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CONTRIBUTION TO THE AFFINE DEFORMATION OF SURFACES

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The analytic conditions for a correspondence between two surfaces sustaining conjugate net in an unimodular affine 4-dimensional space to be a deformation of second order were derived in the paper [3]. In the following a geometrical characterization of these correspondences – partly in connection with the deformation of the systems of tangent planes, partly in connection with the deformation of Laplace congruences being determined by the surfaces above mentioned – will be given.

1.

1.1. Let (A) be a surface immersed in a 4-dimensional unimodular affine space A_4 and generated by the point $A = A(u, v)$. The admissible couples (u, v) are taken from an open neighborhood of C^2 (C = complex numbers). Let (A) be a surface sustaining conjugate net. To each point of the surface we associate a frame consisting of the point A and linearly independent vectors I_1, I_2, I_3, I_4 such that

$$[I_1 I_2 I_3 I_4] = 1 . \tag{1.1}$$

The fundamental equations of the moving frame are

$$\mathrm{d}A = \sum_{j=1}^{4} \omega_j I_j , \quad \mathrm{d}I_j = \sum_{k=1}^{4} \omega_{jk} I_k \quad (j = 1, 2, 3, 4) , \tag{1.2}$$

the forms ω fulfilling the structure equations of an affine space

$$\mathrm{d}\omega_j = \sum_{k=1}^{4} \omega_k \wedge \omega_{kj} , \quad \mathrm{d}\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} \quad (i, j = 1, 2, 3, 4) . \tag{1.3}$$

Taking a suitable specialization of the frame (see [3], equations (1.14)), the equations

(1.2) are of the form

$$
\begin{aligned}
(1.4) \qquad dA &= \omega_1 I_1 + \omega_2 I_2 , \\
dI_1 &= \omega_{11} I_1 + \alpha_1 \omega_2 I_2 + \omega_1 I_3 , \\
dI_2 &= \alpha_2 \omega_1 I_1 + \omega_{22} I_2 + \omega_2 I_4 , \\
dI_3 &= \omega_{31} I_1 + \omega_{32} I_2 + \omega_{33} I_3 + \beta_2 \omega_1 I_4 , \\
dI_4 &= \omega_{41} I_1 + \omega_{42} I_2 + \beta_1 \omega_2 I_3 + \omega_{44} I_4 .
\end{aligned}
$$

Moreover, the following relations hold

$$
\begin{aligned}
(1.5) \qquad \omega_{11} + \omega_{22} + \omega_{33} + \omega_{44} &= 0 , \\
2\omega_{11} - \omega_{33} &= \alpha_2 \omega_2 , \\
2\omega_{22} - \omega_{44} &= \alpha_1 \omega_1 .
\end{aligned}
$$

$$
(1.6) \qquad \omega_1 \wedge \omega_2 \neq 0 , \quad \alpha_1 \alpha_2 \beta_1 \beta_2 \neq 0 .
$$

1.2. Let us consider a surface (B) immersed in a 4-dimensional affine space A'_4 and generated by the point $B = B(u', v')$. Let us take the same suppositions on (B) as those made on (A). Let the frame of (B) be consisted of the point B and the vectors J_1, J_2, J_3, J_4 such that

$$
(1.1') \qquad [J_1 J_2 J_3 J_4] = 1 .
$$

We denote all expressions connected with (B) by an apostroph. As the frame associated with (B) is specialized in the same way as that associated with (A), the fundamental system of differential equations is of the form

$$
\begin{aligned}
(1.4') \qquad dB &= \omega'_1 J_1 + \omega'_2 J_2 , \\
dJ_1 &= \omega'_{11} J_1 + \alpha'_1 \omega'_2 J_2 + \omega'_1 J_3 , \\
dJ_2 &= \alpha'_2 \omega'_1 J_1 + \omega'_{22} J_2 + \omega'_2 J_4 , \\
dJ_3 &= \omega'_{31} J_1 + \omega'_{32} J_2 + \omega'_{33} J_3 + \beta'_2 \omega'_1 J_4 , \\
dJ_4 &= \omega'_{41} J_1 + \omega'_{42} J_2 + \beta'_1 \omega'_2 J_3 + \omega'_{44} J_4 ,
\end{aligned}
$$

where

$$
\begin{aligned}
(1.5') \qquad \omega'_{11} + \omega'_{22} + \omega'_{33} + \omega'_{44} &= 0 , \\
2\omega'_{11} - \omega'_{33} &= \alpha'_2 \omega'_2 , \\
2\omega'_{22} - \omega'_{44} &= \alpha'_1 \omega'_1 .
\end{aligned}
$$

The correspondence C between the surfaces (A), (B) is determined by the relations

$$\omega_1' = \lambda_{11}\omega_1 + \lambda_{12}\omega_2\,, \quad \omega_2' = \lambda_{21}\omega_1 + \lambda_{22}\omega_2 \tag{1.7}$$

$$\begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} \neq 0$$

so that the points A, $B = CA$ with equal parameters (u, v) are corresponding one to the other. In the next, the above mentioned correspondence will be marked by $C : (A) \to (B)$.

The correspondence $C : (A) \to (B)$ is called *conjugate* in case it is given by relations

$$\omega_1' = \lambda_1\omega_1\,, \quad \omega_2' = \lambda_2\omega_2\,, \quad \lambda_1\lambda_2 \neq 0\,. \tag{1.8}$$

We shall use the specification

$$\tau_{ij} = \omega_{ij}' - \omega_{ij}\,, \quad \omega_{ij}(\delta) = e_{ij}\,, \quad t_{ij} = e_{ij}' - e_{ij}\,,$$

where δ is such a symbol of differentiation that $\delta u = \delta v = 0$. We derive from (1.8) (see [3], eq. (1.20))

$$\delta\lambda_1 = -\lambda_1 t_{11}\,, \quad \delta\lambda_2 = -\lambda_2 t_{22}\,, \quad \delta(\lambda_1\lambda_2) = 0\,. \tag{1.9}$$

1.3. The surface (A) will be considered as the surface of the projective space P_4 arising from the space A_4 by its projective extension. Then each vector in A_4 is an improper point of this space. These points generate a 3-dimensional improper space N_3 of the affine space A_4. Without danger of misunderstanding, we shall speak about the points I_1, I_2 and so on meaning the improper points determined by the vectors above mentioned. We shall do a similar supposition concerning the surface (B).

The tangent plane $[AI_1I_2]$ of the surface (A) at the point A meets the improper space N_3 in the straight line $[I_1I_2]$. When moving the point A on the surface (A), the straight line $[I_1I_2]$ generates the line congruence L. In accordance with our suppositions the congruence L has two different focal surfaces. Similarly L' is the marking of the line congruence $[J_1J_2]$.

Suppose, $C : (A) \to (B)$ to be the correspondence between the surfaces (A), (B). Now, the corespondence $\gamma : L \to L'$ is determined by a natural way so that improper lines of tangent planes at the points A, $B = CA$ of the surfaces (A), (B) are corresponding. Especially, $C : (A) \to (B)$ being conjugate then $\gamma : L \to L'$ is developable. (See [2], concerning developable correspondences.)

Finally, the straight lines $[AI_1]$, $[AI_2]$ generate two congruences having one common focal surface (A) and one common focal plane in the tangent plane $[AI_1I_2]$ of the surface (A) at the point A which is the common focus of the rays of the congruences above mentioned. We shall mark by L_1 and L_2 the congruences $[AI_1]$ or $[AI_2]$ respectively. In accordance with our suppositions both the congruences L_1, L_2 are non-parabolic with character $m = 3$. (See [1], p. 12.)

Similar consideration can be held concerning the surface (B). We denote by L_1' or L_2' the line congruences $[BJ_1]$ or $[BJ_2]$ respectively. Let $C:(A)\to(B)$ be the correspondence between the surfaces (A), (B). Now, the correspondence $C_1 : L_1 \to \to L_1'(C_2 : L_2 \to L_2')$ is determined so that the rays $[AI_1]$, $[BJ_1]$ ($[AI_2]$, $[BJ_2]$) of the congruences L_1, L_1' (L_2, L_2') passing through the points A, $B = CA$ are corresponding.

2.

In this paragraph, we shall deal with the deformation of systems of tangent planes.

2.1. Let (A) be a surface immersed into an unimodular affine 4-dimensional space A_4 determined together with the system of frames $\{A, I_1, I_2, I_3, I_4\}$ by differential equations (1.4). Similarly the surface (B) is determined by the equations (1.4'). Let $C:(A)\to(B)$ be the correspondence so that the point $B = CA$ of the surface (B) corresponds to the point A of the surface (A). By means of C the correspondence between the tangent planes $[AI_1I_2]$, $[BJ_1J_2]$ at the corresponding points A, $B = CA$ is determined in a natural way.

The correspondence $C:(A)\to(B)$ is called *an affine deformation of the system of tangent planes (briefly t-deformation) of order k*, if for each point A of the surface (A) there exists an affinity $T: A_4 \to A_4'$ such that the structures $\{T[AI_1I_2]\}$, $\{[BJ_1J_2]\}$ have an analytic contact of order k. We say that T realizes the affine t-deformation C.

2.2. At first we shall consider the case $k = 1$. The conditions for the correspondence C to be an affine t-deformation of first order are consisting in the existence of an affinity T so that it holds

$$(2.1)\qquad \begin{aligned} T[AI_1I_2] &= \sigma[BJ_1J_2]\,,\\ T\,\mathrm{d}[AI_1I_2] &= \sigma\,\mathrm{d}[BJ_1J_2] + \vartheta[BJ_1J_2]\,,\end{aligned}$$

where $\sigma \neq 0$ and ϑ is a convenable Pfaff's form.

With regard to the first equation (2.1) we can suppose, the affinity T to be given by equations

$$(2.2)\qquad \begin{aligned} TA &= B + a_1J_1 + a_2J_2\,,\\ TI_1 &= a_{11}J_1 + a_{12}J_2\,,\\ TI_2 &= a_{21}J_1 + a_{22}J_2\,,\\ TI_\mu &= \sum_{\nu=1}^{4} a_{\mu\nu}J_\nu\,,\quad \mu = 3, 4\,.\end{aligned}$$

Moreover, it holds

$$(2.3)\qquad a_{11}a_{22} - a_{12}a_{21} = \sigma\,,\quad a_{33}a_{44} - a_{34}a_{43} = \sigma^{-1}\,.$$

Making use of equations (1.4), we compute

(2.4) $$d[AI_1I_2] = (\omega_{11} + \omega_{22})[AI_1I_2] - \omega_1[AI_2I_3] + \omega_2[AI_1I_4]$$

and similarly

(2.4') $$d[BJ_1J_2] = (\omega'_{11} + \omega'_{22})[BJ_1J_2] - \omega'_1[BJ_2J_3] + \omega'_2[BJ_1J_4].$$

By means of affinity (2.2), using the equations (2.4), (2.4'), we get

$$Td[AI_1I_2] - \sigma\, d[BJ_1J_2] - \vartheta[BJ_1J_2] = U_1[BJ_1J_2] + U_2[BJ_3J_1] + U_3[BJ_3J_2] + U_4[BJ_1J_4] + U_5[BJ_4J_2] + U_6[J_1J_2J_3] + U_7[J_1J_2J_4],$$

where we denote

(2.5) $$\begin{aligned} U_1 &= (a_{22}a_{31} - a_{21}a_{32})\,\omega_1 + (a_{11}a_{42} - a_{12}a_{41})\,\omega_2 - \sigma(\tau_{11} + \tau_{22}) - \vartheta,\\ U_2 &= a_{33}a_{21}\omega_1 - a_{11}a_{43}\omega_2,\\ U_3 &= a_{22}a_{33}\omega_1 - a_{12}a_{43}\omega_2 - \sigma\omega'_1,\\ U_4 &= -a_{21}a_{34}\omega_1 + a_{11}a_{44}\omega_2 - \sigma\omega'_2,\\ U_5 &= a_{34}a_{22}\omega_1 - a_{12}a_{44}\omega_2,\\ U_6 &= a_{33}(a_2a_{21} - a_1a_{22})\,\omega_1 + a_{43}(a_1a_{12} - a_2a_{11})\,\omega_2,\\ U_7 &= a_{34}(a_2a_{21} - a_1a_{22})\,\omega_1 + a_{44}(a_1a_{12} - a_2a_{11})\,\omega_2. \end{aligned}$$

With respect to the second condition (2.1) by comparison of the expressions (2.5) with zero and by using of (2.3), we obtain ($a_{11} \neq 0$)

(2.6) $$a_{43} = a_{21} = a_{12} = a_{34} = a_1 = a_2 = 0,$$

(2.7) $$\omega'_1 = \lambda_1\omega_1, \quad \omega'_2 = \lambda_2\omega_2,$$

where it is

(2.8) $$\lambda_1 = a_{22}a_{33}\sigma^{-1}, \quad \lambda_2 = a_{11}a_{44}\sigma^{-1}.$$

Further, it holds

(2.9) $$\lambda_1\lambda_2 = \sigma^{-2}.$$

From the equations (2.8), (2.3), we get

(2.10) $$a_{22} = \frac{\sigma}{a_{11}}, \quad a_{33} = \lambda_1 a_{11}, \quad a_{44} = \frac{\lambda_2\sigma}{a_{11}}.$$

Further, we have

(2.11) $$\vartheta = -\sigma(\tau_{11} + \tau_{22}) + \frac{a_{31}}{a_{11}}\sigma\omega_1 + a_{11}a_{42}\omega_2.$$

Thus the following lemma can be formulated.

Lemma 1. *The correspondence $C:(A)\to(B)$ is an affine t-deformation of the first order if and only if it is conjugate. The most general affinity $T: A_4 \to A'_4$ realizing this t-deformation of the first order exists and it is of the form*

$$TA = B \tag{2.12}$$

$$M = \begin{Vmatrix} a_{11} & 0 & 0 & 0 \\ 0 & \dfrac{\sigma}{a_{11}} & 0 & 0 \\ a_{31} & a_{32} & \lambda_1 a_{11} & 0 \\ a_{41} & a_{42} & 0 & \dfrac{\lambda_2\sigma}{a_{11}} \end{Vmatrix},$$

where M means the matrix of the coefficients a_{ij} in (2.2) and σ is determined by (2.9).

According to the Proposition 3. in [1], p. 18, we have

Lemma 2. *Let $C:(A)\to(B)$ be an affine t-deformation of the first order. Then the correspondence $\gamma: L\to L'$ is a projective deformation of the first order.*

2.3. We next consider the case $k = 2$. The necessary conditions for $C:(A)\to(B)$ to be an affine t-deformation of second order are to be computed at first.

Let $C:(A)\to(B)$ be t-deformation of second order. Then the affinity T exists so that it holds

$$\begin{aligned} T[AI_1I_2] &= \sigma[BJ_1J_2], \\ T\mathrm{d}[AI_1I_2] &= \sigma\,\mathrm{d}[BJ_1J_2] + \vartheta[BJ_1J_2], \\ T\mathrm{d}^2[AI_1I_2] &= \sigma\,\mathrm{d}^2[BJ_1J_2] + 2\vartheta\,\mathrm{d}[BJ_1J_2] + \vartheta_1[BJ_1J_2]. \end{aligned} \tag{2.13}$$

With respect to the results obtained in the previous section we can suppose that the affinity T is of the form (2.12). Moreover, it holds (2.7), (2.9), (2.11). Making use of equations (1.4), we get from (2.4)

$$\begin{aligned} \mathrm{d}^2[AI_1I_2] = {} & V_{12}[AI_1I_2] + V_{32}[AI_3I_2] + V_{14}[AI_1I_4] + \omega_1^2[I_1I_3I_2] + \\ & + \omega_2^2[I_2I_1I_4] + 2\omega_1\omega_2[AI_3I_4] + (\beta_2\omega_1^2 - \alpha_1\omega_2^2)[AI_4I_2] + \\ & + (\alpha_2\omega_1^2 - \beta_1\omega_2^2)[AI_3I_1]. \end{aligned} \tag{2.14}$$

Analogous expression (2.14′) not being written would be obtained for $\mathrm{d}^2[BJ_1J_2]$. Further, we denote

$$\begin{aligned} V_{12} &= \mathrm{d}(\omega_{11} + \omega_{22}) + (\omega_{11} + \omega_{22})^2 + \omega_1\omega_{31} + \omega_2\omega_{42}, \\ V_{32} &= \mathrm{d}\omega_1 + \omega_1(\omega_{11} + 2\omega_{22} + \omega_{33}), \\ V_{14} &= \mathrm{d}\omega_2 + \omega_2(\omega_{22} + 2\omega_{11} + \omega_{44}). \end{aligned} \tag{2.15}$$

Now we have

$$(2.16)\qquad T\,\mathrm{d}^2[AI_1I_2] - \sigma\,\mathrm{d}^2[BJ_1J_2] - 2\vartheta\,\mathrm{d}[BJ_1J_2] - \vartheta_1[BJ_1J_2] =$$
$$= \Phi_{12}[BJ_1J_2] + \Phi_{32}[BJ_3J_2] + \Phi_{14}[BJ_1J_4] + \Phi_{42}[BJ_4J_2] +$$
$$+ \Phi_{31}[BJ_3J_1] + (a_{11} - \lambda_1)\,\sigma\lambda_1\omega_1^2[J_1J_3J_2] +$$
$$+ \left(\frac{\sigma}{a_{11}} - \lambda_2\right)\sigma\lambda_2\omega_2^2[J_2J_1J_4]\,,$$

where

$$(2.17)\,\Phi_{12} = \sigma(V_{12} - V'_{12}) + a_{31}\frac{\sigma}{a_{11}}V_{32} + a_{11}a_{42}V_{14} + a_{41}(\beta_2\omega_1^2 - \alpha_1\omega_2^2)\frac{\sigma}{a_{11}} -$$
$$- a_{32}a_{11}(\alpha_2\omega_1^2 - \beta_1\omega_2^2) + 2(a_{31}a_{42} - a_{32}a_{41})\,\omega_1\omega_2 - 2(\omega'_{11} + \omega'_{22})\,\vartheta - \vartheta_1\,,$$
$$\Phi_{32} = \sigma(\lambda_1V_{32} - V'_{32}) + 2a_{11}a_{42}\lambda_1\omega_1\omega_2 - 2\vartheta\omega'_1\,,$$
$$\Phi_{14} = \sigma(\lambda_2V_{14} - V'_{14}) + 2\lambda_2a_{31}\frac{\sigma}{a_{11}}\omega_1\omega_2 - 2\vartheta\omega'_2\,,$$
$$\Phi_{42} = (\beta_2\omega_1^2 - \alpha_1\omega_2^2)\,\lambda_2\left(\frac{\sigma}{a_{11}}\right)^2 - \sigma(\beta'_2\omega_1'^2 - \alpha'_1\omega_2'^2) - 2a_{32}\lambda_2\frac{\sigma}{a_{11}}\omega_1\omega_2\,,$$
$$\Phi_{31} = (\alpha_2\omega_1^2 - \beta_1\omega_2^2)\,\lambda_1a_{11}^2 - \sigma(\alpha'_2\omega_1'^2 - \beta'_1\omega_2'^2) + 2\lambda_1a_{11}a_{41}\omega_1\omega_2\,.$$

Comparing the coefficients on the right-hand side of (2.16) with zero we obtain the necessary conditions for $C:(A)\to(B)$ to be an affine t-deformation of second order. These conditions are sufficient, too.

There is

$$(2.18)\qquad a_{11} = \lambda_1\,,\quad a_{31} = a_{32} = a_{41} = a_{42} = 0\,,\quad \sigma = \lambda_1\lambda_2\,,$$

$$(2.19)\qquad \beta_2\lambda_2^2 = \beta'_2\lambda_1^3\,,\quad \alpha_1 = \lambda_1\alpha'_1\,,\quad \alpha_2 = \lambda_2\alpha'_2\,,\quad \lambda_1^2\beta_1 = \lambda_2^3\beta'_1\,.$$

With respect to the last equation (2.18) it results from (2.9)

$$\sigma^3 = 1$$

so that

$$(2.20)\qquad \sigma = 1$$

can be chosen. Then it is also

$$(2.21)\qquad \lambda_1\lambda_2 = 1$$

and moreover

$$\vartheta = 0$$

$$\vartheta_1 + (\lambda_1\omega'_{31} - \omega_{31})\,\omega_1 + (\lambda_2\omega'_{42} - \omega_{42})\,\omega_2 = 0\,.$$

We can summarize:

Theorem 1. *Let $C : (A) \to (B)$ be a correspondence between the surfaces (A) and (B). Then it is an affine t-deformation of second order if and only if C is conjugate and if it holds*

$$(2.22) \quad \alpha_1 = \lambda_1 \alpha_1' , \quad \alpha_2 = \lambda_2 \alpha_2' , \quad \beta_1 = \lambda_2^5 \beta_1 , \quad \beta_2 = \lambda_1^5 \beta_2' , \quad \lambda_1 \lambda_2 = 1 .$$

The affinity T realizing this t-deformation exists and it is of the from

$$(2.23) \qquad TA = B$$

$$M = \begin{Vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & \lambda_2^2 \end{Vmatrix} .$$

In accordance with the equations (1.9) the frames can be specialized so that

$$\lambda_1 = 1 .$$

Now, we find out that the triplets $[(A), C, (B)]$, $C : (A) \to (B)$ being an affine t-deformation of second order, are determined by the same system of equations as in case of the special affine deformation of second order $C_0 : (A) \to (B)$ (see [3], Theorem 4.). Thus we obtain the following geometrical characterization of the correspondences C_0.

Theorem 2. *The correspondence $C : (A) \to (B)$ is an affine t-deformation of second order if and only if C is a special affine deformation of second order $C_0 : (A) \to (B)$.*

3.

In this paragraph, we shall consider the deformation of line congruences L_1, L_1' or L_2, L_2'.

3.1. Suppose the surfaces (A) and (B) to be given by the equations (1.4) or (1.4') respectively. Let the correspondence C be given by relations (1.7).

Before calculating the conditions for the correspondence $C_1 : L_1 \to L_1'$ or $C_2 : L_2 \to L_2'$ to be a deformation the following relations are to be written. By means of (1.4) we obtain

$$(3.1) \quad \begin{aligned} d[AI_1] &= \omega_{11}[AI_1] + \alpha_1\omega_2[AI_2] + \omega_1[AI_3] - \omega_2[I_1I_2] , \\ d[AI_2] &= \omega_{22}[AI_2] + \alpha_2\omega_1[AI_1] + \omega_2[AI_4] + \omega_1[I_1I_2] , \\ d[AI_3] &= \omega_1[I_1I_3] + \omega_2[I_2I_3] + \omega_{31}[AI_1] + \omega_{32}[AI_2] + \\ &\quad + \omega_{33}[AI_3] + \beta_2\omega_1[AI_4] , \end{aligned}$$

$$\begin{aligned} d[AI_4] &= \omega_1[I_1I_4] + \omega_2[I_2I_4] + \omega_{41}[AI_1] + \omega_{42}[AI_2] + \\ &+ \beta_1\omega_2[AI_3] + \omega_{44}[AI_4], \\ d[I_1I_2] &= (\omega_{11} + \omega_{22})[I_1I_2] + \omega_1[I_3I_2] + \omega_2[I_1I_4]. \end{aligned}$$

Differentiating the first two equations (3.1), we get

$$\begin{aligned} \text{(3.2)} \quad d^2[AI_1] &= V_1[AI_1] + V_2[AI_2] + V_3[AI_3] + V_4[AI_4] + V_5[I_1I_2] + \\ &+ \omega_1^2[I_1I_3] - \omega_2^2[I_1I_4] + 2\omega_1\omega_2[I_2I_3], \\ \text{(3.3)} \quad d^2[AI_2] &= W_1[AI_1] + W_2[AI_2] + W_3[AI_3] + W_4[AI_4] + W_5[I_1I_2] + \\ &+ 2\omega_1\omega_2[I_1I_4] - \omega_1^2[I_2I_3] + \omega_2^2[I_2I_4], \end{aligned}$$

where we denote

$$\begin{aligned} \text{(3.4)} \quad V_1 &= d\omega_{11} + \omega_{11}^2 + \alpha_1\alpha_2\omega_1\omega_2 + \omega_1\omega_{31}, \\ V_2 &= d\alpha_1\omega_2 + \alpha_1\, d\omega_2 + \alpha_1\omega_2(\omega_{11} + \omega_{22}) + \omega_1\omega_{32}, \\ V_3 &= d\omega_1 + \omega_1(\omega_{11} + \omega_{33}), \\ V_4 &= \alpha_1\omega_2^2 + \beta_2\omega_1^2, \\ V_5 &= \alpha_1\omega_1\omega_2 - d\omega_2 - \omega_2(2\omega_{11} + \omega_{22}), \end{aligned}$$

$$\begin{aligned} \text{(3.5)} \quad W_1 &= d\alpha_2\omega_1 + \alpha_2\, d\omega_1 + \alpha_2\omega_1(\omega_{11} + \omega_{22}) + \omega_2\omega_{41}, \\ W_2 &= d\omega_{22} + \omega_{22}^2 + \alpha_1\alpha_2\omega_1\omega_2 + \omega_2\omega_{42}, \\ W_3 &= \alpha_2\omega_1^2 + \beta_1\omega_2^2, \\ W_4 &= d\omega_2 + \omega_2(\omega_{22} + \omega_{44}), \\ W_5 &= d\omega_1 - \alpha_2\omega_1\omega_2 + \omega_1(\omega_{11} + 2\omega_{22}). \end{aligned}$$

Analogous expressions concerning the congruences L_1', L_2' are not written.

Suppose $C_1 : L_1 \to L_1'$ to be the deformation of first order. (See [1], p. 17.) Then the affinity ${}^1T: A_4 \to A_4'$ exists so that it holds

$$\begin{aligned} \text{(3.6)} \quad {}^1T[AI_1] &= {}^1\sigma[BJ_1], \\ {}^1T d[AI_1] &= {}^1\sigma\, d[BJ_1] + {}^1\vartheta[BJ_1]. \end{aligned}$$

According to the first equation (3.6) we can suppose the affinity ${}^1T$ to be of the form

$$\text{(3.7)} \quad {}^1TA = B + {}^1a_1J_1, \quad {}^1TI_1 = {}^1\sigma J_1, \quad {}^1TI_\mu = \sum_{\nu=1}^{4} {}^1a_{\mu\nu}J_\nu, \quad \mu = 2, 3, 4.$$

By a similar way as in the foregoing paragraph we find out that it holds

Lemma 3. *The correspondence $C_1 : L_1 \to L_1'$ is a deformation of first order if and* *only if it is developable. The affinity* ${}^1T$ *realizing this deformation exists and it is of the form*

$$(3.8) \qquad {}^1TA = B$$

$${}^1M = \begin{Vmatrix} {}^1\sigma & 0 & 0 & 0 \\ {}^1a_{21} & \lambda_2 & 0 & 0 \\ {}^1a_{31} & 0 & {}^1\sigma\lambda_1 & 0 \\ {}^1a_{41} & {}^1a_{42} & {}^1a_{43} & \dfrac{1}{({}^1\sigma)^2\,\lambda_1\lambda_2} \end{Vmatrix},$$

where it is

$$(3.9) \qquad {}^1\sigma = \frac{\alpha_1}{\alpha'_1}.$$

3.2. The conditions for $C_1 : L_1 \to L'_1$ to be a deformation of second order are consisting in the existence of the affinity ${}^1T$ so that there is

$$(3.10) \qquad \begin{aligned} {}^1T[AI_1] &= {}^1\sigma[BJ_1], \\ {}^1T\mathrm{d}[AI_1] &= {}^1\sigma\,\mathrm{d}[BJ_1] + {}^1\vartheta[BJ_1], \\ {}^1T\mathrm{d}^2[AI_1] &= {}^1\sigma\,\mathrm{d}^2[BJ_1] + 2({}^1\vartheta)\,\mathrm{d}[BJ_1] + {}^1\vartheta_1[BJ_1]. \end{aligned}$$

Using the results from the previous section we verify the validity of the following theorem.

Theorem 3. *Let $C : (A) \to (B)$ be the correspondence between the surfaces (A), (B). Let $C_1 : L_1 \to L'_1$ be the induced correspondence between Laplace congruences L_1, L'_1. The correspondence $C_1 : L_1 \to L'_1$ is a deformation of second order if and only if it is developable and if it holds*

$$(3.11) \qquad \alpha_1 = \lambda_1\alpha'_1, \quad \alpha_2 = \lambda_2\alpha'_2, \quad \beta_2 = \lambda_1^5\beta'_2, \quad \lambda_1\lambda_2 = 1.$$

With respect to the last equation (3.11) and considering (1.9) it is suitable to specialize the frames so that $\lambda_1 = 1$. Now it can be checked by comparison of our result with the equations (2.16)–(2.22) in [3] that it holds

Theorem 4. *The correspondence $C_1 : L_1 \to L'_1$ is a deformation of second order if and only if the correspondence $C : (A) \to (B)$ is the special deformation $C_0 : (A) \to \to (B)$.*

3.3. Let us attend briefly to the correspondence $C_2 : L_2 \to L'_2$. Let the affinity being considered here be

$$(3.12) \qquad {}^2TA = B + \sum_{\nu=1}^{4} {}^2a_\nu J_\nu, \quad {}^2TI_\mu = \sum_{\nu=1}^{4} {}^2a_{\mu\nu}J_\nu, \quad \mu = 1, 2, 3, 4.$$

The conditions for an analytic contact of second order are

$$(3.13)\qquad \begin{aligned} {}^2T[AI_2] &= {}^2\sigma[BJ_2]\,, \\ {}^2T\,\mathrm{d}[AI_2] &= {}^2\sigma\,\mathrm{d}[BJ_2] + {}^2\vartheta[BJ_2]\,, \\ {}^2T\,\mathrm{d}^2[AI_2] &= {}^2\sigma\,\mathrm{d}^2[BJ_2] + 2({}^2\vartheta)\,\mathrm{d}[BJ_2] + {}^2\vartheta_2[BJ_2]\,. \end{aligned}$$

The formal passage from the congruence L_1 to the congruence L_2 can be carried out so that the indexes are to be changed according to the following substitution

$$\left|\begin{matrix} L_1 : 1 & 2 & 3 & 4 \\ \downarrow L_2 : 2 & 1 & 4 & 3 \end{matrix}\right|\downarrow$$

and moreover we are to write W instead of V.

Thus we obtain in recapitulation:

1) The correspondence $C_2 : L_2 \to L'_2$ is a deformation of the first order if and only if it is developable.

The tangent affinity has the form

$$(3.14)\qquad {}^2TA = B$$

$${}^2M = \left\| \begin{matrix} \lambda_1 & {}^2a_{12} & 0 & 0 \\ 0 & {}^2\sigma & 0 & 0 \\ {}^2a_{31} & {}^2a_{32} & \dfrac{1}{{}^2\sigma\lambda_1\lambda_2} & {}^2a_{34} \\ 0 & {}^2a_{42} & 0 & {}^2\sigma\lambda_2 \end{matrix} \right\|$$

where it is

$$(3.15)\qquad {}^2\sigma = \frac{\alpha_2}{\alpha'_2}\,.$$

2) We verify analogously to Theorem 3. that the correspondence $C_2 : L_2 \to L'_2$ is a deformation of second order if and only if it is developable and it holds

$$(3.16)\qquad \alpha_1 = \lambda_1\alpha'_1\,, \quad \alpha_2 = \lambda_2\alpha'_2\,, \quad \beta_1 = \lambda_2^5\beta_1\,, \quad \lambda_1\lambda_2 = 1\,.$$

Now we obtain again that the special deformation $C_0 : (A) \to (B)$ is equivalent with the deformation of second order $C_2 : L_2 \to L'_2$. It results now

Theorem 5. *Let the correspondence* $C_1 : L_1 \to L'_1$ $(C_2 : L_2 \to L'_2)$ *be a deformation of second order. Then the correspondence* $C_2 : L_2 \to L'_2$ $(C_1 : L_1 \to L'_1)$ *is a deformation of second order.*

Considering Theorems 4. and 5. and taking in mind the well known symmetry in Laplace succession of surfaces, we have

Theorem 6. *Let* $\ldots, (S_{-1}), (S), (S_1), \ldots; \ldots, (S'_{-1}), (S'), (S'_1), \ldots$ *be Laplace successions of surfaces in* A_4, A'_4 *respectively. Let* $\ldots, L_{-1}, L_1, \ldots; \ldots, L'_{-1}, L'_1, \ldots$ *be the successions of congruences of tangent lines of conjugate nets of the surfaces above mentioned. Let us denote by* $\gamma_i : (S_i) \to (S'_i)$ *the induced correspondences between the Laplace transforms and by* $C_i : L_i \to L'_i$ $(i = \pm 1, \pm 2, \pm 3, \ldots)$ *the induced correspondences between the congruences of tangent lines. If* $C : (S) \to (S')$ *is a special deformation of second order, then every* γ_i *is a special deformation of second order and every* C_i *is a deformation of second order.*

4.

In this paragraph, it will be required for the affinities T, ${}^1T$, ${}^2T$ realizing t-deformation of first order and deformation of first order of congruences L_1, L'_1 or L_2, L'_2 respectively, to coincide.

4.1. Let us suppose that the correspondence $C : (A) \to (B)$ is a t-deformation of first order. In accordance with the results from the previous paragraphs (Lemmas 1. and 3.) the correspondences $C_1 : L_1 \to L'_1$ and $C_2 : L_2 \to L'_2$ are deformations of first order. The affinities T, ${}^1T$, ${}^2T$ realizing these deformations are given by the equations (2.12), (3.8), (3.14) and generally they are different.

Let us require for the affinities T, ${}^1T$ to coincide. This can be expressed by equation

$$M = {}^1M \,. \tag{4.1}$$

When comparing the elements in main diagonals we obtain equations from which by using (3.9) the quantities σ, ${}^1\sigma$, a_{11} can be excluded. Thus we get

$$\lambda_1\lambda_2^3\alpha_1^2 = \alpha_1'^2 \,. \tag{4.2}$$

In a similar manner, coinciding of affinities T, ${}^2T$ can be required. Let us summarize:

Lemma 4. *Let* $C : (A) \to (B)$ *be a t-deformation of first order. Let* $C_1 : L_1 \to L'_1$ $(C_2 : L_2 \to L'_2)$ *be a deformation of first order. Let both the deformations be realized by a common tangent affinity. Then it holds*

$$\alpha_1'^2 = \lambda_1\lambda_2^3\alpha_1^2 \quad (\alpha_2'^2 = \lambda_1^3\lambda_2\alpha_2^2) \,.$$

4.2. Let us require for the affinities ${}^1T$, ${}^2T$ to coincide. This can be expressed by equation

$${}^1M = {}^2M \,. \tag{4.3}$$

Making use of (3.9) and (3.15), we obtain from (4.3)

$$\alpha_1 = \lambda_1\alpha'_1 \,, \quad \alpha_2 = \lambda_2\alpha'_2 \,, \tag{4.4}$$

and

$$(\lambda_1 \lambda_2)^3 = 1$$

so that

$$\lambda_1 \lambda_2 = 1 \tag{4.5}$$

can be chosen.

Let us observe explicitly that the common tangent affinity is given by the equations

$$T^* A = B \tag{4.6}$$

$$M^* = \begin{Vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ a_{31} & 0 & \lambda_1^2 & 0 \\ 0 & a_{42} & 0 & \lambda_2^2 \end{Vmatrix},$$

where (4.5) holds.

Comparing (4.4), (4.5) with the equations (2.12) in [3] and taking in mind Lemma 3., we obtain the following characterization of an affine deformation of second order.

Theorem 7. *Let $C : (A) \to (B)$ be the correspondence between the surfaces (A) and (B). Let $C_i : L_i \to L'_i$ $(i = 1, 2)$ be the induced correspondence between the Laplace congruences L_i, L'_i. The correspondence $C : (A) \to (B)$ is an affine deformation of second order if and only if $C_i : L_i \to L'_i$ $(i = 1, 2)$ are deformations of the first order realized by a common affinity.*

Let us remark finally, when it is required for all the affinities T, ${}^1T$, ${}^2T$ to coincide, we obtain the same result being expressed in the previous Theorem.

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