Časopis pro pěstování matematiky

František Zítek

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Časopis pro pěstování matematiky, Vol. 88 (1963), No. 3, 283--294

Persistent URL: `http://dml.cz/dmlcz/117462`

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FRANTIŠEK ZÍTEK, Praha

(Received November 20, 1961)

The aim of this paper is to complete the results of the first part (§§ 1—3) of Kimme's paper [1], the main emphasis being on properties of the limit process such as continuity, absolute continuity, etc. The inspiration has been provided by the parallel between the classical theory of limit laws, and the theory of random functions of intervals (see [4], [5]).

1. INTRODUCTION

Following E. G. KIMME [1], we shall start from a fixed probability space $(\Omega, \mathscr{B}, \mathbf{P})$ having all the properties that are needed; all random variables we shall consider here will be defined on this space. We shall consider double sequences of random variables

$$\{X_{nk}\}\,, \quad k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots, \tag{1.1}$$

where $k_n \to \infty$ when $n \to \infty$. For any fixed n the random variables $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are assumed to be stochastically independent (en bloc). For each t, $0 \leqq t \leqq 1$, let us put

$$k_n(t) = [tk_n] \tag{1.2}$$

and then define

$$X_n(t) = \sum_{k=1}^{k_n(t)} X_{nk}\,. \tag{1.3}$$

If $k_n(t) = 0$ for some t, the corresponding $X_n(t)$ is defined to be a random variable such that $\mathbf{P}\{X_n(t) = 0\} = 1$. In particular, we have $\mathbf{P}\{X_n(0) = 0\} = 1$ for all n.

For n fixed, $X_n(t)$ is a random function with independent increments defined on the interval $\langle 0, 1\rangle$.

E. G. Kimme [1] has studied problems of convergence of sequences of random functions of the type (1.3) to corresponding limit random functions with independent increments. We shall continue his work here with special reference to the continuity properties of the limit random functions. In doing so, we can take advantage of the parallel existing between the theory of limit laws for sequences of the type (1.1) and the theory of random functions of intervals which has been developed in our earlier papers [4], [5] and [6] (see also [7]): we shall assume that the basic notions and results of [4], [5], [6] are already known.

Random functions defined on $\langle 0, 1\rangle$ will therefore be expressed in two parallel ways: i) as point-functions $X(t)$, $0 \leqq t \leqq 1$; ii) as functions of intervals $X(t_1, t_2)$, $0 \leqq t_1 \leqq t_2 \leqq 1$, with the obvious relation

$$X(t_1, t_2) = X(t_2) - X(t_1)\,. \tag{1.4}$$

We now can write

$$X(t_1, t_2) = \sum_{k=k_n(t_1)+1}^{k_n(t_2)} X_{nk}\,, \tag{1.5}$$

where, of course, $\mathbf{P}\{X_n(t_1, t_2) = 0\} = 1$ whenever $k_n(t_1) = k_n(t_2)$.

We also use the standard notation for the corresponding distribution and characteristic functions:

$$F_{nk}(x) = \mathbf{P}\{X_{nk} \leqq x\}\,, \quad \varphi_{nk}(s) = \int e^{isx}\, \mathrm{d}F_{nk}(x)\,,$$

and

$$F_n(t; x) = \mathbf{P}\{X_n(t) \leqq x\}\,, \quad F(t; x) = \mathbf{P}\{X(t) \leqq x\}\,,$$

$$F_n(t_1, t_2; x) = \mathbf{P}\{X_n(t_1, t_2) \leqq x\}\,, \quad F(t_1, t_2; x) = \mathbf{P}\{X(t_1, t_2) \leqq x\}\,.$$

In the sequel, the term "random function" always means "*random function with independent increments*", as other types of random functions will not be considered.

Given a sequence (1.1) we shall say that it is *convergent* if there exists a random function $X(t)$ – called the *limit* of the sequence (1.1) – defined on $\langle 0, 1\rangle$ and such that for every pair (t_1, t_2), $0 \leqq t_1 \leqq t_2 \leqq 1$, we have

$$\lim_{n\to\infty} F_n(t_1, t_2; x) = F(t_1, t_2; x) \tag{1.6}$$

for all x that are points of continuity of the function $F(t_1, t_2; x)$.[1])

Condition (c) will be said to hold for a given sequence (1.1) if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} \mathbf{P}\{|X_{nk}| \geqq \varepsilon\} = 0\,. \tag{1.7}$$

Another equivalent form of this condition is the following: for any $\varepsilon > 0$, $\sigma > 0$ there exists an $N > 0$ such that $n > N$, $|s| \leqq \sigma$, $1 \leqq k \leqq k_n$, implies

$$|\varphi_{nk}(s) - 1| < \varepsilon\,. \tag{1.8}$$

We now shall consider convergent sequences of the type (1.1) satisfying condition (c) and study some properties of their limits.

[1]) It has been shown (see [1]) that this kind of convergence is sufficiently general in the case of random functions with independent increments.

2. CONVERGENCE TO A CONTINUOUS LIMIT

Let a sequence (1.1) be given; for $k = 1, 2, \ldots, k_n$, $n = 1, 2, \ldots$, $-\infty < y < \infty$, let

$$\alpha_{nk} = \alpha_{nk}(\tau) = \int_{|x|<\tau} x \, \mathrm{d}F_{nk}(x)\,, \quad 0 < \tau = \text{const}\,, \tag{2.1}$$

$$\gamma_{nk} = \alpha_{nk} + \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, \mathrm{d}F_{nk}(x + \alpha_{nk})\,, \tag{2.2}$$

$$G_{nk}(y) = \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \, \mathrm{d}F_{nk}(x + \alpha_{nk})\,. \tag{2.3}$$

Further let for $0 \leqq t \leqq 1$

$$\gamma_n(t) = \sum_{k=1}^{k_n(t)} \gamma_{nk}\,, \quad G_n(t; y) = \sum_{k=1}^{k_n(t)} G_{nk}(y)\,, \tag{2.4}$$

where $\gamma_n(t) = 0$, $G_n(t; y) \equiv 0$, whenever $k_n(t) = 0$.

Our condition (c) (which we always assume to hold) implies then (see [1], p. 215)

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} |\alpha_{nk}(\tau)| = 0\,, \tag{2.5}$$

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} |\gamma_{nk}| = 0\,, \tag{2.6}$$

$$\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} |G_{nk}(\infty)| = 0\,. \tag{2.7}$$

It follows then from Kimme's Theorem 1 (see [1], p. 211) that the convergence of (1.1) is equivalent to the convergence of the onedimensional distribution functions $F_n(t; x)$ to $F(t; x)$ (for every t, $0 \leqq t \leqq 1$, at all continuity points x of the function $F(t; x)$). Hence the following condition given by Kimme is necessary and sufficient for the convergence of (1.1):

Condition (K): There exist a real function $\gamma(t)$, $0 \leqq t \leqq 1$, and a bounded real function $G(t, y)$, $0 \leqq t \leqq 1$, $-\infty < y < \infty$, such that

$$\lim_{y\to-\infty} G(1; y) = 0 \tag{2.8}$$

and

$$\lim_{n\to\infty} \gamma_n(t) = \gamma(t)\,, \tag{Kα}$$

for $0 \leqq t \leqq 1$,

$$\lim_{n\to\infty} G_n(t, y) = G(t, y) \tag{Kβ}$$

for $0 \leqq t \leqq 1$ and for all y that are continuity points of $G(t, y)$,

$$\lim_{n\to\infty} G_n(t, \infty) = \lim_{y\to\infty} G(t, y)\,. \tag{Kγ}$$

Remark 1. The relation (2.8) included here in the formulation of condition (K) has been considered by Kimme as an obvious property of the function $G(t, y)$. However, it is easy to give an example of sequence (1.1) which is not convergent even though it satisfies all Kimme's conditions, but with $G(t, y) = 1$ for all $t \in \langle 0, 1\rangle$, $y \in (-\infty, \infty)$.

A random function $X(t)$ defined on $\langle 0, 1\rangle$ is called *continuous at* $t_0 \in \langle 0, 1\rangle$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for $t \in \langle 0, 1\rangle$, $|t - t_0| < \delta$, we have

$$\mathbf{P}\{|X(t) - X(t_0)| \geqq \varepsilon\} \leqq \varepsilon . \tag{2.9}$$

If $X(t)$ is continuous at each $t \in \langle 0, 1\rangle$, we shall say that it is continuous in the interval $\langle 0, 1\rangle$.

Remark 2. Since the interval $\langle 0, 1\rangle$ is compact, any random function $X(t)$ continuous in $\langle 0, 1\rangle$ is also uniformly continuous in $\langle 0, 1\rangle$; this means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequalities $0 \leqq t_1 \leqq t_2 \leqq 1$, $t_2 < t_1 + \delta$, always imply

$$\mathbf{P}\{|X(t_1, t_2)| \geqq \varepsilon\} \leqq \varepsilon . \tag{2.10}$$

Kimme's Theorem 4 (see [1], p. 213) states that the random function $X(t)$, the limit of a convergent sequence (1.1), is continuous at $t_0 \in \langle 0, 1\rangle$ if and only if the corresponding functions $\gamma(t)$ and $G(t, \infty)$ are continuous at t_0. Then in this Theorem 5 Kimme shows that the limit $X(t)$ is continuous in $\langle 0, 1\rangle$ if the convergence in (Kα), (Kβ) and the left-hand member of (Kγ) is uniform in t.[2])

This uniform convergence is, of course, only sufficient; in order to obtain a necessary and sufficient condition we have to replace it by the quasi-uniform convergence (see e.g. [3], p. 155, 7.2d). Hence we have

Theorem 1. *In order that a given sequence* (1.1) *be convergent to a limit random function $X(t)$ continuous at $t_0 \in \langle 0, 1\rangle$, it is necessary and sufficient that condition* (K) *holds and that the convergence in* (Kα) *and* (Kγ) *be quasi-uniform at t_0.*

Proof. Since the convergence of (1.1) follows from our condition (K) (see [1], Theorem 4), it will be sufficient to prove that the continuity of $\gamma(t)$ and $G(t; \infty)$ at t_0 is equivalent to the quasi-uniformity of the convergence in (Kα) and (Kγ). Let us first suppose that the convergence in (Kα) is quasi-uniform in t at $t = t_0$. For all t and all n we have

$$|\gamma(t_0) - \gamma(t)| \leqq |\gamma(t_0) - \gamma_n(t_0)| + |\gamma_n(t_0) - \gamma_n(t)| + |\gamma_n(t) - \gamma(t)| . \tag{2.11}$$

First, it follows from (Kα) that $|\gamma(t_0) - \gamma_n(t_0)|$ will be arbitrarily small if n is large enough. Then the fact that the convergence in (Kα) is quasi-uniform implies that for every sufficiently large n there exists a $\delta_1 > 0$ (dependent on n) such that $|\gamma_n(t) - \gamma(t)|$ will be small for $|t - t_0| < \delta_1$. Finally, (2.6) shows that $\max\limits_{1 \leqq k \leqq k_n} |\gamma_{nk}|$ will be small if n is large enough. Let us now choose an n which is "sufficiently large" in the sense of all

[2]) Clearly, the convergence in the right-hand member of (Kγ) is also uniform in t.

these conditions. Then there exists a $\delta_2 > 0$ such that $|t - t_0| < \delta_2$ implies $|k_n(t) - k_n(t_0)| \leqq 1$, and hence $|\gamma_n(t) - \gamma_n(t_0)| \leqq \max |\gamma_{nk}|$. Thus, for $|t - t_0| < \min(\delta_1, \delta_2)$ all three terms of the right-hand side of (2.11) will be small. Therefore, the function $\gamma(t)$ is continuous at t_0.

Let us suppose now that $\gamma(t)$ is continuous at a point $t_0 \in \langle 0, 1 \rangle$. Instead of (2.11) we shall take the analogous inequality

$$(2.12) \qquad |\gamma_n(t) - \gamma(t)| \leqq |\gamma_n(t) - \gamma_n(t_0)| + |\gamma_n(t_0) - \gamma(t_0)| + |\gamma(t_0) - \gamma(t)| .$$

Since $\gamma(t)$ is continuous at t_0, $|\gamma(t_0) - \gamma(t)|$ will be small if $|t - t_0|$ is small enough. Then (Kα) implies that $|\gamma_n(t_0) - \gamma(t_0)|$ will also be small for large n. As to the first term of the right hand side of (2.12), we proceed in the same manner as in proving sufficiency. Thus, for n large and $|t - t_0|$ small $|\gamma_n(t) - \gamma(t)|$ will be small, and therefore the convergence in (Kα) is quasi-uniform at t_0.

The case of (Kγ) and $G(t, \infty)$ is quite analogous and we shall omit the corresponding part of the proof.

If we restrict ourselves to the usual convergence in (K), our condition (c) will be insufficient to guarantee the continuity of the limit random function $X(t)$, but it is possible to formulate a stronger condition:

Condition (cc) will be said to hold for a given sequence (1.1) if for every $\varepsilon > 0$ there exist an $N > 0$ and a δ, $0 < \delta \leqq 1$, such that for all $n > N$ the inequality $r < \delta k_n$ implies

$$(2.13) \qquad \mathbf{P}\{|\sum_{k=j+1}^{j+r} X_{nk}| \geqq \varepsilon\} \leqq \varepsilon , \quad j = 0, 1, \ldots, k_n - r .$$

Theorem 2. *If a convergent sequence* (1.1) *fulfills condition* (cc), *then the corresponding limit random function $X(t)$ is continuous in $\langle 0, 1 \rangle$.*

Proof. Let $t_0 \in \langle 0, 1 \rangle$, $\varepsilon > 0$. Since condition (cc) holds, there exist a $\delta > 0$ and an $N_1 > 0$ such that (2.13) holds for $r < \delta k_n$, $n > N_1$. We shall now prove that for $t \in \langle 0, 1 \rangle$, $|t - t_0| < \delta$, we have

$$(2.14) \qquad \mathbf{P}\{|X(t) - X(t_0)| \geqq \varepsilon\} \leqq \varepsilon ;$$

since t_0 is arbitrary, this will suffice to prove our theorem.

Let $t \in \langle 0, 1 \rangle$, $|t - t_0| < \delta$, be choosen. Since $k_n \to \infty$ as $n \to \infty$, we shall also have $|t - t_0| + k_n^{-1} < \delta$ if n is large enough ($n > N_2$, N_2 depending on t). If $t_0 \leqq t$ (for $t \leqq t_0$ the proof is analogous), $k_n(t_0) \leqq k_n(t)$, and for $n > \max(N_1, N_2)$ we have

$$k_n(t) - k_n(t_0) \leqq tk_n - t_0 k_n + 1 < \delta k_n .$$

Hence by (2.13)

$$\mathbf{P}\{|X_n(t) - X_n(t_0)| \geqq \varepsilon\} \leqq \varepsilon .$$

On passing to the limit we obtain (2.14), q. e. d.

By means of (1.4), the notion of *absolute continuity*, introduced in [4] for random functions of intervals, can also be applied to point-function (see also [7]).

Condition (ac) will be said to hold for a given sequence (1.1) if for every $\varepsilon > 0$ there exist an $N > 0$ and a δ, $0 < \delta \leqq 1$, such that for all $n > N$ the inequality $r < \delta k_n$ implies

$$\mathbf{P}\{|\sum_{i=1}^{r} X_{nj_i}| \geqq \varepsilon\} \leqq \varepsilon\,, \quad 1 \leqq j_1 < j_2 < \ldots < j_r \leqq k_n\,. \tag{3.1}$$

Remark 3. It can easily be seen that (ac) $\Rightarrow$ (cc) $\Rightarrow$ (c).

Theorem 3. *If a convergent sequence* (1.1) *fulfills condition* (ac), *the corresponding limit random function* $X(t)$ *is absolutely continuous in* $\langle 0, 1\rangle$.

The proof of theorem 3 is analogous to that of theorem 2 (see also [7] and [8]). Let $\varepsilon > 0$ be fixed and let δ, N_1 be the two positive numbers corresponding to ε in the sense of condition (ac). Let us consider any finite system of disjoint intervals $(t_j, t'_j) \subset \subset \langle 0, 1\rangle$, $j = 1, 2, \ldots, m$ such that $\sum_{j=1}^{m} (t'_j - t_j) < \delta$; we shall show that

$$\mathbf{P}\{|\sum_{j=1}^{m} X(t_j, t'_j)| \geqq \varepsilon\} \leqq \varepsilon\,, \tag{3.2}$$

(see [4], p. 587). Again we have $\sum_{j=1}^{m} (t'_j - t_j) + mk_n^{-1} < \delta$ if n is large enough ($n > N_2$, N_2 depending on the system of intervals). Thus for $n > N_2$ we have

$$\sum_{j=1}^{m} \{k_n(t'_j) - k_n(t_j)\} \leqq \sum_{j=1}^{m} (k_n t'_j - k_n t_j + 1) = k_n \sum_{j=1}^{m} (t'_j - t_j) + m < \delta k_n\,.$$

Hence for $n > \max(N_1, N_2)$ we obtain from condition (ac)

$$\mathbf{P}\{|\sum_{j=1}^{m} X_n(t_j, t'_j)| \geqq \varepsilon\} \leqq \varepsilon\,. \tag{3.3}$$

On passing to the limit as $n \to \infty$ we obtain (3.2) q. e. d.

Remark 4. Our theorems 2 and 3 confirm once more the parallel between the theory of limit theorems for sums of independent random variables and the theory of random functions of intervals and their (BB)-integrals (see also [7], Part I). The corresponding analogues are Theorem 1 of [5] and Theorem 10 of [4].

Remark 5. Some applications of theorems 2 and 3 on problems of regularity of arrival flows in the queueing theory were given in [8].

Given a sequence (1.1) we can form the "associated sequence" (see [7], p. 838) of random variables X^*_{nk}, $k = 1, 2, \ldots, k_n$; $n = 1, 2, \ldots$, whose distribution laws are determined by the relation

$$\varphi^*_{nk}(s) = \exp\{\varphi_{nk}(s) - 1\}\,. \tag{3.4}$$

Thus we can state the following simple generalization of the theorem quoted on p. 838 in [7]:

Theorem 4. *A sequence* (1.1) *satisfying condition* (ac) *is convergent if and only if the associated sequence is convergent. Their limits then follow the same laws:* $F(t; x) \equiv F^*(t; x)$.

4. DIFFERENTIABILITY

We shall say of a random function $X(t)$, $0 \leqq t \leqq 1$, that is has a *derivative* (a derivative from the right, to be precise) at $t = t_0$ if there exists a characteristic function $\varphi(s)$ such that

$$\varphi(s) = \exp \{ \lim_{h \to 0+} h^{-1}[\varphi(t_0, t_0 + h; s) - 1] \} , \tag{4.1}$$

where

$$\varphi(t_0, t_0 + h; s) = \int e^{isx} \, \mathrm{d}F(t_0, t_0 + h; x)$$

(see [5], § 4). The corresponding derivative will be defined as a random variable $\mathrm{D}X(t)$ (independent on the whole $X(t)$) having $\varphi(s)$ for its characteristic function.

In (1.8) condition (c) was expressed by means of characteristic functions. We could obtain analogous formulations of conditions (cc) and (ac), too (see also [5], lemma 1); they are very useful in some considerations, but conditions (2.13) and (3.1) permit a more intuitive interpretation. We shall now try to get the sufficient conditions that the limit $X(t)$ of a sequence (1.1) have a derivative. Here it is necessary to use characteristic functions.

Condition ($\mathbf{d}_0$) will be said to hold for a given sequence (1.1) if there exists a characteristic function $\varphi_0(s)$ such that

$$\lim_{n \to \infty} [\varphi_{nk}(s)]^{k_n} = \varphi_0(s) , \tag{4.2}$$

the convergence in (4.2) being: (i) locally uniform in s (ii) uniform in k in the domain $k = o(k_n)$.

Theorem 5. *If a convergent sequence* (1.1) *fulfills condition* (d_0), *the limit function* $X(t)$ *admits a derivative* (*from the right*) *at* $t = 0$ *and this derivative* $\mathrm{D}X(0)$ *has* $\varphi_0(s)$ *for its characteristic function.*

Proof. Since $\varphi_0(0) = 1$, there exists an interval $\langle -\sigma, \sigma \rangle$ in which $|\varphi_0(s)| \geqq \frac{1}{2}$. From (4.2) it follows that also $|\varphi_{nk}(s)| \geqq \frac{1}{2}$ for $-\sigma \leqq s \leqq \sigma$, provided n is sufficiently large. Thus we can take logarithmes on both sides of (4.2): $\psi_{nk}(s) = \log \varphi_{nk}(s)$, $\psi_0(s) = \log \varphi_0(s)$; this gives us

$$\lim_{n \to \infty} k_n \, . \, \psi_{nk}(s) = \psi_0(s) , \tag{4.3}$$

where of course, $-\sigma \leqq s \leqq \sigma$, $k = o(k_n)$. Since $\psi_0(s)$ is finite and $k_n \to \infty$, for every fixed s, $-\sigma \leqq s \leqq \sigma$, we obtain $\psi_{nk}(s) \to 0$, i. e. $\varphi_{nk}(s) \to 1$ in the interval $\langle -\sigma, \sigma \rangle$. Hence (see [2], p. 197) $\varphi_{nk}(s) \to 1$ for all s, the convergence being obviously locally uniform in s. But it can easily be shown that it is uniform in k, $k = o(k_n)$, too.

It follows from the above that $\varphi_0(s)$ must be infinitely divisible, so that $\psi_0(s)$ exists for all real s. Hence $\psi_{nk}(s)$ exists and is finite whenever n is large enough.

We can therefore state that for every $\varepsilon > 0$, $\sigma > 0$ there exist an $N > 0$ and a $\delta > 0$ such that for $n > N$, $|s| \leqq \sigma$, $k < \delta k_n$ we have $|k_n \psi_{nk}(s) - \psi_0(s)| < \varepsilon$ and therefore

$$|\psi_{nk}(s) - k_n^{-1} \psi_0(s)| < \varepsilon k_n^{-1} . \tag{4.4}$$

Hence we have for any positive $h < \delta$

$$\left| \sum_{k=1}^{k_n(h)} \psi_{nk}(s) - \frac{k_n(h)}{k_n} \psi_0(s) \right| = \left| \sum_{k=1}^{k_n(h)} [\psi_{nk}(s) - k_n^{-1} \psi_0(s)] \right| \leqq \sum_{k=1}^{k_n(h)} |\psi_{nk}(s) - k_n^{-1} \psi_0(s)| .$$

Now from (4.4) we obtain for $|s| \leqq \sigma$, $n > N$

$$\left| \sum_{k=1}^{k_n(h)} \psi_{nk}(s) - \frac{k_n(h)}{k_n} \psi_0(s) \right| < \varepsilon \frac{k_n(h)}{k_n} . \tag{4.5}$$

Since evidently $\lim\limits_{n \to \infty} k_n(h)/k_n = h$, passage to the limit ($n \to \infty$) in (4.5) yields

$$|\psi(0, h; s) - h \psi_0(s)| \leqq h\varepsilon , \tag{4.6}$$

where of course $\psi(0, h; s) = \log \varphi(0, h; s)$. From (4.6) we have the inequality

$$|h^{-1} \psi(0, h; s) - \psi_0(s)| \leqq \varepsilon \tag{4.7}$$

valid for $|s| \leqq \sigma$, and $0 < h < \delta$. On passing to the limit for $h \to 0+$ we obtain another inequality

$$\left| \lim_{h \to 0+} h^{-1} \psi(0, h; s) - \psi_0(s) \right| \leqq \varepsilon$$

which is valid for arbitrary ε and for $|s| \leqq \sigma$ (σ being independent of ε). We now see that

$$\lim_{h \to 0+} h^{-1} \psi(0, h; s) = \psi_0(s)$$

locally uniformly in s, so that (see [5], Theorem 9)

$$\lim_{h \to 0+} h^{-1}[\varphi(0, h; s) - 1] = \psi_0(s)$$

q. e. d.

For t other than $t = 0$ we can obtain condition (d_t) analogous to (d_0), by replacing in (ii) the domain $k = o(k_n)$ by $k = k_n(t) + o(k_n)$. Then (d_t) will be a sufficient condition for the existence of $DX(t)$.

Remark 6. Because of the analogy with random functions of intervals we have restricted ourselves to derivatives from the right only, but it is not essential.

5. HOMOGENEOUS CASE

Sequences (1.1) where all random variables X_{nk} in each row (n being fixed) follow the same probability law, form an interesting special case. Since the $F_{nk}(x)$ are independent of n, conditions for convergence, continuity, etc. are very simplified. As we see from Kimme's Theorem 6 ([1], p. 218), such a sequence (1.1) is convergent if there exists a single random variable $X = X(1)$ such that

$$\lim_{n \to \infty} [\varphi_n(s)]^{k_n} = \varphi(s) = \varphi(1; s) . \tag{5.1}$$

It is evident that in this case (5.1) and (4.2) are equivalent so that if (1.1) is convergent, it is also differentiable (in the above sense) for any t, $0 \leqq t < 1$, all derivatives $\mathrm{D}X(t)$ having the same distribution function (the same as $X(1)$). The corresponding functions $\gamma(t)$ and $G(t, y)$ are of the form $\gamma(t) = t\gamma$, $G(t, y) = t\, G(y)$.

6. λ-CONVERGENCE

In this short final paragraph we shall only indicate briefly some directions in which the convergence scheme considered can be generalized.

First, it can be seen that the choice of the functions $k_n(t)$ that mediate the passage from random variables X_{nk} to random functions $X_n(t)$ in (1.3), is in a great measure arbitrary. If we use (1.2), it means that all random variables X_{nk}, $k = 1, 2, \ldots, k_n$ play equal rôles in the definition of $X_n(t)$, but this need not be so. Thus we can consider more generally a system λ of real functions $\lambda_n(t)$, $n = 1, 2, \ldots$, $0 \leqq t \leqq 1$, having the following properties:

(α) they only take positive entire values,

(β) they are non-decreasing,

(γ) $\lambda_n(0) = 0 , \quad \lambda_n(1) = k_n$ for all n.

In place of (1.5) we now put

$$X(t_1, t_2) = \sum_{k=\lambda_n(t_1)+1}^{\lambda_n(t_2)} X_{nk} , \quad 0 \leqq t_1 \leqq t_2 \leqq 1 \tag{6.1}$$

with the obvious convention for the case of $\lambda_n(t_1) = \lambda_n(t_2)$.

Given a sequence (1.1) we shall then say that it is convergent if the sums (6.1) converge in the usual sense, i.e. in distribution. The question of whether (1.1) is convergent and whether its limit $X(t)$ has some special property (as for instance continuity) depends now not only on the properties of the sequence (1.1) itself but also on the properties of the system λ used.

One has no difficulties to see how Kimme's condition (K) has to be transformed in order to correspond to this new kind of convergence: it suffices to write $\lambda_n(t)$ in place of $k_n(t)$ everywhere in (Kα), (Kβ) and (Kγ). Thus we can transfer Kimme's Theorem 4 to the case of λ-convergence for any fixed λ.

But a question of much greater interest is that of the conditions under which λ-convergence follows from the usual convergence. In the general case, the situation is rather complicated. An important part is played here by the limit

$$A(t) = \lim_{n\to\infty} \lambda_n(t)/k_n = \lim_{n\to\infty} \lambda_n(t)/\lambda_n(1)\,, \tag{6.2}$$

(provided it does exist). We have, for example, the following theorem.

Theorem 6. *Let* (1.1) *be a convergent sequence satisfying condition* (cc). *Let* λ *be any system with properties* (α), (β), *and* (γ) *and such that the limit* $A(t)$ *given by* (6.2) *exists, is continuous and increasing in* $\langle 0, 1\rangle$. *Then* (1.1) *is* λ*-convergent (to a* λ*-limit,* $X^{(\lambda)}(t)$, *say) and for the distribution functions* $F^{(\lambda)}(t; x) = \mathbf{P}\{X^{(\lambda)}(t) \leqq x\}$ *we have*

$$F^{(\lambda)}(t; x) = F(A(t); x)\,, \quad 0 \leqq t \leqq 1\,. \tag{6.3}$$

Proof. We write as usual

$$X_n^{(\lambda)}(t) = \sum_{k=1}^{\lambda_n(t)} X_{nk}\,, \quad X_n(A(t)) = \sum_{k=1}^{k_n(A(t))} X_{nk}\,. \tag{6.4}$$

Let $\varepsilon > 0$ be given. From condition (cc) we see that there exist an $N_1 = N_1(\varepsilon)$ and an $\eta > 0$ such that for $n > N_1$, $r < \eta k_n$ we have

$$\mathbf{P}\{|\sum_{k=j+1}^{j+r} X_{nk}| \geqq \varepsilon\} \leqq \varepsilon \tag{6.5}$$

for all j, $0 \leqq j \leqq k_n - r$. For this η we can select a number $N_2 > 0$ such that $n > N_2$ implies $\eta k_n > 2$ and another number $N_3 > 0$ such that $|\lambda_n(t) - k_n A(t)| < \frac{1}{2}\eta$ for all t, $0 \leqq t \leqq 1$ and $n > N_3$. This can be done because it follows from the continuity of $A(t)$ that the convergence in (6.2) is uniform in t in the interval $\langle 0, 1\rangle$. For $n > \max(N_1, N_2, N_3)$ and for all $t \in \langle 0, 1\rangle$ we have

$$\begin{aligned}|\lambda_n(t) - k_n(A(t))| &= |\lambda_n(t) - k_n A(t) + k_n A(t) - k_n(A(t))| \leqq \\ &\leqq |\lambda_n(t) - k_n A(t)| + |k_n A(t) - k_n(A(t))| < \tfrac{1}{2}\eta k_n + 1 < \eta k_n\,.\end{aligned}$$

From (6.5) we now obtain

$$\mathbf{P}\{|X_n^{(\lambda)}(t) - X_n(A(t))| \geqq \varepsilon\} \leqq \varepsilon$$

for $n > \max(N_1, N_2, N_3) = N_0(\varepsilon)$. But ε is arbitrary and (1.1) is convergent, hence (1.1) is λ-convergent and (6.3) holds, q. e. d. The continuity of the random function $X^\lambda(t)$ then follows from the continuity of $A(t)$ and of $X(t)$ (which is implied by our Theorem 2).

Remark 7. If in Theorem 6 condition (cc) is replaced by (ac) and if we suppose that $A(t)$ is absolutely continuous, then (1.1) will be λ-convergent (for (ac) $\Rightarrow$ (cc)), its limit $X^{(\lambda)}(t)$ being absolutely continuous.

Remark 8. As well as the continuity properties of $X^{(\lambda)}(t)$, the existence of derivatives also depends on corresponding properties of the function $A(t)$: if (1.1) is λ-convergent, $\mathrm{D}X^{(\lambda)}(0)$ will exist if (d_0) holds and if $A'(0+) < \infty$ exists. Sufficient conditions for non-zero t can be obtained analogously.

Remark 9. Relations (6.3) and (6.4) also give corresponding relations for γ and G:

$$\gamma^{(\lambda)}(t) = \gamma(\Lambda(t)), \quad G^{(\lambda)}(t, y) = G(\Lambda(t), y). \tag{6.6}$$

In the special case where the $F_{nk}(x)$ do not depend on k (see § 5), if the sequence (1.1) is convergent, it is also λ-convergent for any system λ such that $\Lambda(t)$ exists and is continuous; the limit random function $X^{(\lambda)}(t)$ is then continuous, too. For γ and G we then have

$$\gamma^{(\lambda)}(t) = \gamma\, \Lambda(t), \quad G^{(\lambda)}(t, y) = \Lambda(t)\, G(y). \tag{6.7}$$

Remark 10. In the extensive paper [9] J.V. Prochorov has also considered problems of convergence of random variables to limit random functions on $\langle 0, 1\rangle$ (see [9], § 3.2, p. 214 ssq). But Prochorov uses another type of convergence which is stronger that the convergence we have used here. Our convergence corresponds to the convergence in the sense of condition δ) of Prochorov's theorem 3.2 (see [9], p. 218). Our condition (cc) follows from his conditions (3.15) and (3.16) (see [9], p. 215) if we put in (3.15) a δ such that $C(\lambda, \delta) \leqq \lambda$; this can always be done as we see by (3.16). Prochorov considers from the beginning a convergence corresponding to our λ-convergence (and therefore more general than that used by Kimme). Prochorov's condition (3.13) – which, interpreted literally, is an obvious consequence of his choice of the numbers $t_{n,k}$ – is evidently meant to express a postulate corresponding to our assumption that $\Lambda(t)$ is increasing.

Another possibility of generalization of the convergence scheme consists in allowing infinite k_n. We could then study convergence to a random function defined on $\langle 0, \infty)$ or a modified λ-convergence with

$$\lambda_n(0) = 0, \quad \lambda_n(1) = \infty; \tag{γ'}$$

in place of (γ).

References

[1] *E. G. Kimme*: On the convergence of sequences of stochastic processes. Trans. Amer. Math. Soc. *84* (1957), 208–229.

[2] *M. Loève*: Probability Theory. Princeton 1960.

[3] *R. Sikorski*: Funkcje rzeczywiste I. Warszawa 1958.

[4] *F. Zítek*: Fonctions aléatoires d'intervalle. Czechoslovak Math. Journal *8* (1958), 583–609.

[5] *F. Zítek*: Fonctions aléatoires d'intervalle, II. Czechoslovak Math. Journal *10* (1960), 457–473.

[6] *F. Zítek*: Fonctions aléatoires d'intervalle, III. Czechoslovak Math. Journal *12* (1962), 366–374.

[7] *F. Zítek*: Sur certaines propriétés infinitésimales des fonctions aléatoires. Transactions of the Second Prague Conference on Information Theory, Statistical Decision Functions, Random Processes; Praha 1960, 837–843.

[8] *F. Zítek*: Convergence des suites de processus stochastiques. Applications à la théorie des files d'attente. Bulletin IIS *39* (1962), 2.

[9] *Ю. В. Прохоров:* Сходимость случайных процессов и предельные теоремы теории вероятностей. Теория вероятностей и ее применения *1* (1956), 177–238.

Výtah

O KONVERGENCI POSLOUPNOSTÍ STOCHASTICKÝCH PROCESŮ

FRANTIŠEK ZÍTEK, Praha

Budiž dána posloupnost (1.1) náhodných proměnných X_{nk}; při pevném n jsou X_{nk} stochasticky nezávislé. Vztahem (1.3) je pak definována posloupnost náhodných funkcí $X_n(t)$, $0 \leqq t \leqq 1$, s nezávislými přírůstky. V práci [1] studoval E. G. Kimme otázky konvergence (v distribuci) posloupností tohoto typu k limitním náhodným funkcím $X(t)$ s nezávislými přírůstky.

Jak bylo ukázáno již v [7], analogie mezi teorií limitních zákonů pro součty nezávislých náhodných veličin a teorií náhodných funkcí intervalu (viz [4], [5], [6]) dovoluje přenášeti některé výsledky z jedné teorie do druhé. Toho je zde využito k formulaci podmínek, kladených na posloupnost (1.1), postačujících k tomu, aby limitní náhodná funkce $X(t)$ byla spojitá nebo absolutně spojitá, anebo aby měla derivaci. Nakonec je uvažováno i jisté zobecnění daného konvergenčního schématu.

Резюме

О СХОДИМОСТИ ПОСЛЕДАВОТЕЛЬНОСТЕЙ СЛУЧАЙНЫХ ПРОЦЕССОВ

ФРАНТИШЕК ЗИТЭК (František Zítek), Прага

Пусть дана последовательность (1.1) случайных величин X_{nk}; при фиксированном n все X_{nk} стохастически независимы. Соотношение (1.3) определяет тогда последовательность случайных функций $X_n(t)$, $0 \leqq t \leqq 1$, с независимыми приращениями. В работе [1] изучал Кимме сходимость (по распределению) последовательностей этого типа к предельным случайным функциям $X(t)$ с независимыми приращениями.

Как показано уже в [7] аналогия между теорией предельных законов для сумм независимых случайных величин и теорией случайных функций интервала (см. [4], [5], [6]) позволяет перенести некоторые результаты из одной теории в другую. В настоящей статье это обстоятельство используется для того, чтобы указать условия, налагаемые на последовательность (1.1), при выполнении которых предельная функция $X(t)$ будет непрерывной, или абсолютно непреривной, или дифференцируемой. Наконец рассмотривается еще некоторое обобщение данной схемы.