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Pavla Vrbová

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ON CONTINUITY OF LINEAR TRANSFORMATIONS COMMUTING WITH GENERALIZED SCALAR OPERATORS IN BANACH SPACE

PAVLA VRBOVÁ, Praha

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1. INTRODUCTION

In the present paper we give a modification of methods having been presented in the paper of B. E. JOHNSON and A. M. SINCLAIR [5]. The question is, under which condition a linear transformation S commuting with a linear continuous operator T in a (complex) Banach space X is continuous. Similarly as in the paper mentioned above we shall deal with operator T having a suitable spectral decomposition. More exactly: suppose that there exists, for every closed subset F of the complex plane $\mathbf{C}$, a closed linear subspace $\mathscr{E}(F)$ in X such that the following conditions are fulfilled:

(1) $$\mathscr{E}(\emptyset) = \{0\}\,, \quad \mathscr{E}(\mathbf{C}) = X\,;$$

(2) $$\bigcap_{n=1}^{\infty} \mathscr{E}(F_n) = \mathscr{E}\Big(\bigcap_{n=1}^{\infty} F_n\Big)\,;$$

(3) if $\{G_j\}_{j=1}^{m}$ is a finite open covering of the complex plane, then

$$X = \mathscr{E}(\overline{G}_1) + \ldots + \mathscr{E}(\overline{G}_m)\,;$$

(4) $$T\mathscr{E}(F) \subset \mathscr{E}(F) \quad \text{and} \quad \sigma(T \mid \mathscr{E}(F)) \subset F\,.$$

For the sake of completness we recall now some definitions.

Definition. Let $x \in X$. A complex number λ is an element of $\varrho_T(x)$ if there is a vector-valued analytic function $x(.)$ defined in a neighbourhood G_λ of λ such that $(\mu I - T)\, x(\mu) = x$ for all $\mu \in G_\lambda$. The spectrum $\sigma_T(x)$ is the complement of $\varrho_T(x)$. Obviously $\sigma_T(x) \subset \sigma(T)$.

Definition. An operator $T \in \mathscr{L}(X)$ (the algebra of all linear continuous operators of X) is said *to have the single-valued extension property* if for every open subset G

of the complex plane and for any vector valued analytic function $f: G \to X$ the equality $(\lambda I - T) f(\lambda) \equiv 0$ on G implies $f \equiv 0$.

For every operator T having the single-valued extension property $\sigma_T(x) = \emptyset$ if and only if $x = 0$.

It has been shown in [2] that each operator of the present class has the single-valued extension property and that the present class of operators is nothing else than the class of decomposable operators (in sense of [2]) with

$$\mathscr{E}(F) = \{x : \sigma_T(x) \subset F\} . \tag{5}$$

We shall use the usual notation $X_T(F) = \mathscr{E}(F)$. Let $L \in \mathscr{L}(X)$ be such that $TL = LT$. Then it is easy to prove that $LX_T(F) \subset X_T(F)$ for every F closed.

Hence, consider now a linear transformation S commuting with our operator T and such that $S\, X_T(F) \subset X_T(F)$ for every F closed.

Denote by σ_S the linear subspace of X consisting of all elements x such that there exists a sequence $x_n \to 0$ with $Sx_n \to x$. The subspace σ_S is closed. According to the closed graph theorem the transformation S is continuous if and only if $\sigma_S = 0$.

Since we have, for an arbitrary finite open covering, the decomposition of X, it is natural to take into account only the subspaces on which S is not continuous. It is easy to see that each such subspace must have a non-trivial intersection with σ_S. We shall consider, therefore, the subspace $X_T(F)$ such that $\sigma_S \subset X_T(F)$. If λ is not an element of F, then there exists a closed neighbourhood G of λ with $G \cap F = \emptyset$ and $S \mid X_T(G)$ is continuous by the closed graph theorem. This fact leads quite naturally to the following

Definition. We shall call a number λ *a discontinuity value* if the operator $S \mid X_T(F)$ is discontinuous for every closed neighbourhood F of λ.

Obviously every discontinuity value is an element of the set F such that $\sigma_S \subset X_T(F)$.

Further, from the definition it follows immediately that the set of all discontinuity values is closed and contained in $\sigma(T)$.

Lemma. $\sigma_S \subset X_T(K)$ *where K is the set of all discontinuity values.*

Proof. Let $\lambda \notin K$, let F_0 be a closed neighbourhood of λ such that $S \mid X_T(F_0)$ is continuous. Let $\{G_0, G_1\}$ be an open covering of the complex plane, $\bar{G}_0 \subset F_0$, $\lambda \notin \bar{G}_1$. Take an $x \in X$, let $x_n \to 0$ with $Sx_n \to x$. Since we have, for every $x \in X$, the decomposition $x = x_1 + x_2$ where $x_1 \in X_T(\bar{G}_0)$, $x_2 \in X_T(\bar{G}_1)$, we can find sequences $x_n^1 \to 0$, $x_n^2 \to 0$ such that $x_n = x_n^1 + x_n^2$, $x_n^1 \in X_T(\bar{G}_0)$, $x_n^2 \in X_T(\bar{G}_1)$. We have $Sx_n = Sx_n^1 + Sx_n^2$. Since $S \mid X_T(\bar{G}_0)$ is continuous it follows $Sx_n^1 \to 0$, $Sx_n^2 \to x$ and $x \in X_T(\bar{G}_1)$, i.e. $\sigma_S \subset X_T(\bar{G}_1)$. We have obtained the following implication: if $\lambda \notin K$ then there is a closed F_λ such that $\lambda \notin F_\lambda$ and $\sigma_S \subset X_T(F_\lambda)$. By (5) the family of subspaces $X_T(F)$ is closed with respect to the intersection and we have

$$\sigma_S \subset \bigcap_{\lambda \notin K} X_T(F_\lambda) = X_T\big(\bigcap_{\lambda \notin K} F_\lambda\big) \subseteq X_T(K) .$$

However, for the proof of the main theorem we have taken generalized scalar operators for which it is easy to characterize the structure of spaces $X_T(\{\lambda\})$.

2. PRELIMINARIES

2.1. Definition. Denote by $(C^\infty(R_2), \tau)$ the Fréchet space of all infinitely differentiable complex functions $\varphi(x_1, x_2)$ defined on R_2 with the family of pseudonorms

$$|\varphi|_{k,m} = \sum_{p_1+p_2=0}^{m} \sup_{(x_1,x_2)\in K} \left| \frac{\partial^{p_1+p_2}\varphi(x_1, x_2)}{\partial^{p_1}x_1\, \partial^{p_2}x_2} \right|$$

for every compact set K and $p_1, p_2, m \geqq 0$.

2.2. Definition. A continuous linear operator T in a Banach space X is said to be *a generalized scalar operator* if there exists a continuous linear mapping $\mathcal{U}$: : $(C^\infty(R_2), \tau) \to \mathcal{L}(X)$ such that

$$\mathcal{U}_{\varphi\psi} = \mathcal{U}_\varphi \mathcal{U}_\psi \quad \text{for} \quad \varphi, \psi \in C^\infty(R_2),$$

$$\mathcal{U}_1 = I, \quad \mathcal{U}_a = T \quad \text{where} \quad a(\lambda) = \lambda.$$

We shall use some properties of generalized scalar operators contained in [1] (Theorem 2, Propositions 1, 2, 3) which we mention without proving them.

2.3. Proposition. *Every generalized scalar operator T has the single valued extension property. If we denote $X_T(F) = \{x : \sigma_T(x) \subset F\}$ for $F = \bar{F}$, then $X_T(F)$ is a closed invariant subspace with respect to T such that $\sigma(T \mid X_T(F)) \subset F$.*

2.4. Proposition. *Let $x \in X$, let φ_1, φ_2 be two functions from $C^\infty(R_2)$ such that $\varphi_1 \equiv$ $\equiv 1$ in a neighbourhood of $\sigma_T(x)$ and* supp $\varphi_2 \cap \sigma_T(x) = \emptyset$. *Then $\mathcal{U}_{\varphi_1}x = x$ and $\mathcal{U}_{\varphi_2}x = 0$.*

2.5. Proposition. *Let $x \in X$. Then $\mathcal{U}_\varphi x \in X_T$* (supp φ) *for every $\varphi \in C^\infty(R_2)$. Further* supp $\mathcal{U} = \sigma(T)$.

Remark. Every generalized scalar operator T is an element of the class of operators having been considered in the introduction.

Indeed, proposition 2.3 asserts that (1) and (4) is satisfied for each $X_T(F)$. (2) is obviously satisfied and to prove (3) take an open covering $\{G_j\}_{j=1}^m$ of the complex plane. There exist functions $\varphi_j \in C^\infty(R_2)$ such that $0 \leqq \varphi_j \leqq 1$, supp $\varphi_j \subset \bar{G}_j$ $(j = 1, 2, \ldots, m)$ and $\sum_{j=1}^m \varphi_j \equiv 1$ in a neighbourhood of $\sigma(T)$. Since supp $\mathcal{U} = \sigma(T)$ we may write, for every x, that $x = \sum_{j=1}^m \mathcal{U}_{\varphi_j} x$ where $\mathcal{U}_{\varphi_j} x \in X_T$ (supp φ_j) $\subset X_T(\bar{G}_j)$ for $j = 1, 2, \ldots, m$ and (3) holds.

Every linear operator in the finite dimensional space as well as every spectral operator of the finite type are generalized scalar operators. For other examples see [1].

It will be useful to characterize the spaces $X_T(\{\lambda\})$.

2.6. Proposition. *Let Q be a polynomial with the roots $\mu_1, \ldots, \mu_n$. Then $\{x : : Q(T)\, x = 0\} \subset X_T(\{\mu_1, \ldots, \mu_n\})$.*

Proof. Let λ be a complex number and let $x, y \in X$ be such that $x = (\lambda I - T)\, y$. Obviously $\sigma_T(x) \subset \sigma_T(y)$. We shall show that $\sigma_T(y) \subset \sigma_T(x) \cup \{\lambda\}$ or equivalently $\varrho_T(x) \cap \{\mathbf{C} \setminus \lambda\} \subset \varrho_T(y)$. Take a $\mu \neq \lambda$ and $\mu \in \varrho_T(x)$. There exists an analytic function $x(\gamma)$ defined in a neighbourhood G_μ of $\mu (\lambda \notin G_\mu)$ with $x = (\gamma I - T)\, x(\gamma)$ for $\gamma \in G_\mu$. Put $y(\gamma) = [1/(\gamma - \lambda)]\, (y - x(\gamma))$. The function $y(\gamma)$ is analytic in G_μ and $(\gamma I - T)\, y(\gamma) = y$. This means of course that $\mu \in \varrho_T(y)$.

Let $Q(T)\, z = x$. The induction with respect to the degree of the polynomial Q yields $\sigma_T(z) \subset \sigma_T(x) \cup \{\mu_1, \ldots, \mu_n\}$. Particularly if $x = 0$ then we obtain the result desired.

2.7. Proposition. *If $\{\lambda_1, \ldots, \lambda_k\}$ is a finite set of complex numbers, then there is a polynomial $P(.)$ with the roots $\lambda_1, \ldots, \lambda_k$ such that*

$$P(T) \mid X_T(\{\lambda_1, \ldots, \lambda_k\}) = 0\,.$$

Proof. Denote $\mathcal{U}'_\varphi = \mathcal{U}_\varphi \mid X_T(\{\lambda_1, \ldots, \lambda_k\})$. It is easy to see that $T' = T \mid X_T(\{\lambda_1, \ldots \ldots, \lambda_k\})$ is a generalized scalar operator and $\mathcal{U}'$ is its distribution. Let n be the order of the distribution $\mathcal{U}$, let f be a continuous linear functional defined on $\mathcal{L}(X)$. Put $P(\lambda) = [(\lambda - \lambda_1) \,.\, (\lambda - \lambda_2) \ldots (\lambda - \lambda_k)]^{n+1}$. Then $\mathcal{V}_\varphi = f\mathcal{U}'_\varphi$ is a continuous linear functional on $(C^\infty(R_2), \tau)$, supp $\mathcal{V} \subset$ supp $\mathcal{U}' \subseteq \{\lambda_1, \ldots, \lambda_k\}$ and the order of $\mathcal{V}$ does not exceed the order of $\mathcal{U}'$. Since $P(\lambda)$ is zero on supp $\mathcal{V}$ and all derivatives up to n are zero as well, it follows by [3], theorem 1.5.4. that $\mathcal{V}_P = f\mathcal{U}'_P = 0$ for each f so that $P(T) \mid X_T(\{\lambda_1, \ldots, \lambda_k\}) = \mathcal{U}'_P = 0$.

Remark. From 2.6 and 2.7 it follows that $X_T(\{\lambda_1, \ldots, \lambda_k\}) = X_T(\{\mu_1, \ldots, \mu_j\})$ $(j \leqq k)$ where μ_j are all eigenvalues of T from the set $\{\lambda_1, \ldots, \lambda_k\}$.

3. LINEAR TRANSFORMATIONS COMMUTING WITH GENERALIZED SCALAR OPERATORS

Let T be a generalized scalar operator and let S be a linear transformation such that $S\, X_T(F) \subset X_T(F)$ for $F = \bar{F}$.

3.1. Lemma. *The set of discontinuity values is either empty or it has only a finite number of elements.*

Proof. To prove the lemma, we shall suppose that there is a sequence of distinct discontinuity values $\{\lambda_i\}_{i=1}^{\infty}$ and a closed sets F_i such that $\lambda_i \in \operatorname{Int} F_i$ and $F_i \cap \overline{\bigcup_{j \neq i} F_j} = \emptyset$ for every $i \in N$. Take further $\varphi_i \in C^{\infty}(R_2)$ with $\operatorname{supp} \varphi_i \cap \overline{\bigcup_{j \neq i} F_j} = \emptyset$ and $\varphi_i \equiv 1$ in a neighbourhood of F_i. The restriction of S to each of $X_T(F_i)$ is a discontinuous operator so that there exists, for each $i \in N$, an element $\xi_i \in X_T(F_i)$ such that

$$|\xi_i| < \frac{1}{2^i}, \tag{1}$$

$$|S\xi_i| > i\,|\mathcal{U}_{\varphi_i}|. \tag{2}$$

Now put $\eta = \sum_{i=1}^{\infty} \xi_i$. We can write, for each $i \in N$,

$$S\eta = S\xi_i + S\sum_{j \neq i} \xi_j .$$

If a $j \neq i$ is given, then $\xi_j \in X_T(F_j) \subset X_T(\overline{\bigcup_{j \neq i} F_j})$ and $\sum_{j \neq i} \xi_j \in X_T(\overline{\bigcup_{j \neq i} F_j})$.

By the assumption all $X_T(F)$ are invariant with respect to S so that $S\xi_i \in X_T(F_i)$ and $S\sum_{j \neq i} \xi_j \in X_T(\overline{\bigcup_{j \neq i} F_j})$. Using 2.4 we obtain

$$\mathcal{U}_{\varphi_i} S \sum_{j \neq i} \xi_j = 0 , \quad \mathcal{U}_{\varphi_i} S\xi_i = S\xi_i .$$

We have, for any $i \in N$, the estimate

$$|\mathcal{U}_{\varphi_i}| \cdot |S\eta| \geqq |\mathcal{U}_{\varphi_i} S\eta| = |S\xi_i| > i\,|\mathcal{U}_{\varphi_i}|$$

and this is a contradiction.

We shall show now that the existence of the distribution $\mathcal{U}$ is not essential and we can prove the same result for wider class of operators.

3.2. Definition. A decomposable operator T is said *to be a strongly decomposable operator* if the equality

$$\mathscr{E}(F) = \mathscr{E}(F) \cap \mathscr{E}(\bar{G}_1) + \ldots + \mathscr{E}(F) \cap \mathscr{E}(\bar{G}_m)$$

holds for every finite open covering $\{G_j\}_{j=1}^{m}$ of the complex plane and for every subspace $\mathscr{E}(F)$.

The problem if there exists a decomposable operator which is not a strongly decomposable one is still open.

We shall use again the notation $X_T(F) = \mathscr{E}(F)$.

Lemma 3.1.' *Let T be a strongly decomposable operator. Then the set of discontinuity values is empty or it has only a finite number of elements.*

Proof. Take the same sequence of discontinuity values as in 3.1. Let i be fixed. Since T is strongly decomposable, we have, for every $x \in X_T(\overline{\bigcup_{i=1}^{\infty} F_i})$, a unique representation $x = x_1^i + x_2^i$ where $x_1^i \in X_T(F_i)$, $x_2^i \in X_T(\overline{\bigcup_{j \neq i} F_j})$. The operator $R_1^i x = x_1^i$ is linear, continuous and $R_1^i \neq 0$. The transformation $S \mid X_T(F_i)$ is a discontinuous operator and we can find a $\xi_i \in X_T(F_i)$ with $|\xi_i| < 1/2^i$ and $|S\xi_i| > i|R_1^i|$. Put $\eta = \sum_{i=1}^{\infty} \xi_i$. Then

$$R_1^i S\eta = R_1^i S\xi_i + R_1^i S \sum_{j \neq i} \xi_j = S\xi_i .$$

We have, for each $i \in N$,

$$|R_1^i| \, . \, |S\eta| \geqq i|R_1^i| .$$

With regard to the properties of generalized scalar operators we can reformulate the lemma from the introduction as follows:

3.3. Lemma. *Either $\sigma_S = \{0\}$ or there exists a finite set of eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ of T with the property*

$$\sigma_S \subset X_T(\{\lambda_1, \ldots, \lambda_k\}) .$$

Proof. First we shall find the minimal subspace $X_T(F)$ containing σ_S. Denote by $\mathfrak{A}$ the family of all closed F such that $\sigma_S \subset X_T(F)$. Put $Y = \bigcap_{F \in \mathfrak{A}} X_T(F)$. It follows immediately from 2.3 that $Y = X_T(\bigcap_{F \in \mathfrak{A}} F)$ and $\sigma(T \mid Y) \subset F$ for each $F \in \mathfrak{A}$. To prove the lemma, it is sufficient to show that $\sigma(T \mid Y)$ consists of discontinuity values only. Indeed, if the set of discontinuity values is empty, then $\sigma_S \subset Y = X_T(\sigma(T \mid Y)) = X_T(\emptyset) = \{0\}$. If the set of discontinuity values consists of elements $\lambda_1, \ldots, \lambda_k$, then $\sigma_S \subset X_T(\{\lambda_1, \ldots, \lambda_k\})$.

In view of the remark in the end of the preceding section we may assume that all $\lambda_1, \ldots, \lambda_k$ are eigenvalues of T.

Take a λ which is not a discontinuity value. In such case there is a closed neighbourhood F_0 of λ such that $S \mid X_T(F_0)$ is continuous. We can find functions $\varphi_1, \varphi_2 \in C^\infty(R_2)$ such that $\varphi_1 + \varphi_2 \equiv 1$, $\varphi_1 \equiv 1$ in a neighbourhood of λ and supp $\varphi_1 \subset F_0$.

We shall show that there exists a closed set F for which $\lambda \notin F$ and $\sigma_S \subset X_T(F)$. To prove that, take an $x \in \sigma_S$. Let $\{x_n\}$ be a sequence such that $x_n \to 0$ and $Sx_n \to x$.

We have

$$Sx_n = S\mathscr{U}_{\varphi_1} x_n + S\mathscr{U}_{\varphi_2} x_n .$$

Since supp $\varphi_1 \subset F_0$, it follows that $\mathscr{U}_{\varphi_1} x_n \in X_T(F_0)$ and $S\mathscr{U}_{\varphi_1} x_n \to 0$ by the assumption that $S \mid X_T(F_0)$ is continuous. From this fact $S\mathscr{U}_{\varphi_2} x_n \to x$; x being a limit of elements of X_T (supp φ_2), it is an element of X_T (supp φ_2) as well.

3.4. Definition. A complex number λ is said *to be a critical eigenvalue* of T if λ is an element of the point spectrum of T and the range $R(\lambda I - T)$ is of infinite codimension, i.e. a Hamel basis in the quotient space $X/R(\lambda I - T)$ is not a finite set.

Consider now a T having a critical eigenvalue. Then there exists a discontinuous S such that $TS = ST$ and $SX_T(F) \subset X_T(F)$ for every F closed. To prove this we shall apply the example given in [4], lemma 2.1.

Let λ be a critical eigenvalue, let $y \in X$ be a corresponding eigenvector $Ty = \lambda y$. $R(\lambda I - T)$ has not a finite codimension. Using a Hamel basis in $X/R(\lambda I - T)$ we can construct a discontinuous linear functional f defined on X with the property $f(x) = 0$ for $x \in R(\lambda I - T)$. The linear transformation S defined by the formula

$$Sx = y f(x)$$

is obviously discontinuous and from the equality $(\lambda I - T) S = S(\lambda I - T) = 0$ it follows that S commutes with T.

According to the definition we have, for every x, $\sigma_T(Sx) \subset \sigma_T(y) = \{\lambda\}$. Providing that $\lambda \in \sigma_T(x)$ we have $\sigma_T(Sx) \subset \sigma_T(x)$. If $\lambda \notin \sigma_T(x)$, then there is an x_λ such that $x = (\lambda I - T) x_\lambda \in R(\lambda I - T)$ and $Sx = y \,.\, f(x) = 0$ so that $\sigma_T(Sx) = \emptyset \subset \sigma_T(x)$. We have obtained $\sigma_T(Sx) \subset \sigma_T(x)$ for every $x \in X$ and this is obviously equivalent to $SX_T(F) \subset X_T(F)$ for $F = \bar{F}$.

Now, knowing the properties of space $X_T(\{\lambda\})$ in case of generalized scalar operators, we can prove the following

3.5. Theorem. *Let T be a generalized scalar operator in a Banach space X which has no critical eigenvalue. Let S be a linear transformation such that*

1) $TS = ST$,

2) $SX_T(F) \subset X_T(F)$ for $F = \bar{F}$.

Then S is continuous.

Proof. In the preceding lemma we showed that either $\sigma_S = \{0\}$ and S is continuous or that there exists a $k \geqq 1$ and elements $\lambda_1, \ldots, \lambda_k$ of the point spectrum of T such that $\sigma_S \subset X_T(\{\lambda_1, \ldots, \lambda_k\})$. By 2.7 there exists a polynomial $P(.)$ with $P(T) \mid \sigma_S = 0$. Denote by q the quotient map from X onto X/σ_S and by $P(T)'$ the corresponding operator to $P(T)$ from X/σ_S into X both being continuous. By the closed graph theorem we see that qS is a continuous operator so that $P(T) S = P(T)' qS$ is continuous as well.

Since each $R(\lambda_i I - T)$ $(i = 1, 2, \ldots, k)$ has a finite codimension, it is easy to see that $P(T) X$ has also a finite codimension. In this case there exists a finite dimensional vector space Z such that we can find, for each $x \in X$, a unique representation $x = x_1 + x_2$ with $x_1 \in P(T)$ and $x_2 \in Z$. It is not difficult to prove that the maps

$R_1x = x_1$, $R_2x = x_2$ are continuous and the space $P(T)X$ is closed. See also [4]. Now we have

$$Sx = SR_1x + SR_2x .$$

Since $S \mid P(T)X$ and $S \mid Z$ are continuous, S is a continuous operator on the whole X as well.

We have obtained the above result by a slight modification of the methods in [5]. However, the assumption that $\{0\}$ is the only T-divisible subspace can be replaced by the assumption that all $X_T(F)$ are invariant with respect to S, which is weaker.

3.6. Definition. A subspace Y is called *T-divisible* if for every complex number λ there is $(\lambda I - T)Y = Y$.

Let Z be a subspace of X invariant with respect to T. Similarly as in [5] we denote by $\overset{\infty}{\bigcap}(\lambda \in M)(\lambda I - T)Z$ the constant value of the transfinite sequence $Z(\alpha)$ defined by

1) $Z(0) = Z$,
2) $Z(\alpha + 1) = \bigcap\limits_{\lambda \in M} (\lambda I - T) Z(\alpha)$,
3) $Z(\alpha) = \bigcap\limits_{\beta \prec \alpha} Z(\beta)$ for limit ordinals.

We can always find such transfinite sequence with eventual constant value. If we put $Z = X$ and $M = \mathbf{C}$, then $\overset{\infty}{\bigcap}(\lambda \in \mathbf{C})(\lambda I - T)X$ is the largest T-divisible subspace in X. For other properties see also [5]. It is easy to see that every $\overset{\infty}{\bigcap}(\lambda \in M)(\lambda I - T)X$ is invariant with respect to any linear transformation commuting with T. Further, $X_T(F) \subset \overset{\infty}{\bigcap}(\lambda \notin F)(\lambda I - T)X$ and particularly $x \in \overset{\infty}{\bigcap}(\lambda \notin \sigma_T(x))(\lambda I - T)X$.

3.7. Proposition. *Let T be a generalized scalar operator for which $\{0\}$ is the only T-divisible subspace.*

Then, for every closed F, the subspace $X_T(F)$ is invariant with respect to any linear transformation S such that $ST = TS$.

Proof. Take an $x \in X$ and a $\varphi \in C^\infty(R_2)$ with the properties $0 \leqq \varphi \leqq 1$ and $\varphi \equiv 1$ in a neighbourhood of $\sigma_T(x)$. We shall show that $\mathcal{U}_{1-\varphi}Sx = 0$. We have

$$x \in \overset{\infty}{\bigcap}(\lambda \notin \sigma_T(x))(\lambda I - T)X .$$

Since the subspace on the right hand side is invariant with respect to S, we obtain

$$Sx \in \overset{\infty}{\bigcap}(\lambda \notin \sigma_T(x))(\lambda I - T)X .$$

On the other hand we can easily show that

$$\mathcal{U}_{1-\varphi} \bigcap^{\infty}(\lambda \notin \sigma_T(x))\,(\lambda I - T)\,X \subset \bigcap^{\infty}(\lambda \notin \sigma_T(x))\,(\lambda I - T)\,\mathcal{U}_{1-\varphi}X\,.$$

Take a $\lambda \in \sigma_T(x)$. By 2.5 the subspace $\mathcal{U}_{1-\varphi}X \subset X_T(\operatorname{supp}(1 - \varphi))$ and thus the operator $(\lambda I - T)$ is one-to-one on $\mathcal{U}_{1-\varphi}X$. Let us show that $(\lambda I - T)$ maps $\mathcal{U}_{1-\varphi}X$ onto itself.

Let $x = \mathcal{U}_{1-\varphi}z$, then there is a $y \in X_T(\operatorname{supp}(1 - \varphi))$ such that $x = (\lambda I - T)\,y$. Put $\psi(\lambda) = 0$ and $\psi(\mu) = (1 - \varphi(\mu))/(\lambda - a(\mu))$ for $\mu \neq \lambda$, so that $\psi \in C^\infty(R_2)$. Denote $u = \mathcal{U}_\psi z - y$. Since

$$(\lambda I - T)\,u = \mathcal{U}_{1-\varphi}z - (\lambda I - T)\,y = 0\,,$$

it is $\sigma_T(u) \subset \{\lambda\}$. On the other hand $\sigma_T(y) \subset \operatorname{supp}(1 - \varphi)$, $\sigma_T(\mathcal{U}_\psi z) \subset \operatorname{supp}(\psi) = \operatorname{supp}(1 - \varphi)$ and thus $\sigma_T(u) \subset \operatorname{supp}(1 - \varphi)$. But $\sigma_T(u) \subset \operatorname{supp}(1 - \varphi) \cap \{\lambda\} = \emptyset$ and $u = 0$. We have obtained $y = \mathcal{U}_\psi z$. Let $\varphi_0 \in C^\infty(R_2)$, $\varphi_0 \equiv 1$ in a neighbourhood of $\sigma_T(y)$ such that $\operatorname{supp}\varphi_0 \cap \sigma_T(x) = \emptyset$. Then $y = \mathcal{U}_{\varphi_0}y = \mathcal{U}_{1-\varphi}\mathcal{U}_{\varphi_0/(\lambda - a)}z \in \mathcal{U}_{1-\varphi}X$.

So we can write

$$\bigcap^{\infty}(\lambda \notin \sigma_T(x))\,(\lambda I - T)\,\mathcal{U}_{1-\varphi}X = \bigcap^{\infty}(\lambda \in \mathrm{C})\,(\lambda I - T)\,\mathcal{U}_{1-\varphi}X\,.$$

Since $\{0\}$ is the only T-divisible subspace, we have $\mathcal{U}_{1-\varphi}Sx = 0$ and $Sx = \mathcal{U}_{1-\varphi}Sx + \mathcal{U}_\varphi Sx = \mathcal{U}_\varphi Sx \in X_T(\operatorname{supp}\varphi)$ for every $\varphi \in C^\infty(R_2)$ such that $\varphi \equiv 1$ in a neighbourhood of $\sigma_T(x)$. From this fact it follows obviously $\sigma_T(Sx) \subset \sigma_T(x)$.

Open problem: Is there a generalized scalar operator or a spectral operator of the finite type having a non-trivial divisible subspace? [1])

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Author's address: Praha 1, Žitná 25 (Matematický ústav ČSAV v Praze).

[1]) The problem was solved by the author and the results will be published.