider any circuit in L_2. A circuit in G which is neither in L_1 nor in L_2 evidently cannot have a diagonal arc in G'' not contained in G. We have proved that by adjoining the edge h no pair of topologically crossing diagonal arcs of any circuit of G is obtained. Now consider a circuit C' in G'' not contained in G. Evidently it consists of an arc R_1 from a to u_1 in L_1, the edge h and an arc R_2 from u_2 to a in L_2. Any diagonal arc of C' joins either two vertices of R_1, or two vertices of R_2. As L_1 is a lobe, there exists an arc R'_1 joining a and u_1 in L_1 and having no vertex in common with R_1 except for a and u_1. The arcs R_1, R'_1 form a circuit C_1 in L_1; any diagonal arc of C joining two vertices of R_1 is also a diagonal arc of C_1 and any two such arcs which would be topologically crossing in G'' would be topologically crossing also in G. Analogously for R_2. Finally, a diagonal arc of C joining two vertices of R_1 and a diagonal arc of C joining two vertices of R_2 cannot evidently be topologically crossing. Therefore G'' has also Hadwiger number 3, which is a contradiction with the maximality of G.

Lemma 3. *Let G be a graph of Hadwiger number 3 with n vertices. Let u be its vertex and G_0 the graph obtained from G by deleting u. Let G_0 be a connected graph with p lobes. Then u is joined in G at most with $p + 1$ vertices.*

Proof. First assume that three vertices of a lobe L of G_0 are joined with u in G. Then there exists a circuit in this lobe containing all of them; it can be contracted onto a triangle, whose vertices are these three vertices. This triangle together with u and the edges joining u with its vertices form a complete graph with four vertices and $\eta(G_0) \geqq 4$, which is a contradiction. Therefore u can be joined at most with two vertices of the same lobe. Now let u be joined with two vertices v_1, v_2 of a lobe L, none of which is a cut-vertex. If there exists at least on vertex v_3 in G_0 which is joined with u in G and different from v_1 and v_2, then let a be the cut-vertex belonging to L and separating v_3 from v_1 and v_2. In L there exists a circuit containing v_1, v_2 and a. The subgraph of G consisting of this circuit, of the edges uv_1, uv_2, uv_3 and of an arc joining a and v_3 in G_0 (none of whose edges is in L) can be contracted onto a complete graph with four vertices. Therefore if G_0 has cut-vertices, at most three vertices of G_0 are joined with u in G and two of them belong to one lobe, not being cut-vertices, the graph G has not the assumed property. We shall continue by induction with respect to p. For $p = 1$ the assertion holds, because G_0 consists of one lobe and we have proved that no three vertices of one lobe can be joined with u in G. Let $r \geqq 2$, let the assertion hold for $p < r$. If we delete one lobe L except for the cut vertices in it from G_0 so that the resulting graph G_1 is connected (this is always possible), then G_1 has $r - 1$ lobes and u is joined in G with at most r vertices of G_1. Now at most one vertex of L which is no cut-vertex may be joined with u. The lobe L contains only one cut-vertex which is in G_1 (because it is a common vertex of L and some other lobe), thus at most $r + 1$ vertices of G_0 can be joined in G with u.

Now we shall prove

Theorem 1. *Let $\lambda_k(n)$ be the maximal number of edges of a graph G of Hadwiger number k with n vertices. Then*

$$\lambda_3(n) = 2n - 3$$

for any positive integer $n \geqq 2$.

Proof. We shall prove the assertion by induction. The graphs with two or three vertices evidently cannot be contracted onto a complete graph with four vertices. The maximal number of edges of a graph with $n = 2$ vertices is $2n - 3 = 1$, the maximal number of edges of a graph with $n = 3$ is $2n - 3 = 3$. For $n = 4$ only the complete graph with 4 vertices has Hadwiger number 4, no other can be contracted onto it.

Thus the graph of Hadwiger number 3 with four vertices and the maximal possible number of edges is the graph obtained from the complete graph with four vertices by deleting one edge. Now let $n = r \geqq 5$ and let the assertion hold for $2 \leqq n < k$. Let G be a graph with k vertices and $\lambda_3(r)$ edges for which $\eta(G) = 3$. Delete one vertex u from G and denote the obtained graph by G_0. According to Lemma 2 G_0 is connected. According to Lemma 3 the number of vertices of G_0 joined by edges with u in G is at most $p + 1$, where p is the number of lobes of G_0. Let the lobes of G_0 be $L_1, \ldots, L_p$, let l_i be the number of vertices of L_i for $i = 1, \ldots, p$. For the number $r - 1$ of vertices of G_0 we have

$$r - 1 = \sum_{i=1}^{p} l_i - p + 1 . \tag{1}$$

Any lobe of G_0 is a graph with Hadwiger number not exceeding 3 (because this property is evidently hereditary). According to the induction assumption the number of edges of L_i does not exceed $2l_i - 3$ for $i = 1, \ldots, p$. For the number m_0 of edges of G_0 we have

$$m_0 \leqq \sum_{i=1}^{p} (2l_i - 3) = 2\sum_{i=1}^{p} l_i - 3p .$$

As u is joined with not more than $p + 1$ vertices of G_0, for the number m of edges of G we have

$$m \leqq m_0 + p + 1 \leqq 2\sum_{i=1}^{p} l_i - 2p + 1 .$$

From (1) we have

$$\sum_{i=1}^{p} l_i = r + p - 2 ,$$

therefore

$$m \leqq 2r - 3 .$$

We have proved that $2n - 3$ is the upper bound for the number of edges of a graph with Hadwiger number 3 with n vertices. It remains to prove that for every $n \geqq 2$ this bound is attained. For any given $n \geqq 2$ we construct the "fan graph" F_n as follows. The vertices of F_n are $v_1, \ldots, v_n$ and its edges are $v_i v_{i+1}$ for $i = 1, \ldots, n - 1$ and $v_1 v_j$ for $j = 3, \ldots, n$. If $n > 2$, a contraction of any edge leads either to F_{n-1}, or to the graph with two lobes isomorphic to F_r with $2 \leqq r < n$. If $n = 2$, then F_2 is a graph consisting of two vertices and one edge. Thus by induction one can prove that F_n cannot be contracted onto a complete graph with four vertices, q.e.d.

In the end we shall consider also $\lambda_1(n)$ and $\lambda_2(n)$. Any graph containing at least one edge can be contracted onto a complete graph with two vertices. Thus $\eta(G) = 1$ if and only if G contains no edges and

$$\lambda_1(n) = 0 .$$

Any graph containing at least one circuit can be contracted onto a complete graph with three vertices. Thus $\eta(G) = 2$ if and only if G is a forest with at least in edge and

$$\lambda_2(n) = n - 1 .$$

Comparing $\lambda_1(n)$, $\lambda_2(n)$, $\lambda_3(n)$ leads us to a conjecture.

Conjecture. For the maximal number $\lambda_k(n)$ of edges of a graph of Hadwiger number k with n vertices we have

$$\lambda_k(n) = (k - 1)\, n - \binom{k}{2}$$

for any two positive integers $k, n \geqq 2$.

II.

The other problem which will be studied here is the following one:

Which is the maximal number of edges of a connected undirected graph with n vertices, none of whose spanning trees has more than k terminal edges?

We shall denote this number by $\tau(n, k)$. We shall give the solution for some special cases, namely for $k = 2$, $k = 3$, $k = n - 3$, $k = n - 2$, $k = n - 1$. We study graphs without loops and multiple edges.

Evidently we can define neither $\tau(n, 1)$ nor $\tau(n, n)$, because a spanning tree of a graph with n vertices has at least two and at most $n - 1$ terminal edges.

Before investigating concrete values of k, we shall introduce an auxiliary concept.

If G_0 is a connected subgraph of G, then the degree of G_0 in G is the number of vertices of G not belonging to G_0 which are joined with a vertex of G_0. If G_0 consists only of one vertex, its degree is equal to the degree of this vertex.

Now we shall prove a lemma.

Lemma 4. *Let G be a connected undirected graph. Then the maximal number of terminal edges of a spanning tree of G is equal to the maximal degree of a connected subgraph of G.*

Proof. Let G_0 be a connected subgraph of G with the maximal degree k. Let $u_1, \ldots, u_k$ be the vertices not belonging to G_0 and joined by edges with vertices of G_0. Choose a spanning tree T_0 of G_0. Then for any $i = 1, \ldots, k$ choose an edge e_i joining u_i with a vertex of G_0. The graph T_0' consisting of all vertices of G_0, vertices $u_1, \ldots, u_k$, all edges of G_0 and all edges $e_1, \ldots, e_k$ is a tree in which $e_1, \ldots, e_k$ are terminal edges. This tree T_0' is a subtree of a spanning tree T of G which has also at least k terminal edges. (Evidently the number of terminal edges of a subtree of a tree T is less than or equal to the number of terminal edges of T.) On the other hand, let l

be the maximal number of terminal edges of a spanning tree of G. Let T_1 be a spanning tree of G with l terminal edges. Let G_1 be the subgraph of G generated by all vertices which are not terminal in T_1. Then G_0 has the degree l.

Now we shall prove theorems on the numbers $\tau(n, k)$.

Theorem 2. $\tau(n, 2) = n$ *for every* $n \geqq 3$.

This assertion is evident; we leave the proof to the reader.

Theorem 3. $\tau(n, 3) = n + 2$ *for every* $n \geqq 4$.

Proof. Let G be a graph with n vertices ($n \geqq 4$) such that none of its spanning trees has more than three vertices. At first assume that G has a Hamiltonian circuit C consisting of the vertices $u_1, \ldots, u_n$ and the edges u_iu_{i+1} for $i = 1, \ldots, n - 1$ and u_nu_1. Assume that there exists an edge u_iu_j where $|i - j| \geqq 3$ (the difference is taken modulo n). Without any loss of generality let $i = 1$; then $j \neq 2, j \neq 3, j \neq n - 1$, $j \neq n$. Let T_0 be a subgraph of G consisting of the vertices $u_1, u_2, u_{j-1}, u_j, u_{j+1}, u_n$ and of the edges $u_1u_2, u_1u_n, u_1u_j, u_{j-1}u_j, u_ju_{j+1}$; it is a tree in which all edges except u_1u_j are terminal, therefore with four terminal edges. The tree T_0 is a subtree of some spanning tree T of G which has at least four vertices, which is a contradiction. Therefore any edge not belonging to C is u_iu_{i+2} for some i, $1 \leqq i \leqq n$ (the sum $i + 2$ is taken modulo n). Let there exist an edge u_1u_3 (without any loss of generality) and some other edge u_ju_{j+2} (where $j \neq 1$). Evidently $j \neq 3$, $j \neq n - 1$, because otherwise u_j or u_{j+2} would have the degree at least four. Assume $4 \leqq j \leqq n - 2$. The there exists a subgraph T_1 of G consisting of the vertices $u_1, \ldots, u_{j+2}$ and of the edges u_1u_3, u_ju_{j+2} and u_iu_{i+1} for $i = 2, \ldots, j$. It is a tree with four terminal edges $u_1u_3, u_2u_3, u_ju_{j+1}, u_ju_{j+2}$ and we obtain a similar contradiction as in the preceding case. Therefore an edge of G not belonging to C and different from u_1u_3 can be only u_2u_4 or u_nu_2; they cannot exist both, because u_2 would have the degree at least four. Therefore G has at most $n + 2$ edges. Now assume that G has no Hamiltonian circuit. Let C_0 be the circuit of the maximal length l in G, let its vertices be $v_1, \ldots, v_l$ and its edges v_iv_{i+1} for $i = 1, \ldots, l - 1$ and v_lv_1. Let there exist two vertices w_1, w_2 not belonging to C and joined by edges with vertices of C. If the length of C is at least 5, we can choose an edge e of C such that w_1 and w_2 are joined with the vertices v_i, v_j which are consequently not incident with e. The tree whose edges are all edges of C except e and v_iw_1, v_jw_2 (we may have $v_i = v_j$) is a subtree of G with four terminal edges. Thus if the length of C is at least 5, there may exist only one vertex w not belonging to C and joined with a vertex of C. For the edges joining two vertices of C and not belonging to C the same holds as in the case of a Hamiltonian circuit. So assume that there are two such edges; let one of them (without any loss of generality) be v_1v_3 and the other v_lv_2. There exist two subtrees of G with three terminal edges not containing w, namely T_1 with the edges v_iv_{i+1} for $i = 2, \ldots, l - 1$ and v_1v_3 and T_2

with the edges v_iv_{i+1} for $i = 3, \ldots, l - 1$, v_lv_1, v_lv_2. If w is joined with some v_i, where $4 \leqq i \leqq l - 1$, then by adding the vertex w and the edge u_iw to T_1 or to T_2 we obtain a tree with four terminal edges. If w is joined with v_3 or v_l, then by adding w and v_3w or v_lw to T_1 or T_2 respectively we obtain also a tree with four terminal edges. If w is joined with v_1 or v_2, then v_1 or v_2 has the degree at least four. We have proved that if there are two edges joining vertices of C and not belonging to C (for C of the length at least 5), then C must be a Hamiltonian circuit of G. Now assume that there exists one edge joining two vertices of C and not belonging to C; analogously to the case when C is Hamiltonian this edge is (without any loss of generality) v_1v_3. Then there exist two subtrees of G not containing vertices outside of C with three terminal edges, namely T_1 with the edges v_iv_{i+1} for $i = 4, \ldots, l - 1$, v_lv_1, v_1v_2, v_1v_3 and T_2 with the edges v_iv_{i+1} for $i = 2, \ldots, l - 1$, v_1v_3. If w is joined with v_i for $4 \leqq i \leqq$ $\leqq l - 1$, then by adding w and v_iw to T_1 or T_2 we obtain a tree with four terminal edges. If w is joined with v_1 or v_3, then by adding w and v_1w or v_3w to T_1 or T_2 respectively we obtain also a tree with four terminal edges. Thus w can be joined only with u_2. If there are two vertices x_1, x_2 joined with w and not belonging to C, then by adding the edges v_2w, wx_1, wx_2 to T_1 or T_2 we obtain again a tree with four terminal edges. Thus w can be joined only with one vertex w_1 not belonging to C; analogously w_1 can be joined only with one vertex w_2 not belonging to C and different from w etc.; therefore the subgraph of G generated by v_2 and all vertices not belonging to C is an arc. We have proved that the subgraph generated by the vertices of C has at most $l + 2$ edges, if C is Hamiltonian, or at most $l + 1$ edges, if there are some vertices not belonging to C. In the former case C is Hamiltonian and $l = n$, thus $l + 2 = n + 2$. In the latter case the number of vertices not belonging to C is $n - l$ and, as they generate an arc, the number of edges joining vertices not belonging to C is $n - l - 1$ and there is one edge joining a vertex not belonging to C, namely w, with a vertex of C, namely v_2. The total number of edges of G is at most $n + 2$. From the proof it follows that this bound can be always attained. It remains to discuss the case when the length of the longest circuit in G is less than 5. If it is 3, then any circuit of G is a lobe of G, therefore any lobe of G is either a triangle, or a bridge. Assume that two lobes L_1, L_2 of G are triangles. If they have a common vertex, it has the degree at least 4, which is impossible. Otherwise we take an arc joining a vertex v_1 of L_1 with a vertex v_2 of L_2 and having no edge in common with L_1 and L_2. The tree consisting of this arc, of two edges from L_1 incident with v_1 and of two edges of L_2 incident with v_2 has four terminal edges, namely the edges of L_1 and L_2 incident with v_1 or v_2. Therefore G can have at most one lobe which is a triangle, the others being bridges. The cyclomatic number of G is at most 1, thus G has at most n edges. If the length of the longest circuit in G is 4, then any lobe of G is either a bridge or a triangle, or it consists of a system of at least two edge-disjoint arcs of the lengths 1 or 2 joining two vertices a and b. Analogously to the preceding case we can prove that there is at most one lobe which is not a bridge. According to the assumption it cannot be a triangle, thus it is of the last type. The number of paths joining a and b

can be at most three, otherwise a and b would have the degree greater than three. If they are two or three, the cyclomatic number of G is 1 or 2 respectively, and the number of vertices of G is n or $n + 1$, respectively.

Theorem 4. $\tau(n, n-3) = \frac{1}{2}n^2 - \frac{5}{2}n + 5$ *for every* $n \geqq 5$.

Proof. Let G be a graph with n vertices ($n \geqq 5$) such that none of its spanning trees has more than $n - 3$ terminal edges. Investigate the complement $\bar{G}$ of G. The graph $\bar{G}$ has the following properties:

(a) the degree of any vertex of $\bar{G}$ is at least two;
(b) the diameter of $\bar{G}$ is at most two;
(c) the complement G of $\bar{G}$ is connected.

If $\bar{G}$ had not the property (a), there would exist some vertex u of $\bar{G}$ of the degree 0 or 1. This vertex would have the degree $n - 1$ or $n - 2$ in G, therefore the star with the center u would be a subtree of G with more than $n - 3$ terminal edges. If $\bar{G}$ had not the property (b), then there would exist two vertices u_1, u_2 of $\bar{G}$ with the distance greater than two. There would not exist any vertex joined with both u_1 and u_2 and these two vertices also would not be joined together. This means that in G any vertex would be joined at least with one of the vertices u_1, u_2 and there would exist the edge u_1u_2. For any vertex of G different from u_1 and u_2 we choose one edge joining it with u_1 or u_2; these edges together with u_1u_2 would form a spanning tree of G with $n - 2$ terminal edges. The condition (c) follows from the text of the problem, because only connected graphs have spanning trees.

We can construct a graph G_0 satisfying the conditions (a), (b), (c) and having $2n - 5$ edges. This is the graph whose vertex set is $u_1, u_2, v_1, \ldots, v_{n-4}, w_1, w_2$ and whose edges are u_1v_i and u_2v_i for $i = 1, \ldots, n - 4$, and further u_1w_1, w_1w_2, u_2w_2. This graph G_0 contains n vertices and $2n-5$ edges. We shall prove that there does not exist any graph with less than $2n - 5$ edges satisfying the conditions (a), (b), (c). Assume that there exist a graph G_1 with n vertices and less than $2n - 5$ edges ($n \geqq 5$) satisfying the conditions. At least one of the vertices of G_1 must have the degree less than four; in the opposite case G_1 would contain at least $2n$ edges. Thus also the vertex connectivity degree of G_1 is at most 3. Let R be a cut set of G_1 with the minimal number of vertices. At first assume that $|R| = 1$, thus $R = \{a\}$, where a is some cut vertex. If u, v are two vertices of G_1 separated by a, then they must be both joined with a, because their distance cannot be greater than two and any arc joining them must contain a. As these vertices were chosen arbitrarily, this implies that a is joined with all other vertices of G_1. Then a is joined with no other vertex in the complement of G_1 and is an isolated vertex; therefore this complement is not connected, which contradicts (c). Assume $|R| = 2$, thus $R = \{a_1, a_2\}$. Let $K_1, \ldots, K_t$ be the connected components of the graph obtained from G_1 by deleting the vertex set R and all edges incident to it. Assume that in K_1 (without any loss of generality) there exists a vertex u_1 joined with a_1 and not with a_2 and a vertex u_2 joined with a_2 and not with a_1. Let v be a vertex of some K_i for $i \neq 1$. It must have the distance at most 2 from both u_1 and u_2, therefore it must be joined with both a_1 and a_2. As v was chosen quite arbitrarily, any vertex of $\bigcup_{i=2}^{l} K_i$ must be joined with both a_1 and a_2. Let m be the total number of vertices of $\bigcup_{i=2}^{l} K_i$; then the number of edges not incident with vertices of K_1 is at least $2m$. The component K_1 contains $n - m - 2$ vertices. It must be connected, thus it contains at least $n - m - 3$ edges. Each vertex of K_1 must be joined with some vertex of R, therefore there are at least $n - m - 2$ edges joining vertices of K_1 with vertices of R. The graph G_1 has then at least $2m + (n - m - 2) + (n - m - 3) = 2n - 5$ edges. Now assume that in K_1 there is a vertex u_1 joined with a_1 and not with a_2, but all vertices of K_1 are joined with a_1. Then in K_i for each $i = 2, \ldots, l$ also all vertices are joined with a_1 and there may also exist in it some vertices joined with a_1 and not with a_2. Let M be the set of vertices of G_1 not belonging to R joined with a_1 and not joined with a_2. Let M_i for $i = 1, \ldots, l$ be the intersection of M with the vertex set of K_i. Consider a connected component of the subgraph of G_1 generated by the set M_i; let p be its number of vertices. As this component C is connected, it contains at least $p - 1$ edges. As any of its vertices is joined with a_1, we have further p edges incident with vertices of this component. This component C is a subgraph of some K_i and evidently a proper subgraph; otherwise no vertex of K_i would be joined with a_2 and a_1 would be a cut vertex separating vertices of K_i from other vertices of G_1. Therefore there exists at least one edge joining a vertex of C with some other vertex of K_i. We have at least $2p$ edges incident with vertices of C and with no other vertices of M. Therefore if $|M| = q$, then there exist $2q$ edges incident with vertices of M (this number was obtained as a sum over all such components C). Any of the vertices not belonging to $M \cup R$ are joined with both a_1 and a_2. As the number of vertices not belonging to $M \cup R$ is $n - q - 2$, we have $2n - 2q - 4$ edges joining these vertices with the vertices of R. Thus G_1 has at least $2q + (2n - 2q - 4) = 2n - 4$ edges. If all vertices not belonging to R are joined with both a_1 and a_2, the graph G_1 has evidently also at least $2n - 4$ edges.

Finally assume that $|R| = 3$, thus $R = \{a_1, a_2, a_3\}$. We shall prove that in each of the components $K_1, \ldots, K_l$, except at most one, either there exists a vertex joined with all vertices of R, or there exist two vertices, each of which is joined with two vertices of R. Assume that K_1 has not this property; i.e. that at most one vertex of K_1 is joined with two vertices of R, any other vertex being joined exactly with one vertex of R. If each vertex of K_1 is joined only with one vertex of R, there must exist three vertices u_1, u_2, u_3 of K_1 so that u_i is joined with a_i for $i = 1, 2, 3$, and with no other vertex of R (otherwise the vertex connectivity degree of G_1 would be less than three). Any vertex of K_i for $i = 2, \ldots, l$ must have the distance at most two from all three vertices u_1, u_2, u_3, therefore it must be joined with all the vertices a_1, a_2, a_3. If there exists a vertex v of K_1 joined with two vertices a_1, a_2 (without any loss of generality) of R and not with a_3 and all other vertices are joined only with one vertex of R each, then there exists a vertex u_3 of K_1 joined with a_3 and with no other vertex of R. Any vertex of K_i $(i = 2, \ldots, l)$ must have the distance from both v and u_3 at most 2, therefore it must be joined with a_3 and one of the vertices a_1, a_2. If this K_i contains only one vertex, it must be joined with all vertices of R, because we have assumed that the vertex connectivity degree of G_1 is 3 and therefore each vertex has the degree at least 3. If K_i contains two different vertices w_1, w_2, any of them must be joined with a_3 and one of the vertices a_1, a_2. Any of the components K_i $(i = 1, \ldots, l)$ must contain at least $k_i - 1$ edges, where k_i is the number of its vertices, and there are at least k_i edges joining its vertices with vertices of R; therefore there are at least $2k_i - 1$ edges incident with vertices of K_i. But if for some K_i this number is exactly $2k_i - 1$, this means that any vertex of K_i is joined exactly with one vertex of R; then any vertex of K_j for $j \neq i$ is joined with all vertices of R. Then the graph G_1 contains at least $3(n - k_i - 3) + 2k_i - 1 = 3n + k_i - 10$ vertices, which is more than $2n - 5$, because $n \geqq 5$. If exactly one vertex of K_i is joined with two vertices of R and any other vertex of K_i is joined only with one vertex of R, then there are at least $2k_i$ edges incident with vertices of K_i and any vertex of K_j for $j \neq i$ must be joined at least with two vertices of R; if such K_j consists only of one vertex, it is joined with all vertices of R, otherwise there exists at least one edge of K_j. Thus there are at least $2k_j + 1$ edges incident with vertices of K_j for $j \neq i$ (k_j is the number of vertices of K_j) and the total number of edges of G_1 is at least $2n - 5$. If in each K_i either there are two vertices joined with two vertices of R, or there is a vertex joined with all vertices of R, then there are $2k_i + 1$ edges incident with vertices of K_i and G_1 has at least $2n - 4$ edges. We have proved that there does not exist any graph satisfying (a), (b), (c) and having less than $2n - 5$ edges. The existence of such a graph with exactly $2n - 5$ edges had been proved before. The graph G with the property that none of its spanning trees has more than $n - 3$ terminal edges and with the maximal possible number of edges is a complement of such a graph. Therefore its number of edges is $\frac{1}{2}n(n - 1) - - (2n - 5) = \frac{1}{2}n^2 - \frac{5}{2}n + 5$, q.e.d.

Theorem 5. *$\tau(n, n - 2) = \frac{1}{2}n^2 - n$ for n even, $\tau(n, n - 2) = \frac{1}{2}n^2 - n - \frac{1}{2}$ for n odd, $n \geqq 4$.*

Proof. The only tree with n vertices and $n - 1$ terminal edges is a star. A star can be a spanning tree of a graph G if and only if G contains a vertex u joined with all other vertices, i.e. of the degree $n - 1$. Therefore we look for a graph G with n vertices with the maximal number of edges, in which no vertex has the degree $n - 1$. For n even such a graph is a regular graph of the degree $n - 2$; it contains $\frac{1}{2}n^2 - n$ edges. For n odd such a graph does not exist, but there exists a graph, one of whose vertices has the degree $n - 3$ while all others have the degree $n - 2$. This is evidently the required graph and its number of edges is $\frac{1}{2}n^2 - n - \frac{1}{2}$.

Theorem 6. $\tau(n, n-1) = \frac{1}{2}n^2 - \frac{1}{2}n$ *for every* $n \geqq 3$.

Proof is easy, it is left to the reader.

Remark. The English terminology of the graph theory used in this paper is that of [1].

References

[1] *O. Ore:* Theory of Graphs. Providence 1962.

[2] *В. Г. Визинг:* Некоторые нерешенные задачи в теории графов. Успехи мат. наук *23* (1968), 117–134.

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