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Časopis pro pěstování matematiky, roč. 96 (1971), Praha

AN INEQUALITY FOR UNIVALENT FUNCTIONS DUE TO DVOŘÁK[1])

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1. In a recent note DVOŘÁK established the following result [1].

Theorem A. *Let* $f(z) = z + a_2 z^2 + \ldots$ *be analytic and univalent in the unit disc D. Then* $f(z)$ *satisfies the inequality*

$$\text{Re}\,\sqrt{(f(z)/z)} > \tfrac{1}{2} \tag{1}$$

for $|z| < r_0'$ *where* r_0' *is the smallest positive root of the equation*

$$r \log \frac{1 + r}{1 - r} = 2\,.$$

A computation shows

$$r_0' = 0{\cdot}83355\ldots \tag{2}$$

In this note we obtain the exact value of r_0'.

Theorem B. *Let* $f(z) = z + a_2 z^2 + \ldots$ *be analytic and univalent in the unit disc D. Then* $f(z)$ *satisfies* (1) *for* $|z| < r_0$ *where* r_0 *is the smallest positive root of the equation*

$$\left[S^{-1}\left(\tfrac{1}{2}\log\frac{1+r}{1-r}\right)\right]^2 + \left[E^{-1}\left(\frac{\sqrt{(1-r^2)}}{4}\log\frac{1+r}{1-r}\right)\right]^2 = \left[\tfrac{1}{2}\log\frac{1+r}{1-r}\right]^2, \tag{3}$$

where $S^{-1}(x)$ *and* $E^{-1}(x)$ *are the inverse of* $S(x) = [x/\sin x]$ *and* $E(x) = xe^{-x}$ *respectively. This result is sharp. A computation shows*

$$r_0 = 0{\cdot}83559\ldots \tag{4}$$

Proof. It is easy to see that the condition (1) is equivalent to the inequality

$$\left|\sqrt{(z/f(z))} - 1\right| < 1\,. \tag{5}$$

Now GRUNSKY has shown that for normalized univalent functions in the unit disc we must have the sharp inequality

$$\left|\log\left(f(z)/z\right) + \log\left(1 - |z|^2\right)\right| \leqq \log\frac{1+|z|}{1-|z|} \tag{6}$$

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for all z in D [3; p. 113]. From (6) we obtain

$$\left|\log\sqrt{(z/f(z))} - \tfrac{1}{2}\log(1 - |z|^2)\right| \leqq \tfrac{1}{2}\log\frac{1+|z|}{1-|z|}. \tag{7}$$

We now set $w = \log\sqrt{(z/f(z))}$, $A = \frac{1}{2}\log(1 - |z|^2)$, $B = \frac{1}{2}\log[(1 + |z|)/(1 - |z|)]$ in (5) and (7) to obtain

$$|e^w - 1| < 1 \tag{8}$$

and

$$|w - A| < B, \tag{9}$$

respectively.

We are now going to show how A and B must be related in order that the inequality (8) should hold subject to the condition (9). We set $W = e^w = Re^{i\Theta}$ in (8) and (9) to obtain

$$R < 2\cos\Theta \tag{10}$$

and

$$(\log R - A)^2 + \Theta^2 < B^2, \tag{11}$$

respectively. The relations (10) and (11) define domains in the $W-$ plane that correspond to the domains defined by (8) and (9) in the $w-$ plane. If $|z| = r$ is small, it is clear that the domain (11) lies in the domain (10). As $|z| = r$ increases, the boundary of (11) eventually makes contact with that of (10) *before* r reaches 1.

Let us consider this first point of contact. At such a point we must have

$$\log R = \log(2\cos\Theta) = A + \sqrt{(B^2 - \Theta^2)} \tag{12}$$

and

$$\frac{dR}{d\Theta} = -2\sin\Theta = \frac{-\Theta}{\sqrt{(B^2 - \Theta^2)}}\, e^{A+\sqrt{(B^2-\Theta^2)}}. \tag{13}$$

If we eliminate Θ from (12) and (13), then we obtain

$$\tfrac{1}{2}Be^A = \sqrt{(B^2 - \Theta^2)}\, e^{-\sqrt{(B^2-\Theta^2)}}. \tag{14}$$

Now (13) and (14) yield

$$\frac{\Theta}{\sin\Theta} = B. \tag{15}$$

If we let $E^{-1}(x)$ and $S^{-1}(x)$ denote the inverse of $E(x) = xe^{-x}$ and $S(x) = x/\sin x$, respectively, then (14) and (15) yield

$$[E^{-1}(\tfrac{1}{2}Be^{A})]^2 + [S^{-1}(B)]^2 = B^2 ,$$

from which we obtain (3).

This result is sharp because the relations (6) and (7) are sharp. This completes our proof.

We note that our result (4) is at variance with another recent result due to Dvořák [2; p. 180].

2. Dvořák also obtained the following result [2; p. 187].

Theorem C. *If $g(z) = z + a_3z^3 + \ldots$ is an analytic univalent odd function in the unit disc D, then*

$$Re(g(z)/z) > \tfrac{1}{2} \tag{16}$$

holds for $|z| < r_1'$, where r_1' is the smallest positive root of the equation

$$\sqrt{(r)} \log \frac{1 + \sqrt{r}}{1 - \sqrt{r}} = 2 .$$

A computation shows that

$$r_1' = 0{\cdot}913 \ldots$$

We obtain the following *sharp* result.

Theorem D. *Let $g(z) = z + a_3z^3 + \ldots$ be analytic, univalent and odd in the unit disc D. Then the inequality (16) holds for $|z| < r_1$, where r_1 is the smallest positive root of the equation*

$$\left[S^{-1}\left(\tfrac{1}{2}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right]^2 + \left[E^{-1}\left(\tfrac{1}{2}\sqrt{(1-r)}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right]^2 = \left(\tfrac{1}{2}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)^2 .$$

This result is sharp. Moreover, a computation shows that

$$r_1 = 0{\cdot}914 \ldots$$

Proof. If we get $f(z^2) = [g(z)]^2$, then $f(z)$ is analytic and univalent in the unit disc D. We then apply Theorem B. This completes the proof.

References

[1] *Dvořák*, Časopis pro pěstování matematiky, *63* (1934), 9—16 (Czech).

[2] *Dvořák*, „Über schlichte Funktionen, I", Časopis pro pěstování matematiky, *92* (1967), 162—189.

[3] *Golusin*, „Geometrische Funktionentheorie", Berlin, 1957.

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