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Václav Doležal

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THE EXISTENCE OF A CONTINUOUS BASIS OF A CERTAIN LINEAR SUBSPACE OF E_r WHICH DEPENDS ON A PARAMETER

VÁCLAV DOLEŽAL, Praha

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In the article a theorem concerning the existence of a continuous basis of the space of all solutions $x \in E_r$ of the equation $A(t)\,x = 0$ is given.

Let $A(t)$ be an $r \times r$ matrix which is continuous on $\langle 0, \infty)$ and let $S_t \subset E_r$ be the linear space of all solutions x of the equation $A(t)\,x = 0$ for a chosen $t \geqq 0$; the question is whether there is a fixed set of continuous vectors $P_t = \{x_1(t), x_2(t), \ldots, \ldots, x_k(t)\}$ such that P_t is a basis of S_t for any $t \geqq 0$. The answer is contained in the following theorem:

Theorem. *Let $A(t)$ be an $r \times r$ matrix which has a continuous n-th derivative everywhere in $\langle 0, \infty)$, $n \geqq 0$; moreover, let an integer $h < r$ exist such that* $\operatorname{rank} A(t) = h$ *for every $t \in \langle 0, \infty)$. Then there is an $r \times r$ matrix $M(t)$ which possesses a continuous n-th derivative in $\langle 0, \infty)$ such that* $\det M(t) \neq 0$ *in $\langle 0, \infty)$ and $A(t)\,M(t) = [B(t) \mid 0]$, where $B(t)$ is an $r \times h$ matrix with* $\operatorname{rank} B(t) = h$ *for every $t \in \langle 0, \infty)$.*

Obviously, the last $r - h$ columns of the matrix $M(t)$ constitute the sought set P_t.

Proof. Choose a $\tilde{T} > 0$. Since $A(t)$ is continuous, a minor of $A(t)$ with order h exists which is different from zero on an interval $\langle 0, \delta)$. By the same argument, for each $t \in \langle \delta/2, \tilde{T} \rangle$ there is an open interval J_t containing t such that a minor of $A(t)$ with order h exists which is different from zero on J_t. The system of all intervals $\{J_t\}$, $t \in \langle \delta/2, \tilde{T} \rangle$, however, covers $\langle \delta/2, \tilde{T} \rangle$; consequently, by Borel's theorem, there is a finite subsystem $\{\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_k\}$ of $\{J_t\}$ with the same property. From this it follows that there is a sequence of closed intervals $I_i = \langle t_i, t_i^* \rangle$, $i = 1, 2, \ldots$ which has the properties:

a) $t_1 = 0$, $t_i < t_{i+1} < t_i^* < t_{i+1}^*$, $i = 1, 2, \ldots$, $t_i \to \infty$,

b) for every i there is a minor $A_i(t)$ of the matrix $A(t)$ with order h such that $|\det A_i(t)| \geqq c_i > 0$ for $t \in I_i$.

Using this fact it can be easily verified that for every $i = 1, 2, \dots$ there is an $r \times r$ matrix $M_i(t)$ such that

1) $M_i(t)$ is defined on I_i, possesses a continuous n-th derivative there and $\det M_i(t) = \tilde{c}_i \neq 0$ on I_i,
2) $A(t)\, M_i(t) = [B_i(t) \mid 0]$, where $B_i(t)$ is an $r \times h$ matrix with rank $B_i(t) = h$ on I_i.

Indeed, for every i there are constant regular $r \times r$ matrices C_i, D_i such that

$$C_i\, A(t)\, D_i = \begin{bmatrix} A_{11}^{(i)}(t) & A_{12}^{(i)}(t) \\ A_{21}^{(i)}(t) & A_{22}^{(i)}(t) \end{bmatrix},$$

where $A_{11}^{(i)}(t)$ is an $h \times h$ matrix fulfilling the inequality $|\det A_{11}^{(i)}(t)| \geqq c_i > 0$ for every $t \in I_i$. Thus putting

$$M_i(t) = D_i \begin{bmatrix} I & (-A_{11}^{(i)}(t))^{-1} A_{12}^{(i)}(t) \\ 0 & I \end{bmatrix},$$

where I denotes the unit matrix, we can verify that the matrix $M_i(t)$ has the properties stated above.

Consider now two neighboring intervals I_i and I_{i+1}. Denoting $K_i = (t_{i+1}, t_i^*) \subset \subset I_i \cap I_{i+1}$, choose a number $\tau_i \in K_i$. Then we have $A(\tau_i)\, M_i(\tau_i) = [B_i(\tau_i) \mid 0]$, $A(\tau_i)\, M_{i+1}(\tau_i) = [B_{i+1}(\tau_i) \mid 0]$; consequently, there is a constant regular $r \times r$ matrix F_i such that

$$M_i(\tau_i) = M_{i+1}(\tau_i)\, F_i\,, \tag{1}$$

and F_i has the form

$$F_i = \begin{bmatrix} F_{11}^{(i)} & 0 \\ F_{21}^{(i)} & F_{22}^{(i)} \end{bmatrix},$$

$F_{11}^{(i)}$ being an $h \times h$ matrix.

Let $\eta(t)$ be a function which possesses a continuous n-th derivative on K_i and fulfills the inequality $0 \leqq \eta(t) \leqq 1$, $t \in K_i$, and define the matrix $H_i(t)$ on K_i by

$$H_i(t) = M_i(t) + \eta(t)\,(M_{i+1}(t)\, F_i - M_i(t))\,. \tag{2}$$

Obviously, $H_i(t)$ has a continuous n-th derivative on K_i and due to the form of F_i we have $A(t)\, H_i(t) = [\tilde{B}_i(t) \mid 0]$ on K_i, $\tilde{B}_i(t)$ being an $r \times h$ matrix. Moreover, $H_i(\tau_i) = M_i(\tau_i)$.

Next, denoting the elements of $M_i(t)$ by $m_{jk}(t)$, $j, k = 1, 2, \dots, r$, consider the expression

$$\Phi(t, \xi) = |\det [m_{jk}(t) + \xi_{jk}]| \tag{3}$$

as a function of $r^2 + 1$ variables $t \in K_i$ and $\xi_{jk} \in (-a, a)$, $j, k = 1, 2, \dots, r$. Then we have $\Phi(\tau_i, 0) = |\det M_i(\tau_i)| = |\tilde{c}_i| \neq 0$. Since $\Phi(t, \xi)$ is a continuous function of its variables, there is an open interval $\bar{K}_i \subset K_i$ which contains τ_i and a number $\delta > 0$ such that

$$(4) \qquad \frac{|\tilde{c}_i|}{2} < \Phi(t, \xi) < \frac{3|\tilde{c}_i|}{2}$$

for every $t \in \bar{K}_i$ and $\xi_{jk} \in (-\delta, \delta)$, $j, k = 1, 2, \ldots, r$.

On the other hand, since the matrix $Q_i(t) = M_{i+1}(t) F_i - M_i(t)$ is continuous on K_i and $Q_i(\tau_i) = 0$, there is an open interval $K_i^* \subset K_i$ containing τ_i such that for every element $q_{jk}^{(i)}(t)$, $j, k = 1, 2, \ldots, r$ of $Q_i(t)$ we have $|q_{jk}^{(i)}(t)| < \delta$ whenever $t \in K_i^*$. Consequently, using (2), we have

$$(5) \qquad \frac{|\tilde{c}_i|}{2} < |\det H_i(t)| < \frac{3|\tilde{c}_i|}{2}$$

for every $t \in \bar{K}_i \cap K_i^*$.

Thus, denote $\bar{K}_i \cap K_i^* = (\tilde{t}_{i+1}, \tilde{\tilde{t}}_i)$ and choose a function $\eta(t)$ which has a continuous n-th derivative and satisfies the conditions $\eta(t) = 0$ for $t \in \langle t_i, \tilde{t}_{i+1} \rangle$, $0 < \eta(t) < 1$ for $t \in (\tilde{t}_{i+1}, \tilde{\tilde{t}}_i)$, $\eta(t) = 1$ for $t \in \langle \tilde{\tilde{t}}_i, t_{i+1}^* \rangle$. Putting then

$$\bar{H}_i(t) = (1 - \eta(t))\, \bar{M}_i(t) + \eta(t)\, \bar{M}_{i+1}(t)\, F_i\,,$$

where $\bar{M}_k(t) = M_k(t)$ on I_k, $\bar{M}_k(t) = 0$ elsewhere, $k = i, i+1$, the matrix $\bar{H}_i(t)$ is defined on the entire interval $\langle t_i, t_{i+1}^* \rangle = I_i \cup I_{i+1}$, possesses a continuous n-th derivative there and by (5) fulfills the conditions $\det \bar{H}_i(t) \neq 0$, $A(t)\, \bar{H}_i(t) = [\bar{B}_i(t) \mid 0]$ on $I_i \cup I_{i+1}$, where $\bar{B}_i(t)$ is an $r \times h$ matrix.

From the above considerations it follows that there is a sequence of closed intervals $\bar{I}_i = \langle \bar{t}_i, \bar{\bar{t}}_i \rangle$, $i = 1, 2, \ldots$, where $\bar{I}_i \subset I_i$, $\bar{t}_1 = 0$, $\bar{t}_i < \bar{t}_{i+1} < \bar{\bar{t}}_i < \bar{\bar{t}}_{i+1}$, $i = 1, 2, \ldots$, $\bar{t}_i \to \infty$, which has the following property: Defining successively matrices $\bar{M}_i(t)$ on $\langle 0, \infty)$ by

$$(6) \qquad \begin{aligned} \bar{M}_1(t) &= M_1(t) \text{ on } \bar{I}_1\,, & \bar{M}_{i+1}(t) &= M_{i+1}(t)\, F_i \text{ on } \bar{I}_{i+1} \\ &= 0 \quad \text{elsewhere}\,, & &= 0 \quad \text{elsewhere}, \end{aligned}$$

$i = 1, 2, \ldots$, where each matrix F_i can be obtained from matrices $\bar{M}_i(\tau_i)$, $M_{i+1}(\tau_i)$, $\tau_i \in \bar{I}_i \cap \bar{I}_{i+1}$ as indicated above, and functions $\bar{\eta}_i(t)$, $i = 1, 2, \ldots$ with a continuous n-th derivative by $\bar{\eta}_1(t) = 1$ on $\langle 0, \bar{t}_2 \rangle$, $0 < \bar{\eta}_1(t) < 1$ on $(\bar{t}_2, \bar{\bar{t}}_1)$, $\bar{\eta}_1(t) = 0$ on $\langle \bar{\bar{t}}_1, \infty)$, and

$$\begin{aligned} &\bar{\eta}_i(t) = 1 \text{ on } \langle \bar{\bar{t}}_{i-1}, \bar{t}_{i+1} \rangle\,, \quad 0 < \bar{\eta}_i(t) < 1 \text{ on } (\bar{t}_{i+1}, \bar{\bar{t}}_i)\,, \\ &\bar{\eta}_i(t) + \bar{\eta}_{i-1}(t) = 1 \text{ on } (\bar{t}_i, \bar{\bar{t}}_{i-1}) \text{ and } \bar{\eta}_i(t) = 0 \text{ elsewhere}\,, \end{aligned}$$

then the matrix

$$(7) \qquad M(t) = \sum_{i=1}^{\infty} \bar{\eta}_i(t)\, \bar{M}_i(t)$$

has all the properties stated in the Theorem.

The assertion that rank $B(t) = h$ is obvious; hence, the Theorem is proved.

Résumé

EXISTENCE SPOJITÉ BÁZE JISTÉHO LINEÁRNÍHO PODPROSTORU E_r, ZÁVISLÉHO NA PARAMETRU

Václav Doležal, Praha

V článku je dokázána věta o tom, že ke každé čtvercové matici $A(t)$, která je spojitá a má pevnou hodnost na intervalu $\langle 0, \infty)$, existuje pevná soustava spojitých vektorů $P_t = \{x_1(t), x_2(t), \ldots, x_k(t)\}$ tak, že pro každé $t \geqq 0$ je P_t bazí podprostoru všech řešení rovnice $A(t)\, x = 0$.

Резюме

СУЩЕСТВОВАНИЕ НЕПРЕРЫВНОГО БАЗИСА НЕКОТОРОГО ЛИНЕЙНОГО ПОДПРОСТРАНСТВА E_r, ЗАВИСЯЩЕГО ОТ ПАРАМЕТРА

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal), Прага

В статье доказывается теорема о том, что для каждой квадратной матрицы $A(t)$, которая непрерывна и имеет фиксированный ранг на интервале $\langle 0, \infty)$, существует фиксированная система непрерывных векторов $P_t = \{x_1(t), x_2(t), \ldots, \ldots, x_k(t)\}$ так, что для любого $t \geqq 0$ система P_t является базисом подпространства всех решений уравнения $A(t)\, x = 0$.