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GLOBAL DIFFERENTIAL GEOMETRY OF SURFACES IN AFFINE SPACE

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For a surface in the affine space A_3 a certain tensor is defined, this tensor being the fundamental object for surfaces with non-planar points.

S. SASAKI has proved the following theorem (see [1], Theorem 4, p. 81): In the Euclidean space E_3, let two surfaces S, S' and a diffeomorphism $C : S \to S'$ be given. If the first and second tensors are equal at the points $p \in S$ and $C(p) \in S'$ for each $p \in S$, then the surfaces S and S' are globally equal, i. e. there exists an isometry $\mathfrak{J} : E_3 \to E_3$ such that $\mathfrak{J}(p) = C(p)$ for each $p \in S$. In this paper, I define a certain tensor field on a surface of the affine space A_3, and prove a theorem which is analoguous to that of Sasaki.

1. REPRESENTATION OF GROUPS L_n^2 AND L_n^3

Let $L_n \equiv L_n^1$ be the linear group, i. e. the set of matrices $(A_i^{i'})$ with $\det(A_i^{i'}) \neq 0$. F being an n-dimensional vector space with a base e^i, let $\mathscr{T}$ denote the obvious representation of the group L_n in F given by $e^{i'} = A_i^{i'} e^i$. Let L_n^3 be the second extension of the group L_n, i. e. the set of elements $(A_i^{i'}, A_{ij}^{i'}, A_{ijk}^{i'})$ with multiplication as follows:

$$(1.1) \qquad (A_i^{i'}, A_{ij}^{i'}, A_{ijk}^{i'}) \, . \, (A_{i'}^{i''}, A_{i'j'}^{i''}, A_{i'j'k'}^{i''}) = (A_i^{i''}, A_{ij}^{i''}, A_{ijk}^{i''}) \, ;$$
$$A_i^{i''} = A_i^{i'} A_{i'}^{i''} \, , \quad A_{ij}^{i''} = A_{ij}^{i'j'} A_{i'j'}^{i''} + A_{i'}^{i''} A_{ij}^{i'} \, ,$$
$$A_{ijk}^{i''} = A_{ijk}^{i'j'k'} A_{i'j'k'}^{i''} + A_i^{i'} A_{jk}^{j'} A_{i'j'}^{i''} + A_j^{j'} A_{ik}^{i'} A_{i'j'}^{i''} + A_k^{k'} A_{ij}^{i'} A_{i'k'}^{i''} + A_{ijk}^{i'} A_{i'}^{i''} \, ,$$

here we denote $A_{ij}^{i'j'} = A_i^{i'} A_j^{j'}$ etc.

Now let an n-dimensional vector space with a fixed base be given, denote it by E, F, G, H, K, the vectors of the base being denoted by $e^i, \ldots, k^i$. Let

$$(1.2) \qquad M = (\otimes^5 E) \oplus (\otimes^4 F) \oplus (\otimes^4 G) \oplus (\otimes^3 H) \, ,$$

where $\otimes^i K = K \otimes \ldots \otimes K$ (i times), and $\otimes$ ($\oplus$) denotes the tensor product (direct sum). Further, denote by $e^{ij\ldots k} = e^i \otimes e^j \otimes \ldots \otimes e^k$ the base of the space $E \otimes \otimes \ldots \otimes E$.

Proposition 1. *The transformations*

$$
\begin{aligned}
(1.3)\quad e^{ijrst} &= A^{ijrst}_{i'j'r's't'} e^{i'j'r's't'} , \\
f^{irst} &= A^{rst}_{r's't'} A^{i}_{i'j'} e^{i'j'r's't'} + A^{irst}_{i'r's't'} f^{i'r's't'} , \\
g^{ijrs} &= A^{ij}_{i'j'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{s'} A^{s}_{t'r'} + A^{r}_{r'} A^{s}_{s't'}) e^{i'j'r's't'} + A^{ijrs}_{i'j'r's'} g^{i'j'r's'} , \\
h^{irs} &= \{A^{i}_{i'j'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{s'} A^{s}_{t'r'} + A^{r}_{r'} A^{s}_{s't'}) - A^{r}_{i'} A^{s}_{j'} A^{i}_{r's't'}\} e^{i'j'r's't'} + \\
&\quad + A^{i}_{i'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{s'} A^{s}_{t'r'} + A^{r}_{r'} A^{s}_{s't'}) f^{i'r's't'} + \\
&\quad + (A^{rs}_{r's'} A^{i}_{i'j'} - A^{rs}_{i'j'} A^{i}_{r's'}) g^{i'j'r's'} + A^{irs}_{i'r's'} h^{i'r's'}
\end{aligned}
$$

give rise to a representation $\mathscr{S}$ of the group L^3_n in M. If

$$(1.4)\qquad N = (\otimes^3 F) \oplus (\otimes^2 E) \quad or \quad P = (\otimes^4 E) \oplus (\otimes^3 F),$$

then the transformations

$$
\begin{aligned}
(1.5)\qquad f^{rst} &= A^{rst}_{r's't'} f^{r's't'} , \\
e^{rs} &= (A^{r}_{r'} A^{s}_{s't'} + A^{r}_{s'} A^{s}_{t'r'} + A^{r}_{t'} A^{s}_{r's'}) f^{r's't'} + A^{rs}_{r's'} e^{r's'}
\end{aligned}
$$

or

$$
\begin{aligned}
(1.6)\qquad e^{ijrs} &= A^{ijrs}_{i'j'r's'} e^{i'j'r's'} , \\
f^{irs} &= (A^{rs}_{r's'} A^{i}_{i'j'} - A^{rs}_{i'j'} A^{i}_{r's'}) e^{i'j'r's'} + A^{irs}_{i'r's'} f^{i'r's'}
\end{aligned}
$$

give rise to a representation $\mathscr{R}_1$ or $\mathscr{R}_2$ in N or P respectively.

The projection π_1 of the representation $\mathscr{S}$ into the space $\otimes^3 H$ is isomorphic to $\otimes^3 \mathscr{T}$, the projection π_2 into the space $(\otimes^4 F) \oplus (\otimes^3 H)$ is isomorphic to the representation $\mathscr{T} \otimes \mathscr{R}_1$, and, finally, the projection π_3 into the space $(\otimes^4 G) \oplus (\otimes^3 H)$ is isomorphic to the representation $\mathscr{T} \otimes \mathscr{R}_2$.

Let an n-dimensional differentiable manifold V be given, let V^{k+1} be its k-th extension. Let t be an $\mathscr{S}$-tensor on V with values in M. U_α, $U_{\alpha'}$ being two coordinate neighbourhoods in the base V of the principal fibre bundle V^4 and the coordinates of the tensor t being defined by the equation

$$(1.7)\qquad t = t_{ijrst} e^{ijrst} + t_{irst} f^{irst} + t'_{ijrs} g^{ijrs} + t_{irs} h^{irs},$$

we obtain

$$
\begin{aligned}
(1.8)\qquad t_{i'j'r's't'} &= A^{ijrst}_{i'j'r's't'} t_{ijrst} + A^{rst}_{r's't'} A^{i}_{i'j'} t_{irst} + \\
&\quad + A^{ij}_{i'j'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{r'} A^{s}_{s't'} + A^{r}_{s'} A^{s}_{t'r'}) t'_{ijrs} + \\
&\quad + \{A^{i}_{i'j'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{r'} A^{s}_{s't'} + A^{r}_{s'} A^{s}_{t'r'}) - A^{i}_{r's't'} A^{rs}_{i'j'}\} t_{irs} , \\
t^{i'r's't'} &= A^{irst}_{i'r's't'} t_{irst} + A^{i}_{i'} (A^{r}_{t'} A^{s}_{r's'} + A^{r}_{r'} A^{s}_{s't'} + A^{r}_{s'} A^{s}_{t'r'}) t_{irs} , \\
t'_{i'j'r's'} &= A^{ijrs}_{i'j'r's'} t'_{irst} + (A^{rs}_{r's'} A^{i}_{i'j'} - A^{rs}_{i'j'} A^{i}_{r's'}) t_{irs} , \\
t_{i'r's'} &= A^{irs}_{i'r's'} t_{irs} ,
\end{aligned}
$$

where

$$(1.9) \qquad A^i_{i'} = \frac{\partial u^i}{\partial u^{i'}}, \quad A^i_{j'k'} = \frac{\partial^2 u^i}{\partial u^{j'} \partial u^{k'}}, \quad A^i_{i'j'k'} = \frac{\partial^3 u^i}{\partial u^{i'} \partial u^{j'} \partial u^{k'}}.$$

The projections

$$(1.10) \qquad \begin{aligned} \pi_1 t &= t_{irs} h^{irs}, \\ \pi_2 t &= t_{irst} f^{irst} + t_{irs} h^{irs}, \\ \pi_3 t &= t'_{ijrs} g^{ijrs} + t_{irs} h^{irs} \end{aligned}$$

are successively a $(\otimes^3 \mathscr{T})$-tensor, a $(\mathscr{T} \otimes \mathscr{R}_1)$-tensor and a $(\mathscr{T} \otimes \mathscr{R}_2)$-tensor.

2. LOCAL DIFFERENTIAL GEOMETRY OF SURFACES

Let A_3 be an affine space, V_3 its vector space, D a two-dimensional differentiable manifold and $\tau(p, D)$ its tangent vector space at the point p. The mapping $(\boldsymbol{r}, \boldsymbol{n}) : D \to A_3 \times V_3$ with the projections $\boldsymbol{r} : D \to A_3$ and $\boldsymbol{n} : D \to V_3$ is called a *normalized surface* if $(\mathrm{d}\boldsymbol{r})_p$ is an isomorphism between $\tau(p, D)$ and $(\mathrm{d}\boldsymbol{r})_p \, \tau(p, D)$ and we have $\boldsymbol{n}(p) \notin (\mathrm{d}\boldsymbol{r})_p \, \tau(p, D)$ for each $p \in D$.

Let us restrict ourselves to two coordinate neighborhoods U_α, $U_{\alpha'}$ $(U_\alpha \cap U_{\alpha'} \neq \emptyset)$ of the manifold D. In the neighborhood U_α (with the coordinates u^α), the normalized surface is given by the equations

$$(2.1) \qquad \partial_\alpha \boldsymbol{r}_\beta = \Gamma^\varepsilon_{\alpha\beta} \boldsymbol{r}_\varepsilon + b_{\alpha\beta} \boldsymbol{n}, \quad \partial_\alpha \boldsymbol{n} = p^\varepsilon_\alpha \boldsymbol{r}_\varepsilon + q_\alpha \boldsymbol{n}$$

with the integrability conditions

$$(2.2) \qquad \begin{gathered} b_{[\alpha\beta]} = 0, \quad R^\varepsilon_{\gamma\beta\alpha} = -2 b_{\alpha[\beta} p^\varepsilon_{\gamma]}, \\ \nabla_{[\gamma} b_{\beta]\alpha} + b_{\alpha[\beta} q_{\gamma]} = 0, \quad \nabla_{[\beta} q_{\alpha]} + p^\varepsilon_{[\alpha} b_{\beta]\varepsilon} = 0, \\ \nabla_{[\beta} p^\varepsilon_{\alpha]} + q_{[\alpha} p^\varepsilon_{\beta]} = 0. \end{gathered}$$

In the intersection $U_\alpha \cap U_{\alpha'}$, we obtain

$$(2.3) \qquad \begin{gathered} \Gamma^{\gamma'}_{\alpha'\beta'} = A^{\gamma'\alpha\beta}_{\gamma\alpha'\beta'} \Gamma^\gamma_{\alpha\beta} - A^{\alpha\beta}_{\alpha'\beta'} A^{\gamma'}_{\alpha\beta}, \\ b_{\alpha'\beta'} = A^{\alpha\beta}_{\alpha'\beta'} b_{\alpha\beta}, \quad p^{\beta'}_{\alpha'} = A^{\alpha\beta'}_{\alpha'\beta} p^\beta_\alpha, \quad q_{\alpha'} = A^\alpha_{\alpha'} q_\alpha \end{gathered}$$

and the normalized surface $(\boldsymbol{r}, \boldsymbol{n})$ determines globally a linear connection and three tensors on D. Locally, the surface is uniquely determined by the connection and the tensors just mentioned.

Let another normalized surface $(\boldsymbol{s}, \boldsymbol{m}) : D \to A_3 \times V_3$ be given. If $\boldsymbol{r}(p) = \boldsymbol{s}(p) \in A_3$ for each $p \in D$, we say that $(\boldsymbol{s}, \boldsymbol{m})$ arises from $(\boldsymbol{r}, \boldsymbol{n})$ by a change of the normalization. The class of the normalized surfaces, each of them arising from the others by a change of the normalization, is called a *surface*.

In the neighborhood U_a, the change of the normalization of the surface $(\mathbf{r}, \mathbf{n})$ is given by

(2.4) $$\mathbf{n} = \varphi^\varepsilon \mathbf{r}_\varepsilon + \varphi \,.\, {}^*\mathbf{n}\,, \quad \varphi \neq 0\,.$$

For $(\mathbf{r}, {}^*\mathbf{n})$, we obtain

(2.5) $$\begin{aligned} {}^*\Gamma^\gamma_{\alpha\beta} &= \Gamma^\gamma_{\alpha\beta} + \varphi^\gamma b_{\alpha\beta}\,, \quad {}^*b_{\alpha\beta} = \varphi b_{\alpha\beta}\,, \\ {}^*p^\beta_\alpha &= \varphi^{-1}\big(p^\beta_\alpha + \varphi^\beta q_\alpha - \partial_\alpha \varphi^\beta - \varphi^\varepsilon \Gamma^\beta_{\alpha\varepsilon} - b_{\alpha\varepsilon}\varphi^\beta\varphi^\varepsilon\big)\,, \\ {}^*q_\alpha &= q_\alpha - \varphi^\varepsilon b_{\alpha\varepsilon} - \varphi^{-1}\partial_\alpha\varphi\,. \end{aligned}$$

Let us introduce the object

(2.6) $$\varepsilon_{\varrho\sigma} = \big(\mathbf{r}_\varrho, \mathbf{r}_\sigma, \mathbf{n}\big)\,, \quad \varepsilon_{(\varrho\sigma)} = 0\,.$$

Obviously,

(2.7) $${}^*\varepsilon_{\varrho\sigma} = \varphi^{-1}\varepsilon_{\varrho\sigma}\,, \quad \varepsilon_{\varrho'\sigma'} = A^{\varrho\sigma}_{\varrho'\sigma'}\varepsilon_{\varrho\sigma}\,;$$

$\varepsilon_{\varrho\sigma}$ is a globally defined tensor on D. Furthermore, consider the following objects:

(2.8) $$T_{\varrho\sigma\alpha\beta} = \varepsilon_{\varrho\sigma} b_{\alpha\beta}\,,$$

(2.9) $$T_{\varrho\sigma\alpha\beta\gamma} = \varepsilon_{\varrho\sigma}\big(\partial_\gamma b_{\alpha\beta} + \Gamma^\varepsilon_{\alpha\beta} b_{\gamma\varepsilon} + q_\gamma\, b_{\alpha\beta}\big)\,,$$

(2.10) $$T'_{\varrho\sigma\tau\alpha\beta} = \varepsilon_{\varrho\varepsilon}\big(b_{\tau\alpha}\Gamma^\varepsilon_{\sigma\beta} - b_{\sigma\beta}\Gamma^\varepsilon_{\tau\alpha}\big)\,,$$

(2.11) $$T_{\varrho\sigma\tau\alpha\beta\gamma} = T_{\varrho\varepsilon\alpha\beta\gamma}\,\Gamma^\varepsilon_{\sigma\tau} - T_{\varrho\varepsilon\sigma\tau}\big(\partial_\gamma \Gamma^\varepsilon_{\alpha\beta} + \Gamma^\varphi_{\alpha\beta}\Gamma^\varepsilon_{\gamma\varphi} + b_{\alpha\beta}p^\varepsilon_\gamma\big)\,.$$

Obviously,

(2.12) $$\begin{aligned} T_{(\varrho\sigma)\alpha\beta} &= T_{\varrho\sigma[\alpha\beta]} = 0\,, \\ T_{(\varrho\sigma)\alpha\beta\gamma} &= T_{\varrho\sigma[\alpha\beta]\gamma} = T_{\varrho\sigma\alpha[\beta\gamma]} = T_{\varrho\sigma[\alpha|\beta|\gamma]} = 0\,, \\ T'_{\varrho[\sigma\tau]\alpha\beta} &= T'_{\varrho\sigma\tau[\alpha\beta]} = T'_{\varrho\sigma\tau\alpha\beta} - T'_{\varrho\alpha\beta\sigma\tau} = 0\,, \\ T_{\varrho[\sigma\tau]\alpha\beta\gamma} &= T_{\varrho\sigma\tau[\alpha\beta]\gamma} = T_{\varrho\sigma\tau\alpha[\beta\gamma]} = T_{\varrho\sigma\tau[\alpha|\beta|\gamma]} = 0\,. \end{aligned}$$

Proposition 2. *Consider the space $K \otimes M$ and the representation $\mathcal{T} \otimes \mathcal{S}$ of the group L^3_2 in this space. Then*

(2.13) $$\begin{aligned} T = T_{\varrho\sigma\tau\alpha\beta\gamma}\big(k^\varrho \otimes e^{\sigma\tau\alpha\beta\gamma}\big) + T_{\varrho\sigma\alpha\beta\gamma}\big(k^\varrho \otimes f^{\sigma\alpha\beta\gamma}\big) + \\ + T'_{\varrho\sigma\tau\alpha\beta}\big(k^\varrho \otimes g^{\sigma\tau\alpha\beta}\big) + T_{\varrho\sigma\alpha\beta}\big(k^\varrho \otimes h^{\sigma\alpha\beta}\big) \end{aligned}$$

is a $(\mathcal{T} \otimes \mathcal{S})$-tensor globally defined on D. The projections

(2.14) $$\pi_1 T = T_{\varrho\sigma\alpha\beta}\big(k^\varrho \otimes h^{\sigma\alpha\beta}\big)\,,$$

(2.15) $$\pi_2 T = T_{\varrho\sigma\alpha\beta\gamma}\big(k^\varrho \otimes f^{\sigma\alpha\beta\gamma}\big) + T_{\varrho\sigma\alpha\beta}\big(k^\varrho \otimes h^{\sigma\alpha\beta}\big)\,,$$

(2.16) $$\pi_3 T = T'_{\varrho\sigma\tau\alpha\beta}\big(k^\varrho \otimes g^{\sigma\tau\alpha\beta}\big) + T_{\varrho\sigma\alpha\beta}\big(k^\varrho \otimes h^{\sigma\alpha\beta}\big)$$

are successively a $(\otimes^4\mathcal{T})$*-tensor (the so-called asymptotic tensor), a* $(\mathcal{T}\otimes\mathcal{T}\otimes\mathcal{R}_1)$*-tensor and a* $(\mathcal{T}\otimes\mathcal{T}\otimes\mathcal{R}_2)$*-tensor.*

Proposition 3. *For two normalized surfaces* $(\mathbf{r}, \mathbf{n}), (\mathbf{r}, {}^*\mathbf{n}): D \to A_3 \times V_3$, *we have* $T = {}^*T$.

3. DEFORMATION OF SURFACES

Let us consider two normalized surfaces $(\mathbf{r}, \mathbf{n}), (\mathbf{s}, \mathbf{m}): D \to A_3 \times V_3$. We say that the surfaces $\mathbf{r}, \mathbf{s}: D \to A_3$ are in an *affine deformation of the second order* if for each $p \in D$ there is a non-singular affine collineation $\mathfrak{A}(p) = \mathfrak{A}: A_3 \to A_3$ such that $(\mathfrak{A}(p)_\circ \mathbf{r})(p) = \mathbf{s}(p)$ and $\mathfrak{A}(p)(\mathbf{r}(D))$ and $\mathbf{s}(D)$ have an analytic contact of the second order at the point $\mathbf{s}(p)$. In $U_\alpha \subset D$, let $(\mathbf{r}, \mathbf{n})$ be given by (2.1) and $(\mathbf{s}, \mathbf{m})$ by the equations

$$\text{(3.1)} \qquad \partial_\alpha \mathbf{s}_\beta = (\Gamma^\varepsilon_{\alpha\beta} + G^\varepsilon_{\alpha\beta})\,\mathbf{s}_\varepsilon + (b_{\alpha\beta} + B_{\alpha\beta})\,\mathbf{m}\,, \quad \partial_\alpha \mathbf{m} = (p^\varepsilon_\alpha + P^\varepsilon_\alpha)\,\mathbf{s}_\varepsilon + (q_\alpha + Q_\alpha)\,\mathbf{m}\,.$$

Without loss of generality, we may suppose $\varepsilon_{\varrho\sigma} = \bar{\varepsilon}_{\varrho\sigma} = (\mathbf{s}_\varrho, \mathbf{s}_\sigma, \mathbf{m})$. The osculating affine collineation $\mathfrak{A}$ is of the form

$$\text{(3.2)} \qquad \mathfrak{A}\mathbf{r} = \mathbf{s}\,, \quad \mathfrak{A}\mathbf{r}_\alpha = \mathbf{s}_\alpha\,, \quad \mathfrak{A}\mathbf{n} = \pi^\varepsilon \mathbf{s}_\varepsilon + \pi \mathbf{m}\,, \quad \pi \neq 0\,,$$

and we have

$$\text{(3.3)} \qquad \mathfrak{A}\partial_\alpha \mathbf{r}_\beta = \partial_\alpha \mathbf{s}_\beta + \Phi^\varepsilon_{\alpha\beta}\mathbf{s}_\varepsilon + \Phi_{\alpha\beta}\mathbf{m}\,,$$

$$\text{(3.4)} \qquad \Phi^\gamma_{\alpha\beta} = \pi^\gamma b_{\alpha\beta} - G^\gamma_{\alpha\beta}\,, \quad \Phi_{\alpha\beta} = \pi b_{\alpha\beta} - (b_{\alpha\beta} + B_{\alpha\beta})\,.$$

A necessary and sufficient condition for $\mathbf{r}$ and $\mathbf{s}$ to be in a deformation is the existence of $\pi \neq 0$ and π^γ such that

$$\text{(3.5)} \qquad G^\gamma_{\alpha\beta} = \pi^\gamma b_{\alpha\beta}\,, \quad B_{\alpha\beta} = \pi b_{\alpha\beta} - b_{\alpha\beta}\,.$$

From $(2.6_{1,2})$ we obtain: A necessary and sufficient condition for $\mathbf{r}$ and $\mathbf{s}$ to be in a deformation is the existence of normalizations such that

$$\text{(3.6)} \qquad \bar{\varepsilon}_{\varrho\sigma} = \varepsilon_{\varrho\sigma}\,, \quad G^\gamma_{\alpha\beta} = 0\,, \quad B_{\alpha\beta} = 0\,.$$

If $\mathbf{r}$ and $\mathbf{s}$ are in a deformation, we have $\pi_3 T^{(r)} = \pi_3 T^{(s)}$, where $T^{(r)}$ is the tensor T associated with the surface $\mathbf{r}$. Conversely, let $\pi_3 T^{(r)} = \pi_3 T^{(s)}$. From (3.6_1) we obtain $B_{\alpha\beta} = 0$, and then

$$\varepsilon_{\varrho\varphi}(b_{\alpha\beta}G^\varphi_{\sigma\gamma} - b_{\sigma\gamma}G^\varphi_{\alpha\beta}) = 0$$

from $T^{(r)}_{\varrho\sigma\alpha\beta\gamma} = T^{(s)}_{\varrho\sigma\alpha\beta\gamma}$. Choose $\varrho = 1$ or $\varrho = 2$ and let $\tau \neq \varrho$. Then the preceding equation reduces to

$$b_{\alpha\beta}G^\tau_{\sigma\gamma} - b_{\sigma\gamma}G^\tau_{\alpha\beta} = 0\,.$$

But this is a necessary and sufficient condition for the existence of a φ^{γ} such that $G^{\gamma}_{\alpha\beta} = \varphi^{\gamma} b_{\alpha\beta}$. After a convenient change of the normalization (2.4) with $\varphi = 1$ of the surface $\mathbf{r}$ we obtain $*\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + G^{\gamma}_{\alpha\beta}$.

Proposition 4. *A necessary and sufficient condition for the surfaces* $\mathbf{r}, \mathbf{s} : D \to A_3$ *to be in an affine deformation of the second order is* $\pi_3 T^{(r)} = \pi_3 T^{(s)}$.

4. SURFACES WITH NON-ZERO ASYMPTOTIC TENSOR

Let two surfaces $\mathbf{r}, \mathbf{s} : D \to A_3$ with $T^{(r)} = T^{(s)}$ be given. If these surfaces are in an affine deformation of the second order, one may find normalized surfaces $(\mathbf{r}, \mathbf{n})$, $(\mathbf{s}, \mathbf{m}) : D \to A_3 \times V_3$ such that (3.6) holds in every U_α. Let us restrict ourselves to U_α. From $T^{(r)}_{\varrho\sigma\alpha\beta\gamma} = T^{(s)}_{\varrho\sigma\alpha\beta\gamma}$, $T^{(r)}_{\varrho\sigma\tau\alpha\beta\gamma} = T^{(s)}_{\varrho\sigma\tau\alpha\beta\gamma}$ we obtain

$$b_{\alpha\beta} Q_\gamma = 0 \,, \quad \varepsilon_{\varrho\varepsilon} b_{\sigma\tau} b_{\alpha\beta} P^{\varepsilon}_{\gamma} = 0 \,. \tag{4.1}$$

Let the rank of the matrix $(b_{\alpha\beta})$ be $\geqq 1$, and say, $b_{\xi\eta} \neq 0$ for some fixed $\xi, \eta = 1, 2$. In (4.1) take $\alpha = \sigma = \xi$, $\beta = \tau = \eta$. If $\varphi \neq \varrho$, we obtain $Q_\gamma = P^{\varphi}_{\gamma} = 0$.

Proposition 5. *Assume that the surfaces* $\mathbf{r}, \mathbf{s} : D \to A_3$ *have the following properties:* 1° $T^{(r)} = T^{(s)}$, 2° *there is no point of* $\mathbf{r}$ *or* $\mathbf{s}$ *such that all the tangent directions at this point are asymptotic. Then for each* $p \in D$ *there exist a neighborhood* $\mathcal{O}(p) \subset D$ *and an affine collineation* $\mathfrak{A}(p) : A_3 \to A_3$ *such that*

$$(\mathfrak{A}(p)_0 (\mathbf{r} \mid \mathcal{O}(p)) (q) = (\mathbf{s} \mid \mathcal{O}(p)) (q)$$

for each $q \in \mathcal{O}(p)$.

5. GLOBAL DIFFERENTIAL GEOMETRY OF SURFACES

Let us consider the 12-dimensional space R^{12} (R being the real numbers) with the coordinates (r^A, r^A_1, r^A_2, n^A), $A = 1, 2, 3$. Let a set $K \subset R^{12}$ be given by the equations

$$r^1_1 r^2_2 - r^2_1 r^1_2 = r^1_1 r^3_3 - r^3_1 r^1_2 = 0$$

and let $F = R^{12} - K$.

The manifold D may be considered as the base of a fibre bundle B with the fibre type F, the structural group G

$$\bar{x}^A = x^A \; ; \quad \bar{x}^A_\alpha = a^\beta_\alpha x^A_\beta \,, \quad \det(a^\beta_\alpha) \neq 0 \; ; \quad \bar{n}^A = n^A \,,$$

and the projection $\pi : B \to D$. Cover the manifold by the coordinate neighborhoods U_α; we have $\pi^{-1}(U_\alpha) = U_\alpha \times F$. For two neighborhoods U_α, $U_{\alpha'}$ with $U_\alpha \cap U_{\alpha'} \neq \emptyset$, let us introduce the identification

$$\tilde{r}^A = r^A \,, \quad \tilde{r}^A_\alpha = A^{\alpha'}_\alpha r^A_{\alpha'} \,, \quad \tilde{n}^A = n^A \,.$$

In every trivial fibre bundle $\pi^{-1}(U_\alpha)$, define a two-dimensional distribution Δ by the vectors

$$\xi_\alpha = \left(\delta^1_\alpha, \delta^2_\alpha, r^A_\alpha, \Gamma^\varepsilon_{1\alpha} r^A_\varepsilon + b_{1\varepsilon} n^\varepsilon, \Gamma^\varepsilon_{2\alpha} r^A_\varepsilon + b_{2\varepsilon} n^\varepsilon, p^\varepsilon_\alpha r^A_\varepsilon + q_\alpha n^A\right).$$

Following S. Sasaki, one may prove that the distribution Δ is globally defined and involutive. This enables us to formulate and prove the following two propositions.

Proposition 6. *On the manifold D, let a connection $\Gamma^\gamma_{\alpha\beta}$ and tensors $b_{\alpha\beta}$, p^β_α, q_α satisfying* (2.2) *be given. Then there exists a uniquely determined normalized surface $(\mathbf{r}, \mathbf{n})$: $D \to A_3 \times V_3$ such that in every coordinate neighborhood U_α we have* (2.1).

Proposition 7. *Let D be a manifold and $\mathbf{r}, \mathbf{s} : D \to A_3$ be two surfaces with the property that there is no point of $\mathbf{r}$ or $\mathbf{s}$ such that all the tangent directions at this point are asymptotic. A necessary and sufficient condition for the existence of an affine collineation $\mathfrak{A} : A_3 \to A_3$ such that $(\mathfrak{A}_0 \mathbf{r})(p) = \mathbf{s}(p)$ for each $p \in D$ is $T^{(r)} = T^{(s)}$.*

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Résumé

GLOBÁLNÍ DIFERENCIÁLNÍ GEOMETRIE PLOCH AFINNÍHO PROSTORU

ALOIS ŠVEC, Praha

Je nalezen geometrický objekt, určující jednoznačně a globálně plochu trojrozměrného afinního prostoru.

Резюме

ГЛОБАЛЬНАЯ ДИФФЕРЕНЦИАЛЬНАЯ ГЕОМЕТРИЯ ПОВЕРХНОСТЕЙ АФФИННОГО ПРОСТРАНСТВА

АЛОИС ШВЕЦ (Alois Švec), Прага

Находится геометрический объект, определяющий однозначно и глобально поверхность трехмерного аффинного пространства.