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ON DECOMPOSITIONS OF GROUPOIDS

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1. Introduction. In this paper some decomposition theorems for groupoids satisfying certain identities are given. These theorems were motivated by studying examples of groupoids called point algebras and by extracting identities satisfied by them to be used in a more general setting.

We now define *point algebra.* Let S be a non-empty set, n an integer $\geqq 2$ and $j(1), j(2), \ldots, j(n)$ a sequence of (not-necessarily distinct) integers where $1 \leqq j(i) \leqq n$, $i = 1, \ldots, n$. A binary operation, $(.)$, is defined on $S^n (= S \times S \times \ldots \times S)$ as follows: $(x_1, \ldots, x_n) . (y_1, \ldots, y_n) = (1_{j(1)}, \ldots, n_{j(n)})$ for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ in S^n, where each $i_{j(i)}$ is a fixed element either equal to $x_{j(i)}$ or $y_{j(i)}$. The groupoid $(S^n, .)$ is called a *point algebra*. Except for Theorem 2.2, point algebras in this paper will be of the form $(S^3, .)$.

2. On the identity $x^2y^2 = xy$. The first theorem proved here enlarges a class of groupoids characterized as unions of disjoint constant semigroups by EVANS in [1, p. 368]. However, Evans' result provides more information regarding the relationships among the semigroups than does the theorem in this paper. Specifically, Evans proves the following theorem.

Theorem. *A groupoid satisfies the identities $xy . z = xz$, $x . yz = xy$ if and only if it is the union of disjoint constant semigroups C_α where all products xy, with $x \in C_\alpha$, $y \in C_\beta$ are equal and belong to C_α.*

First we prove that the class of groupoids satisfying the identity $x^2y^2 = xy$ properly contains the class satisfying the two identities given in the theorem above. For suppose a groupoid satisfies the identities $xy . z = xz$ and $x . yz = xy$. Then $x^2y^2 = xy^2 = xy$. Now let S be any set such that $|S| > 1$ and consider the following multiplication on S^3. If $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S^3$ then $(x_1, x_2, x_3) . (y_1, y_2, y_3) = (x_3, x_2, y_3)$. It is easily shown that the point algebra $(S^3, .)$ satisfies the identity

$x^2y^2 = xy$ but, in fact, satisfies neither of the identities, $xy \,.\, z = xz$ or $x \,.\, yz = xy$. The following characterization is now given.

Theorem 2.1. *A groupoid G satisfies the identity $x^2y^2 = xy$ if and only if $G = \bigcup_i G_i$, where the G_i are disjoint constant semigroups such that if $x, w \in G_i$, $y, z \in G_j$ then $xy = wz$.*

Proof. The "if" part is obvious. To prove the "only if" part suppose G satisfies the identity $x^2y^2 = xy$. Define the relation ϱ on G by $x\varrho y$ if and only if $x^2 = y^2$. Thus ϱ is clearly an equivalence relation and indeed a congruence relation. For suppose $x\varrho y$ and $w\varrho z$. Then $x^2 = y^2$ and $w^2 = z^2$. Hence $x^2w^2 = y^2z^2$ and $xw = yz$. Let G_u be the equivalence class containing u and let $x, y \in G_u$. Then $x^2 = y^2$. Thus $xy = x^2y^2 = x^2x^2 = x^2$ and so $(xy)^2 = (x^2)^2 = x^2$. Therefore $xy\varrho x$. Hence G_u is a subgroupoid of G. The fact that G_u is a constant semigroup follows from an equation above, which states that $xw = yz$, if $x, y \in G_i$ and $w, z \in G_j$, if $i = j$.

The point algebra G which motivated Theorem 2.1 was the one mentioned previously, namely the one given by the operation $(x_1, x_2, x_3) \,.\, (y_1, y_2, y_3) = (x_3, x_2, y_3)$. G is nonassociative and the ϱ-equivalence classes, described in the proof above, for this example are the sets, $G_{ab} = \{(x, a, b) \mid x, a, b \in S,\ a, b, \text{ fixed}\}$.

It is false in general that if a groupoid G is a union of disjoint constant semigroups that G satisfies the identity $x^2y^2 = xy$. As an example, we note in the groupoid given by the table:

.	a	b	c
a	a	a	a
b	a	a	b
c	a	a	c

that $b^2c^2 \neq bc$. However the following result holds.

Theorem 2.2. *Let $G = (S^n, .)$ be a point algebra such that $G = \bigcup_i G_i$, where the G_i are disjoint constant semigroups. Then G satisfies the identity $x^2y^2 = xy$.*

Proof. As $G = (S^n, .)$ is a point algebra, the binary operation $(.)$ is defined by:

$$(1) \qquad (x_1, \ldots, x_n) \,.\, (y_1, \ldots, y_n) = (1_{j(1)}, \ldots, n_{j(n)})\,.$$

We prove that G satisfies the identities $x^2z = xz$ and $zx^2 = zx$. These together imply $x^2y^2 = xy$. Let $x, z \in G$ where $x = (a_1, \ldots, a_n) \in G_r$ and $z = (z_1, \ldots, z_n)$. Let $k = (k_1, \ldots, k_n) \in G_r$ such that for all $s, t \in G_r$, $st = k$. In particular,

$$(2) \qquad x^2 = (a_1, \ldots, a_n)(a_1, \ldots, a_n) = (k_1, \ldots, k_n) = (k_1, \ldots, k_n)(k_1, \ldots, k_n)\,.$$

We now show that $x^2z = xz$, that is, that

$$(3) \qquad (k_1, \ldots, k_n)(z_1, \ldots, z_n) = (a_1, \ldots, a_n)(z_1, \ldots, z_n).$$

We denote the left side of (3) by $(c_1, \ldots, c_n)$ and the right side by $(d_1, \ldots, d_n)$. Thus we show that $c_u = d_u$, $u = 1, \ldots, n$. Clearly, from (1), for each i such that $i_{j(i)} = y_{j(i)}$, $c_i = z_{j(i)} = d_i$. Now suppose for some i, $i_{j(i)} = x_{j(i)}$ in (1). Then from (3) $c_i = k_{j(i)}$ and $d_i = a_{j(i)}$. From (2), considering the product $(k_1, \ldots, k_n)(k_1, \ldots, k_n) = (k_1, \ldots, k_n)$, we have $k_i = k_{j(i)}$. However, from the product $(a_1, \ldots, a_n)(a_1, \ldots, a_n) = (k_1, \ldots, k_n)$ in (2), $k_i = a_{j(i)}$. Hence $a_{j(i)} = k_{j(i)}$, i.e., $c_i = d_i$. Thus $x^2z = xz$. Similarly, $zx^2 = zx$ and hence for $x, y \in G$, $x^2y^2 = xy$.

3. On the medial law, $xy \,.\, wz = xw \,.\, yz$. A medial (entropic) groupoid is one which satisfies the identity $xy \,.\, wz = xw \,.\, yz$. In the following theorems we prove that groupoids satisfying certain identities are unions of disjoint medial subsemigroups. We actually prove slightly stronger results which imply mediality. The following lemmas will be useful.

Lemma 3.1. *Let G be a groupoid such that $G = \bigcup_i G_i$, where the G_i are disjoint subgroupoids such that for all $t \in G$ and all i, if $x, y \in G_i$ then $tx = ty$. Then if $x \in G$, $v, w, y, z \in G_i$, then $xy \,.\, z = xw \,.\, v$. Furthermore, each G_i satisfies the medial law.*

Proof. $xy \,.\, z = xw \,.\, z = xw \,.\, v$. The fact that each G_i satisfies the medial law is immediate.

Lemma 3.2. *Let G be a groupoid such that $G = \bigcup_i G_i$, where the G_i are disjoint subgroupoids such that for all $t \in G$ and all i, if $x, y \in G_i$, then $xt = yt$. Then if $z \in G$, $x, w, u, y \in G_i$, then $x \,.\, yz = w \,.\, uz$. Furthermore, each G_i satisfies the medial law.*

Proof. $x \,.\, yz = w \,.\, yz = w \,.\, uz$. The fact that each G_i satisfies the medial law is immediate.

Theorem 3.1. *If a groupoid G satisfies the identities $x \,.\, yz = xz$ and $x^2y = xy$ then $G = \bigcup_i G_i$, where the G_i are disjoint subsemigroups of G such that if $x \in G_i$, $w, y \in G_j$ then $xw = xy \in G_j$. Furthermore, if $x \in G$ and $v, w, y, z \in G_i$ then $xy \,.\, z = xw \,.\, v$ and G_i is medial.*

Proof. Define the relation ϱ on G as follows: $x\varrho y$ if and only if $tx = ty$ for all $t \in G$. It is clear that ϱ is an equivalence relation and indeed a congruence relation. For if $x\varrho y$ and $w\varrho z$ then $t \,.\, xw = tw = tz = t \,.\, yz$. Thus $xw\varrho yz$. If $x\varrho y$ then $t \,.\, xy = ty$ and hence $xy\varrho y$. Thus each equivalence class G_i is a subgroupoid of G. If

$x, y, z \in G_i$, i.e., $x\varrho y\varrho z$, then $xy \,.\, z = x^2z = xz = x \,.\, yz$. Hence each G_i is a subsemigroup of G. If $x \in G_i$, $w, y \in G_j$ then $xw = xy\varrho y$, i.e., $xy \in G_j$. The remainder of the theorem follows from Lemma 3.1.

A point algebra which motivated the above theorem is the following. Let $|S| > 1$ and define the following binary operation on S^3. For $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S^3$, $(x_1, x_2, x_3) \,.\, (y_1, y_2, y_3) = (x_1, y_2, x_2)$. Then $G = (S^3, .)$ is non-associative and satisfies $x \,.\, yz = xz$ and $x^2y = xy$. The ϱ-equivalence classes are the $G_a = \{(x, a, y) \mid a, x, y \in S, a, \text{fixed}\}$.

Theorem 3.2. *If a groupoid G satisfies the identities $xy \,.\, z = xz$ and $xy^2 = xy$ then $G = \bigcup_i G_i$, where the G_i are disjoint subsemigroups of G such that if $x, y \in G_i$, $z \in G_j$ then $xz = yz \in G_i$. Furthermore if $z \in G$ and $x, u, w, y \in G_i$ then $x \,.\, yz = w \,.\, uz$ and G_i is medial.*

Proof. By defining a relation ϱ on G by $x\varrho y$ if and only if $xt = yt$, for all $t \in G$, one can show that ϱ is a congruence relation on G and in a proof similar to that of Theorem 3.1 and by applying Lemma 3.2 the proof of this theorem will follow.

A motivating example for the above theorem is the point algebra defined as follows. Let $|S| > 1$ and define on S^3 the product $(x_1, x_2, x_3) \,.\, (y_1, y_2, y_3) = (x_1, y_1, y_3)$ Then $G = (S^3, .)$ is non-associative and satisfies $xy \,.\, z = xz$ and $xy^2 = xy$. The ϱ-equivalence classes defined in the proof above are the $G_a = \{(a, x, y) \mid a, x, y \in S, a, \text{fixed}\}$.

Theorem 3.3. *If a groupoid G satisfies the identities $x \,.\, yz = xy$ and $x^2y = xy$ then $G = \bigcup_i G_i$, where the G_i are disjoint subsemigroups of G such that if $x \in G_i$, $y, z \in G_j$ then $xy = xz \in G_i$. Furthermore, if $x \in G$ and $v, w, y, z \in G_i$ then $xy \,.\, z = xw \,.\, v$ and G_i is medial.*

Proof. Define a relation ϱ on G by $x\varrho y$ if and only if $tx = ty$, for all $t \in G$, and proceed with a proof similar to that of Theorem 3.1.

A motivating example for Theorem 3.3 is the following. Let $|S| > 1$ and define the product $(x_1, x_2, x_3) \,.\, (y_1, y_2, y_3) = (x_1, x_2, y_1)$ on S^3. Then $G = (S^3, .)$ is non-associative and satisfies $x \,.\, yz = xy$ and $x^2y = xy$. The ϱ-equivalence classes from the proof of Theorem 3.3 are the $G_a = \{(a, x, y) \mid a, x, y \in S, a, \text{fixed}\}$.

Theorem 3.4. *If a groupoid G satisfies the identities $xy \,.\, z = yz$ and $xy^2 = xy$ then $G = \bigcup_i G_i$, where the G_i are disjoint subsemigroups of G such that if $x, w \in G_i$, $y \in G_j$ then $xy = wy \in G_j$. Furthermore if $z \in G$ and $x, u, w, y \in G_i$ then $x \,.\, yz = w \,.\, uz$ and G_i is medial.*

Proof. Define a relation ϱ on G by $x\varrho y$ if and only if $xt = yt$, for all $t \in G$, and proceed with a proof similar to that of Theorem 3.2.

A motivating example for Theorem 3.4 is the following point algebra. Let $|S| > 1$ and define the product $(x_1, x_2, x_3) \, . \, (y_1, y_2, y_3) = (x_2, y_2, y_3)$ on S^3. Then $G = (S^3, \, .)$ is non-associative and satisfies $xy \, . \, z = yz$ and $xy^2 = xy$. The ϱ-equivalence classes defined in the proof above are the $G_a = \{(x, a, y) \mid a, x, y \in S, \, a, \text{ fixed}\}$.

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