Vladimir Fedorovitch Krapivin

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# ON APPROXIMATE SOLUTIONS OF INITIAL VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATION WITH QUASILINEAR DIFFERENTIAL OPERATOR AND GENERALIZED VOLTERRÁ OPERATOR[1])

VLADIMIR FEDOROVITCH KRAPIVIN, Moscow

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For integro-differential equations are suggested approximate methods of numerical solutions, based on its substitution on each subinterval by easily integrable ordinary differential equations with constant coefficients. Recurrent and general error estimation methods are presented, and particular examples considered.

**1. Introduction.** A number of problems in electronics leads to the necessity of integrating generally non-linear integro-differential equations; and in a majority of cases, these equations are not integrable by elementary and special functions. To solve them, as a rule, it is necessary to make use of the latest achievements of calculating methods and technics. In many problems the use of well known numerical methods of solving initial value problems even by modern high-speed electronic computers does not lead to desirable results. The existing approximate methods of solving integro-differential equations, as a rule, are based on replacing the derivatives by the finite differences and represent a complicated multistep process, which in practical problems cannot be solved on electron computers in a reasonable time. Therefore in solving practical problems we have to search other means of approximate solutions for integro-differential equations, without using the finite-difference methods.

In the method considered in this paper, the integro-differential equation is substituted in each subinterval of the independent variable by an easily integrable ordinary differential equation with constant coefficients; this method is not a new theoretical idea for it was known to Euler. However, in this paper the error estimations are obtained for the first time, and methods applicable to various problems are developed in detail. In particular, problems in the theory of semi-conductors were solved by this

[1]) This article was reported on the second Czechoslovak conference on differential equations and their applications.

method [1]. It was necessary to solve the boundary value problem for a system of three equations with complicated non-linear boundary conditions on a half line. The boundary value problem was changed to initial value problem with arbitrary initial conditions, and was solved by the method given below.

**2. Method of solution and estimation of error.** Let us consider the equation

$$(1) \qquad L[y] - \lambda W[y] = f(x, y),$$

$L[y]$ being the differential operator

$$(2) \qquad L[y] = \sum_{i=0}^{n} P_i(x, y, y', \ldots, y^{(m_i)})\, y^{(n-i)}, \quad (m_i < n)$$

and $W[y]$ the generalized Volterra operator

$$(3) \qquad W[y] = \int_a^x \sum_{j=0}^{r} K_j(x, \xi)\, y^{(j)}(\xi)\, \mathrm{d}\xi, \quad (r < n)$$

$\lambda$ – a real number, $P_i(x, y, y', \ldots, y^{(m_i)})$ and $f(x, y)$ – continuous functions with respect to their arguments in the finite interval $[a, b]$, $P_0 \neq 0$ and kernels $K_j(x, \xi)$, $j = 0, 1, \ldots, r$ are continuous functions in the region $G\{a \leqq \xi \leqq x \leqq b\}$.

The initial conditions are

$$(4) \qquad y^{(s)}(a) = y_0^{(s)}, \quad s = 0, 1, \ldots, n-1.$$

Assuming that equation (1) with the initial conditions (4) has a unique continuous solution $y(x)$, let us construct an approximate solution $\tilde{y}(x)$ in $[a, b]$. Let us divide the interval $[a, b]$ by a sequence of points $x_0 = a, x_1, \ldots, x_m = b$, $h_k = x_{k+1} - x_k$. On each subinterval $[x_k, x_{k+1}]$, $k = 0, 1, \ldots, m-1$ let us replace equation (1) by the following linear differential equation of the $n$-th order with constant coefficients

$$(5) \qquad \tilde{L}_k[y] = \lambda \tilde{W}_k[y] + f(x_k, \tilde{y}_k)$$

with the initial conditions:

$$(6) \qquad y^{(s)}(x_k) = \tilde{y}_k^{(s)}, \quad s = 0, 1, \ldots, n-1,$$

where

$$(7) \qquad \tilde{L}_k[y] = \sum_{i=0}^{n} P_i(x_k, \tilde{y}_k, \tilde{y}_k', \ldots, \tilde{y}_k^{(m_i)})\, \tilde{y}^{(n-i)},$$

$$(8) \qquad \tilde{W}_k[y] = \sum_{j=0}^{r} (K_{j,k,0}\tilde{y}_0^{(j)}h_0 + K_{j,k,1}\tilde{y}_1^{(j)}h_1 + \ldots + K_{j,k,k}\tilde{y}_k^{(j)}h_k).$$

The general solution of equation (5) is known:

$$\tilde{y} = \tilde{y}(x, c_1^{(k)}, c_2^{(k)}, \ldots, c_n^{(k)}), \tag{9}$$

where the constants are determined from the initial conditions at the beginning of each interval $[x_k, x_{k+1}]$. The calculations are carried out successively beginning with the first interval $(k = 0)$.

Let us estimate the error in solution of equation (1). Let $y(x)$ and $\tilde{y}(x)$ by the exact and the approximate solutions respectively. Let us denote

$$\tilde{P}_{ik} = P_i(x_k, \tilde{y}_k, \tilde{y}'_k, \ldots, \tilde{y}_k^{(m_i)}), \tag{10}$$

$$\tilde{f}_k = f(x_k, \tilde{y}_k), \quad \varepsilon_k = y(x_k) - \tilde{y}(x_k).$$

Let us integrate $n$ times equations (1) and (5) from $x_k$ to $x$, and consider the final results for $x = x_{k+1}$. For the sake of convenience and brevity let us denote

$$\underbrace{\int_{x_k}^{x_{k+1}} \int_{x_k}^{x} \ldots \int_{x_k}^{x}}_{n} \varphi(x)\, \mathrm{d}x \ldots \mathrm{d}x = \int_{\substack{x_k \\ n}}^{x_{k+1}} \varphi(x)\, \mathrm{d}x ;$$

we have

$$y_{k+1} = y_k + y'_k h_k + h_k^2 \sum_{s=2}^{n-1} y_k^{(s)} \frac{h_k^{s-2}}{s!} - \sum_{i=1}^{n} \int_{\substack{x_k \\ n}}^{x_{k+1}} P_i y^{(n-i)}\, \mathrm{d}x + \tag{11}$$

$$+ \int_{\substack{x_k \\ n}}^{x_{k+1}} f(x, y)\, \mathrm{d}x + \lambda \int_{\substack{x_k \\ n}}^{x_{k+1}} W[y]\, \mathrm{d}x,$$

$$\tilde{y}_{k+1} = \tilde{y}_k + \tilde{y}'_k h_k + h_k^2 \sum_{s=2}^{n-1} \tilde{y}_k^{(s)} \frac{h_k^{s-2}}{s!} - \sum_{i=1}^{n} \int_{\substack{x_k \\ n}}^{x_{k+1}} \tilde{P}_{ik} \tilde{y}^{(n-i)}\, \mathrm{d}x + \tag{12}$$

$$+ \int_{\substack{x_k \\ n}}^{x_{k+1}} \tilde{f}_k\, \mathrm{d}x + \lambda \int_{\substack{x_k \\ n}}^{x_{k+1}} \tilde{W}_k[y]\, \mathrm{d}x.$$

From (11) and (12) we get:

$$\varepsilon_{k+1} = \varepsilon_k + \varepsilon'_k h_k + h_k^2 \sum_{s=2}^{n-1} \varepsilon_k^{(s)} \frac{h_k^{s-2}}{s!} + \int_{\substack{x_k \\ n}}^{x_{k+1}} (f - \tilde{f}_k)\, \mathrm{d}x - \tag{13}$$

$$- \sum_{i=1}^{n} \int_{\substack{x_k \\ n}}^{x_{k+1}} [P_i y^{(n-i)} - \tilde{P}_{ik} \tilde{y}^{(n-1)}]\, \mathrm{d}x + \lambda \int_{\substack{x_k \\ n}}^{x_{k+1}} \{W[y] - \tilde{W}_k[y]\}\, \mathrm{d}x.$$

We know that

$$\int_{\substack{x_k \\ n}}^{x_{k+1}} \tilde{f}_k\, \mathrm{d}x = \frac{\tilde{f}_k h_k^n}{n!}, \quad \int_{\substack{x_k \\ n}}^{x_{k+1}} \tilde{W}_k[y]\, \mathrm{d}x = \frac{\tilde{W}_k[y]\, h_k^n}{n!}. \tag{14}$$

Let us denote

$$E_k = \max_j |\varepsilon_k^{(j)}|, \quad h_{\max} = \max_k h_k, \quad p_i = \max_{[a,b]} |P_i|, \quad M_{n-i} = \max_{[a,b]} |y^{(n-i)}|,$$

$$\tilde{L}_i = \max_{[a,b]} |\tilde{P}_{ik}|, \quad N_{n-i} = \max_{[a,b]} |\tilde{y}^{(n-i)}|, \quad F = \max_{[a,b]} |f|,$$

(15)
$$G_0 = \max_{[a,b]} |\tilde{f}_k|, \quad T = \max_G |\tilde{W}_k[y]|,$$

$$s = |b - a| \sum_{j=0}^{r} \max_G |K_j(x, \xi)| M_j \geqq \max_G |W[y]|,$$

$$l = \frac{1}{n!} [\sum_{i=1}^{n} (p_i M_{n-i} + L_i N_{n-i}) + F + G_0 + |\lambda| (T + s)],$$

$$g = 1 + \sum_{s=2}^{n-1} \frac{h_{\max}^{s-1}}{s!},$$

and $M_{n-i} \approx N_{n-i}$. Then from (13) we get the following recurrent error estimation

(16)
$$E_{k+1} \leqq (1 + g h_{\max}) E_k + l h_{\max}^n .$$

Hence we get

(17)
$$E_k \leqq (1 + g h_{\max})^k \varepsilon_0 + \frac{l h_{\max}^{n-1}}{g} [(1 + g h_{\max})^k - 1],$$

where $\varepsilon_0$ is the maximum error in the initial data. Obviously, if $\varepsilon_0 = 0$, then from (17) it follows that if $h_{\max} \to 0$ then $E_k \to 0$, i.e. $\tilde{y}(x_k) \to y(x_k)$.

In case that the equation (1) has the form:

$$L[y] = f(x, y) + \int_a^x F(x, y, y', \ldots, y^{(m_l)}) \, dx, \quad x \in [a, b],$$

then the estimation (17) will read

(18)
$$E_k \leqq (1 + h p^{(0)})^k \varepsilon_0 + \frac{p^{(1)} h^n}{p^{(0)}} [(1 + h p^{(0)})^k - 1],$$

where

$$p^{(0)} = \sum_{s=1}^{n-1} \frac{h^{s-1}}{s!} + \frac{h^{n-1}}{n!} [\tilde{b} + \sum_{i=1}^{n} (p_i + (m_i + 1) \gamma_i \beta_{n-i}) + (b - a) c(m_l + 1)],$$

$$p^{(1)} = \frac{1}{(n + 1)!} (\tilde{a} + \tilde{b} \beta_1 + 2B + \sum_{i=0}^{n} p_{0i}) + \frac{(b - a) q}{n!},$$

$$\tilde{a} = \max_{[a,b]} \left|\frac{\partial f}{\partial x}\right|, \quad \tilde{b} = \max_{[a,b]} \left|\frac{\partial f}{\partial y}\right|, \quad \gamma_i = \max_{s,[a,b]} \left|\frac{\partial_i P}{\partial y^{(s)}}\right|,$$

$$\beta_i = \max_{[a,b]} \{|y^{(i)}|, |\tilde{y}^{(i)}|\}, \quad c = \max_{s,[a,b]} \left|\frac{\partial F}{\partial y^{(s)}}\right|,$$

$$p_{0i} = \beta_{n-i}(l_i + \gamma_i \sum_{s=0}^{m_i} \beta_{s+1}) + 2\alpha_i \beta_{n+1-i},$$

$$q = \tfrac{1}{2}(A + C \sum_{s=0}^{m_l} \beta_{s+1}), \quad A = \max_{[a,b]} \left|\frac{\partial F}{\partial x}\right|,$$

$$l_i = \max_{[a,b]} \left|\frac{\partial P_i}{\partial x}\right|, \quad B = \max_{[a,b]} \{|F|, |\tilde{F}|\}.$$

**3. Solution of equation** $y^{(n)} = f(x, y, y', \ldots, y^{n-1})$. We shall apply the approximate method of solution presented in this paper to integro-differential equations to solve the initial value problem:

$$(19) \qquad y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad (x, y) \in G,$$

$$(20) \qquad y(x_0) = y_0, \quad y^{(j)}(x_0) = y_0^{(j)}, \quad j = 1, \ldots, n-1, \quad (x_0, y_0) \in G,$$

where the function $f$ satisfies the Lipschitz condition

$$(21) \qquad |f(x, y + \delta_0, \ldots, y^{(n-1)} + \delta_{n-1}) - f(x, y, y', \ldots, y^{(n-1)})| \leqq K \sum_{i=0}^{n-1} |\delta_i|.$$

Let us divide the interval $[a, b]$ by a sequence of points $x_0 = a, x_1, \ldots, x_s = b$ into elementary intervals. Let $E = \varepsilon\{x_0, \ldots, x_s\}$. On each interval $[x_\nu, x_{\nu+1}]$, let us solve the initial value problem:

$$(19') \qquad y^{(n)} = f(x, \hat{y}_\nu, \hat{y}'_\nu, \ldots, \hat{y}_\nu^{(n-1)}), \quad (x, \hat{y}_\nu) \in G, \quad \nu = 0, 1, \ldots, s-1$$

$$(20') \qquad y(x^\nu) = \hat{y}_\nu, \quad y^{(j)}(x_\nu) = \hat{y}_\nu^{(j)}, \quad j = 1, \ldots, n-1, \quad (x_\nu, \hat{y}_\nu) \in G.$$

Then, if the function $f$ satisfies condition (21) and $\max [x_{l+1} - x_l] = \max_l (h_e) = h$, the solution of problems (19') (20')

$$\hat{y} = \{y_0, \hat{y}_1, \ldots, \hat{y}_s\}, \quad \hat{y}^{(j)} = \{y_0^{(j)}, \hat{y}_l^{(j)}, \ldots, \hat{y}_s^{(j)}\},$$

$j = 1, \ldots, n-1$ when $h \to 0$ tends to the solution of equations (19), (20) and the estimation for the rate of convergence is as follows:

(22)

$$\max_l |y_r^{(n-l)} - \hat{y}_r^{(n-l)}| \leqq \varepsilon_0(1 + h\alpha_0)^r + \frac{h\alpha_1}{2\alpha_0}[(1 + h\alpha_0)^r - 1], \quad r = 1, 2, \ldots, s,$$

where

$$\alpha_0 = \sum_{i=1}^{n-1} \frac{h^{i-1}}{i!} + K\left(n + \frac{h}{2}\sum_{j=0}^{n-1}\sum_{s=0}^{n-j-2}\frac{h^s}{s!}\right),$$

$$\alpha_1 = K\sum_{j=0}^{n-1}\left[M\frac{h^{n-j-1}}{(n-j-1)!} + \sum_{s=0}^{n-j-2} N_{s+j+1}\frac{h^s}{s!}\right],$$

$$M = \max_{[a,b]}\left|f(x, y, y', \ldots, y^{(n-1)}\right|, \quad N_{s+j+1} = \max_r \left|\tilde{y}_r^{(s+j+1)}\right|.$$

If the initial conditions are exactly given, the error estimation has the form:

$$\max_{1\leqq l\leqq n-1} \left|y_r^{(n-l)} - \tilde{y}_r^{(n-l)}\right| \leqq Dh^2, \quad r = 1, \ldots, s, \tag{23}$$

where

$$N = \max_{1\leqq r\leqq s}\left|N_r\right|, \quad D = K(M + nD)\frac{[1 + he^h(1 + \frac{1}{2}Knh) + Kn]^r - 1}{4[1 + Kh(\frac{1}{2}h + e^{-h})}.$$

**4. Solution of a system of ordinary differential equations.** For the sake of simplicity let us confine ourselves to the important case of equations, having the canonical form

$$y_i^{(m_i)}(t) = f_i(t, y_1, y_1', \ldots, y_1^{(m_1-1)}, \ldots, y_n^{(m_n-1)}), \quad i = 1, \ldots, n. \tag{24}$$

The system (24) can be replaced by an equivalent system of $m = m_1 + \ldots + m_n$ equations of the first order, relative to the derivates for all $m$ unknown functions. Then one of the standard computer programs can be used to solve the last system. Due to the limited memory of computers, difficulties are encountered in solving systems of higher orders and it is not possible to solve then at all when $m \gg 100$. And at the same time, the practical necessities on electronic development constantly demand searching other algorithms to carry out numerical calculations. In the present case, the above mentioned method, without much loss in accuracy, avoids many of the difficulties, is economical in time and convenient for programming.

Let the functions $f_i$, $i = 1, \ldots, n$ be continuous and differentiable with respect to all arguments. Let us suppose that the solution of system (24) with the initial conditions

$$y_i(t_0) = (y_i)_0, \quad y_i'(t_0) = (y_i')_0, \ldots, y_i^{(m_i-1)}(t_0) = (y_i^{(m_i-1)})_0 \tag{25}$$

exists and is unique in $t_0 \leqslant t \leqslant T$.

Divide the interval $[t_0, T]$ into elementary intervals $\Delta k = [t_k, t_{k+1}]$ by a sequence of points $t_0 < t_1 < \ldots < t_l = T$. On each such interval, let us search the solution of system (24) in the form of a series:

$$\tilde{y}_i(t) = \tilde{y}_i(t_k) + \sum_{j=1}^{m_i-1}\frac{(t-t_k)^j}{j!}\tilde{y}^{(j)}(t_k) + \frac{(t-t_k)^{m_i}}{(m_i)!}(\tilde{f}_i)_k, \tag{26}$$

$$(27)\qquad \tilde{y}_i^{(j)}(t_k) = \tilde{y}_i^{(j)}(t_{k-1}) + \sum_{s=1}^{m_i-j-1} \frac{(t_k - t_{k-1})^s}{s!} \tilde{y}_i^{(s+j)}(t_{k-1}) + $$

$$+ \frac{(t_k - t_{k-1})^{m_i-j}}{(m_i - j)!} (\tilde{f}_i)_k, \quad (j = 1, \ldots, m_i - 1).$$

The error of such a solution can be easily estimated, considering the exact expansion of the functions $y_i(t)$ and $y_i^{(j)}(t)$ in a Taylor series:

$$(28)\qquad |\varepsilon_i(t_{k+1})| \leqq |\varepsilon_i(t_k)| + \sum_{j=1}^{m_i-1} \frac{h_k^j}{j!} |\varepsilon_i^{(j)}(t_k)| + M_i \frac{h_k^{m_i+1}}{(m_i + 1)!} +$$

$$+ M_i \frac{h_k^{m_i}}{(m_i)!} \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} |\varepsilon_j^{(s)}(t_k)|,$$

$$(29)\qquad |\varepsilon_i^{(j)}(t_k)| \leqq |\varepsilon_i^{(j)}(t_{k-1})| + \sum_{s=1}^{m_i-j-1} \left|\frac{h_{k-1}^s}{s!}\right| \varepsilon_i^{(s+j)}(t_{k-1}) +$$

$$+ M_i \left( \frac{h_k^{m_i-j+1}}{(m_i - j + 1)!} + \frac{h_k^{m_i-j}}{(m_i - j)!} \sum_{j=1}^{n} \sum_{s=0}^{m_j-1} |\varepsilon_j^{(s)}(t_{k-1})| \right),$$

where

$$M_i = \max_{[t_0, T]} \left\{ \left|\frac{\partial f_i}{\partial t}\right|, \left|\frac{\partial f_i}{\partial y_1}\right|, \ldots, \left|\frac{\partial f_i}{\partial y_n^{(m_n-1)}}\right| \right\}.$$

Formulas (28) and (29) give a recurrent estimation for error. From them it is possible to obtain an error estimation applicable to the entire interval $[t_0, T]$:

$$(30)\qquad E_k \leqq \varepsilon_0(1 + hp_0)^k + \frac{hP_1}{p_0} [(1 + hp_0)^k - 1],$$

where the following notations are introduced

$$h = \max_k h_k, \quad E_k = \max_{i,j} |\varepsilon_i^{(j)}(t_k)|, \quad M = \max_i M_i,$$

$$\nu = \min_s m_s, \quad \mu = \max_s m_s, \quad P_1 = M \frac{h^{\nu-1}}{(\nu + 1)!},$$

$$p_0 = mM \frac{h^{\nu-1}}{\nu!} + \sum_{s=1}^{\mu-1} \frac{h^{s-1}}{s!}.$$

The above method permits to tabulate the solutions with an accuracy sufficient for engineering calculations.

**5. Solutions of equations with known moments on the right hand side.** Let us consider a particular case of equation (1):

$$L[y] = y^{(n)} + \sum_{i=1}^{n} p_i y^{(n-i)} = f(x)\,, \quad x \geqq 0\,, \tag{31}$$

where $p_i$ – constant coefficients, $f(x)$ – a single-valued and differentiable function, $f(x) \to 0$ as $x \to \infty$ and its moments are known:

$$M_\nu f(x) = \int_0^\infty x^\nu f(x)\,\mathrm{d}x < \infty\,, \quad \nu = 0, 1, \ldots, m\,. \tag{32}$$

It is necessary to solve th equation (31) with the following initial conditions:

$$y^{(s)}(x_0) = y_0^{(s)} \quad (s = 0, 1, \ldots, n-1)\,. \tag{33}$$

We shall approximate $f(x)$ in the following manner:

$$f(x) \approx e^{-kx} \sum_{i=0}^{m} a_i x^i = e^{-kx} P_m(x) \tag{34}$$

where $m > 0$ is an integer, $k > 0$ and $a_i$ are constants to be determined. Then, from (32) and (34) we have

$$\tilde{M}_\nu f(x) = \int_0^\infty x^\nu [e^{-kx} \sum_{i=0}^{m} a_i x^i]\,\mathrm{d}x = \sum_{i=0}^{m} a_i \frac{(\nu + i)!}{k^{\nu+i+1}}\,. \tag{35}$$

$k$ is fixed from the conditions of best approximation by (34). Then the equation (31) is replaced by the approximate equation:

$$L[\tilde{y}] = e^{-kx} \sum_{i=0}^{m} a_i x^i\,, \tag{36}$$

which can be easily solved. For the error $\varepsilon(x) = y(x) - \tilde{y}(x)$, we obtain an equation from (31) and (36):

$$L[\varepsilon(x)] = f(x) - e^{-kx} P_m(x) \equiv R_m(x)\,. \tag{37}$$

Solving equation (37), we have

$$|\varepsilon(x)| \leqq \frac{\eta}{n!} \sum_{k=0}^{\infty} \frac{(b-a)^{n+k}}{k!} M^k\,, \tag{38}$$

where

$$K(x, s) = \sum_{i=1}^{n} p_i \frac{(x-s)^{i-1}}{(i-1)!}\,, \quad \eta = \max_{[a,b]} |R_m(x)|\,,$$

$$M = \max_{a \leqq s \leqq x \leqq b} |K(x, s)|\,.$$

**6. Refinements of approximate solutions of Volterra integral equations.** Let us consider Volterra integral equation of the first and of the second kind:

$$\lambda \int_a^x G(x, y)\, \varphi(y)\, \mathrm{d}y = g(x)\,, \tag{39}$$

$$\varphi(x) - \lambda \int_a^x K(x, y)\, \varphi(y)\, \mathrm{d}y = f(x)\,, \tag{40}$$

where $x \in [a, b]$, the kernel $K(x, y)$ and its derivatives $K'_x(x, y)$ are continuous in the region $R\{a \leqq y \leqq x \leqq b\}$, $f(x)$ is a continuously differentiable function in $(a, b)$, the kernel $G(x, y)$ and $g(x)$ are twice continuously differentiable functions of $x$, $G(x, x) \neq 0$. Then, as it is known from [4], equations (39) and (40) have unique solutions $\varphi_1(x)$ and $\varphi_2(x)$ respectively which are continuous and differentiable in $[a, b]$ for any value of $\lambda$. The case when $G(x, x) = 0$ for some point in the interval $[a, b]$ or for the entire interval is considered in the paper [5]. In our case, equation (39) is equivalent to the equation of second kind

$$\varphi(x) + \int_a^x \frac{G'_x(x, y)}{G(x, x)}\, \varphi(y)\, \mathrm{d}y = \frac{g'(x)}{\lambda\, G(x, x)}\,.$$

Therefore, the argument used to find an approximate solution of (40) is valid for equation (39) as well.

In [6, 7] equation (40) is solved by replacing the integral of the equation by a finite sum of some quadratic formula. Applying this approach let us divide the interval $[a, b]$ by a sequence of points $x_0 = a_1 < x_1 < x_2 \ldots x_m = b$ into elementary intervals $\Delta j = [x_j, x_{j+1}]$, and instead of (40) let us write the equation:

$$\varphi(x_j) - \lambda \sum_{i=0}^{j-1} \int_{x_i}^{x_{i+1}} K(x_j, y)\, \varphi(y)\, \mathrm{d}y = f(x_j)\,, \quad (j = 0, 1, \ldots, m)\,. \tag{41}$$

Further, because of the assumptions made on $\varphi(x)$ and $K(x, y)$ we can write

$$\varphi(x) = \varphi(x_i) + \frac{(x - x_i)}{1!}\, \varphi'(x_i) + \frac{(x - x_i)^2}{2!}\, \varphi''(\xi_i)\,, \tag{42}$$

$$(x_i \leqq \xi_i \leqq x \leqq x_{i+1})\,,$$

and assuming the existence and differentiability of $K'_y(x, y)$ we have

$$K(x_j, y) = K(x_j, x_i) + (y - x_i)\, K'_y(x_j, x_i) + \tag{43}$$

$$+ \frac{(y - x_i)^2}{2}\, K''_{yy}(x_j, \eta_i)\,, \quad x_i \leqq \eta_i \leqq y \leqq x_{i+1}$$

substituting (42) and (43) in (41) we get:

$$(44)\quad \varphi(x_j) - \lambda \sum_{i=0}^{j-1} \left\{ K(x_j, x_i)\, \varphi(x_i) + \frac{h_i}{2} [K(x_j, x_i)\, \varphi'(x_i) + K'_y(x_j, x_i)\, \varphi(x_i)] + \right.$$

$$\left. + \frac{h_i^2}{3} K'_y(x_j, x_i)\, \varphi'(x_i) \right\} h_i + R_j = f(x_j)\,,$$

where

$$R_j = -\lambda \sum_{i=0}^{j-1} \left\{ \frac{h_i^3}{2} \varphi''(\xi_i) \left[ \frac{1}{3} K'_y(x_j, x_i) + \frac{h_i}{4} K'_y(x_j, x_i) \right] + \right.$$

$$\left. + \frac{h_i^3}{2} K''_{yy}(x_j, \eta_i) \left[ \frac{1}{3} \varphi(x_i) + \frac{h_i}{4} \varphi'(x_i) \right] + \frac{h_i^5}{20} \varphi''(\xi_i) K''_{yy}(x_j, \eta_i) \right\}.$$

Neglecting the small quantity $R_j$ in (44), we get a recurrent formula for determining $\varphi(x_j)$ from the values of the function $\varphi(x)$ at $x = x_0, x_1, \ldots, x_{j-1}$. By differentiating (40) we get a formula to calculate the values of the derivative $\varphi'(x)$.

$$(45)\qquad \varphi'(x) = f'(x) + \lambda K(x, x)\, \varphi(x) + \lambda \int_a^x K'_x(x, y)\, \varphi(y)\, dy\,.$$

From (45), we have at $x = x_i$

$$(46)\qquad \varphi'(x_i) = f'(x_i) + \lambda K(x_i, x_i)\, \varphi(x_i) + \lambda \sum_{s=0}^{i-1} \int_{x_s}^{x_{s+1}} K'_x(x_i, y)\, \varphi(y)\, dy\,,$$

where

$$\varphi(x_0) = f(a)\,, \quad \varphi'(x_0) = f'(a) + \lambda K(a, a)\, \varphi(a)\,.$$

Neglecting the quantity

$$r_i = \lambda \sum_{s=0}^{i-1} \left\{ K''_{xy}(x_i, Q_s) \frac{h_s^2}{2} \left[ \varphi(x_s) + \frac{2}{3} \varphi'(x_s)\, h_s + \frac{h_s^2}{4} \varphi''(\xi_s) \right] + \right.$$

$$\left. + K'_x(x_i, x_s) \frac{h_s^2}{6} \varphi''(\xi_s) \right\}, \quad (x_s \leqq \xi_s,\, Q_s \leqq y \leqq x_{s+1})\,,$$

equation (46) can be written as:

$$(47)\qquad \tilde{\varphi}'(x_i) = f'(x_i) + \lambda K(x_i, x_i)\, \tilde{\varphi}(x_i) +$$

$$+ \lambda \sum_{s=0}^{i-1} h_s K'_x(x_i, x_s) \left[ \tilde{\varphi}(x_s) + \frac{h_s}{2} \tilde{\varphi}'(x_s) \right].$$

From (44) and (47) we get finally the equation for determining $\tilde{\varphi}(x_j)$:

$$\tilde{\varphi}(x_j) - f(x_j) - \lambda \sum_{i=0}^{j-1} [\tilde{\varphi}(x_i)\, K(x_j, x_i)\, l_{ji} + h_i \gamma_{ji} \delta_i]\, h_i = 0\,, \tag{48}$$

where, for sake of brevity, the following notation are introduced:

$$l_{ji} = 1 + \frac{h_i}{2}\left[\lambda K(x_i, x_i) + \frac{K'_y(x_j, x_i)}{K(x_j, x_i)}\,[1 + \tfrac{2}{3}\lambda h_i\, K(x_i, x_i)]\,,\right.$$

$$\gamma_{ji} = \tfrac{1}{2} K(x_j, x_i)\left[1 + \tfrac{2}{3} h_i \frac{K'_y(x_j, x_i)}{K(x_j, x_i)}\right],$$

$$\delta_i = f'(x_i) + \lambda \sum_{s=0}^{i-1} h_s K'_x(x_i, x_s)\left[\tilde{\varphi}(x_s) + \frac{h_s}{2}\,\tilde{\varphi}'(x_s)\right].$$

Thus, starting the calculation by formula (48) from $j = 1$, we get the values of solution $\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots, \tilde{\varphi}_m$ with the error $\varepsilon(x_i)$. Let us estimate the value of modulus of the error $\varepsilon(x)$ from above. Let us introduce the following notation:

$$M = \max_{[a,b]} |\varphi(x)|\,, \quad N = \max_{[a,b]} |\varphi'(x)|\,, \quad L = \max_{[a,b]} |\varphi''(x)|\,,$$

$$Q = \max_{R} \{|K'_y(x, y)|, |K'_x(x, y)|\}\,, \qquad G = \max_{R} |K(x, y)|\,,$$

$$B = \max_{R} \{|K''_{yy}(x, y)|, |K''_{xy}(x, y)|\}\,, \quad h = \max_{S} |h_s|\,,$$

$$l = \frac{m}{6}\left[B\left(M + \frac{3h}{4} N\right) + L\left(G + \tfrac{3}{4} Q h + \tfrac{3}{10} B h^2\right)\right],$$

$$n = \frac{|\lambda|\, Bm}{2}\left(M + \frac{2h}{3} N + \frac{h^2}{4} L\right) + \frac{h}{6}\, BL\,.$$

Then from (44) and (48) we have

$$|\varepsilon_j| \leqq |\lambda|\, h(G + \tfrac{2}{3} Q h) \sum_{i=0}^{j-1} \left(|\varepsilon_i| + \frac{h}{2}\,|\varepsilon'_i|\right) + |\lambda|\, l h^3\,; \tag{49}$$

similarly from (46) and (47) we have:

$$|\varepsilon'_i| \leqq |\lambda|\, G |\varepsilon_i| + |\lambda|\, Q h \sum_{s=0}^{i-1} \left(|\varepsilon_s| + \frac{h}{2}\,|\varepsilon'_s|\right) + n h^2\,. \tag{50}$$

Formulas (49) and (50) give a recurrent error estimation.

From (50) we get roughly

$$|\varepsilon'_i| \leqq |\lambda|\,(G|\varepsilon_i| + hQ \sum_{s=0}^{i-1} |\varepsilon_s|) + (n + |\lambda|\, QNm)\, h^2\,.$$

Hence and from (49) we get

$$(51)\quad |\varepsilon_j| \leqq |\lambda|\, h(G + \tfrac{2}{3}Qh)\sum_{i=0}^{j-1}\left[\left(1 + \frac{|\lambda|\, Gh}{2}\right)|\varepsilon_i| + \frac{|\lambda|\, Qh^2}{2}\sum_{s=0}^{i-1}|\varepsilon_s|\right] + |\lambda|\, th^3\,,$$

where

$$t = l + \frac{h}{2}\, m(G + \tfrac{2}{3}Qh)(n + |\lambda|\, QNm)\,.$$

Let us denote

$$T = |\lambda|\,(G + \tfrac{2}{3}hQ)\,,\quad b = 1 + \frac{|\lambda|\, hG}{2}\,,$$

$$r = |\lambda|\,(t + QMTm^2)\,.$$

Then from (51) we get:

$$(52)\qquad |\varepsilon_j| \leqq hTB\sum_{i=0}^{j-1}|\varepsilon_i| + rh^3\,.$$

From (52) we have

$$(53)\quad |\varphi_j - \bar{\varphi}_j| \leqq E_j = \varepsilon_0 z_1^j + \frac{rh^3}{1 - Tb_j h} \leqq \varepsilon_0 + \frac{rh^3}{1 - Tb_j h}\,,\quad \left(h \leqq \frac{1}{Tb}\right),$$

where $Z_1$ is the real root of the equation

$$Z^{j+1} - TbhZ^{j-1}/(Z - 1) = 0$$

between $Z = 1$ and $Z = h\, Tb$.

**7. Conclusion.** The methods presented in this paper of solving initial value problems for various types of differential and integrodifferential equations have been often used to solve concrete problems by electronic computers. These methods, being simple in programming, give in reasonable time sufficiently accurate results for such problems which cannot be solved by the well known standard methods.

The idea presented above permits, in principle, very easily to write the recurrent relations for any equation or a system of equations and to estimate the error. More accurate approximations may be used in concrete cases. For example the equation examined in the third section could be substituted on each subinterval by the equation $\tilde{y}^{(n)} = f(x, \tilde{y}, \tilde{y}', \ldots, \tilde{y}^{(s)}, \tilde{y}_k^{(s+1)}, \ldots, \tilde{y}_k^{(n-1)})$, if it can be solved analytically.

Evidently, the error estimations obtained here can be improved.

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*Author's address:* Москва К-9, Проспект Маркса 18, СССР (Институт радиотехники и элекроники АН СССР).