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ON A PROBLEM OF V. PTÁK

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1. INTRODUCTION AND NOTATION

If x, y are reals, $x < y$, then $[x, y]$, (x, y) denote respectively the closed and the open interval with the endpoints x, y, $(x, y] = (x, y) \cup \{y\}$. N will always denote the set of positive integers, $\emptyset$ the empty set etc.

Let T be a positive real number or ∞. The letters u, v, w will always denote mappings of the interval $(0, T)$ into $(0, T)$. For every such mapping $w(x)$ and every nonnegative integer n define

$$w^n(x) = x \quad \text{if} \quad n = 0\,, \qquad w^n(x) = w(w^{n-1}(x)) \quad \text{if} \quad n \in N\,,$$

$$W(x) = w^0(x) + w^1(x) + w^2(x) + \ldots\,,$$

and quite analogously for u, U or v, V instead of w, W.

The function $W(x)$ is a mapping of $(0, T)$ into $(0, \infty) \cup \{\infty\}$. By [1], a function $w(x)$ is said to be small on $(0, T')$, $0 < T' \leqq T$, if $W(x) < \infty$ for all $x \in (0, T')$. A function $w(x)$ is said to be small if it is small on $(0, T)$. The aim of our paper is to give conditions for a function $w(x)$ to be small. V. Pták suggested to study small functions in connection with his results concerning generalizations of the Banach fixed-point theorem and the closed graph theorem.

The main results of this paper are contained in Sections 4 and 5. Sections 2 and 3 contain some lemmas necessary in the proofs of the results in Sections 4 and 5. Section 6 contains examples and counter-examples showing that it is impossible to delete some assumptions in the theorems and lemmas of the previous sections.

All infinite series in the paper consist of nonnegative members and hence their sums always exist; of course, they can be equal to ∞. Analogously all integrals are integrals of nonnegative Lebesgue measurable functions, hence they exist but may be equal to ∞. Measurability of functions is mentioned in theorems and lemmas if necessary but it is not mentioned in their proofs if it is consequence of other assumptions.

2.1. Lemma. *Let* $(a_0, a_1, a_2, \ldots)$ *be a decreasing sequence of positive reals, let* $k > 1$ *and*

$$\sum_{n=1}^{\infty} a_n \,.\, \ln \frac{(a_n - a_{n+1}) + k\,.\,(a_{n-1} - a_n)}{a_n - a_{n+1}} < \infty \,.$$

Then $a_1 + a_2 + a_3 + \ldots < \infty$.

Proof. Denote $c_{n-1} = a_{n-1} - a_n$ for all $n \in N$. Then we have $a_n = (c_n + c_{n+1} + c_{n+2} + \ldots) + a$, where $a = \lim_{n\to\infty} a_n$; the limit obviously exists and is nonnegative. We prove that it is equal to 0. If $a > 0$ then

$$\sum_{n=1}^{\infty} \ln \frac{c_n + k\,.\,c_{n-1}}{c_n} < \infty \,.$$

However, $\lim_{n\to\infty} c_n = 0$ and hence $c_n < c_{n-1}$, i.e.

$$\ln \frac{c_n + k\,.\,c_{n-1}}{c_n} > \ln (1 + k)$$

for infinitely many $n \in N$, which is a contradiction.

Obviously $c_n < c_0$ for almost all $n \in N$; for the sake of simplicity we assume that $c_n < c_0$ for all $n \in N$. Then we have

$$\infty > \sum_{n=1}^{\infty} a_n \,.\, \ln \frac{c_n + k\,.\,c_{n-1}}{c_n} =$$

$$= (c_1 + c_2 + c_3 + \ldots)\,.\,\ln \frac{c_1 + k\,.\,c_0}{c_1} + (c_2 + c_3 + \ldots)\,.\,\ln \frac{c_2 + k\,.\,c_1}{c_2} +$$

$$+ (c_3 + c_4 + \ldots)\,.\,\ln \frac{c_3 + k\,.\,c_2}{c_3} + \ldots =$$

$$= c_1\,.\,\ln \frac{c_1 + k\,.\,c_0}{c_1} + c_2\,.\,\ln \left(\frac{c_1 + k\,.\,c_0}{c_1}\,.\,\frac{c_2 + k\,.\,c_1}{c_2}\right) +$$

$$+ c_3\,.\,\ln \left(\frac{c_1 + k\,.\,c_0}{c_1}\,.\,\frac{c_2 + k\,.\,c_1}{c_2}\,.\,\frac{c_3 + k\,.\,c_2}{c_3}\right) + \ldots =$$

$$= c_1\,.\,\ln \frac{c_1 + k\,.\,c_0}{c_1} + c_2\,.\,\ln \left(\frac{c_1 + k\,.\,c_0}{c_2}\,.\,\frac{c_2 + k\,.\,c_1}{c_1}\right) +$$

$$+ c_3\,.\,\ln \left(\frac{c_1 + k\,.\,c_0}{c_3}\,.\,\frac{c_2 + k\,.\,c_1}{c_1}\,.\,\frac{c_3 + k\,.\,c_2}{c_2}\right) + \ldots \geqq$$

$$\geqq 0\,.\,c_1\,.\,\ln k + 1\,.\,c_2\,.\,\ln k + 2\,.\,c_3\,.\,\ln k + \ldots = (a_2 + a_3 + a_4 + \ldots)\,.\,\ln k\,.$$

Hence $a_2 + a_3 + a_4 + \ldots < \infty$ and then also $a_1 + a_2 + a_3 + \ldots < \infty$, q.e.d.

2.2. Lemma. *Let $(a_1, a_2, a_3, \ldots)$ be a decreasing sequence of positive reals, let $\sum_{n=1}^{\infty} a_n < \infty$ and*

$$a_2/a_1 \leqq a_3/a_2 \leqq a_4/a_3 \leqq \ldots$$

Then

$$\sum_{n=1}^{\infty} a_{n+1} \cdot \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} < \infty .$$

Proof. For an arbitrary non-decreasing sequence $b = (b_1, b_2, b_3, \ldots)$ of positive reals less than 1 and an arbitrary $i \in N$ define

$$F(b) = 1 + b_1 + b_1 b_2 + b_1 b_2 b_3 + \ldots ,$$

$$G(b) = b_1 \cdot \frac{1 - b_1}{1 - b_2} + b_1 b_2 \cdot \frac{1 - b_2}{1 - b_3} + b_1 b_2 b_3 \cdot \frac{1 - b_3}{1 - b_4} + \ldots ,$$

$$F_i(b) = F(b_1, b_2, \ldots, b_i, b_i, b_i, \ldots) ,$$

$$G_i(b) = G(b_1, b_2, \ldots, b_i, b_i, b_i, \ldots) .$$

Up to a finite number of members, $F_i(b)$ and $G_i(b)$ are geometrical series with the quotient b_i, hence they are convergent. By an easy computation we can verify that

$$F_{i+1}(b) - F_i(b) = b_1 b_2 \ldots b_i \cdot \frac{b_{i+1} - b_i}{(1 - b_i) \cdot (1 - b_{i+1})} ,$$

$$G_{i+1}(b) - G_i(b) = b_1 b_2 \ldots b_i \cdot \frac{(2 - b_i) \cdot (b_{i+1} - b_i)}{(1 - b_i) \cdot (1 - b_{i+1})} .$$

Hence for all $i \in N$ we have

$$0 \leqq G_{i+1}(b) - G_i(b) \leqq 2 \cdot (F_{i+1}(b) - F_i(b)) .$$

Comparing term by term the infinite series $F_i(b)$, $F(b)$ we obtain $F_i(b) \leqq F(b)$ for all $i \in N$. On the other hand, $F_i(b)$ is greater than the i-th partial sum of $F(b)$ and hence $\lim_{i\to\infty} F_i(b) = F(b)$. The i-th partial sum of $G(b)$ is less than $G_i(b)$ and hence $\lim_{i\to\infty} G_i(b) \geqq G(b)$. (The limit exists but it may be ∞.)

Now we can prove the lemma. Without loss of generality we may assume $a_1 = 1$. Denote $b_n = a_{n+1}/a_n$ for every $n \in N$. The sequence $b = (b_1, b_2, b_3, \ldots)$ is non-decreasing and $b_n \in (0, 1)$ for all $n \in N$. It holds

$$a_{n+1} \cdot \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} = b_1 b_2 \ldots b_{n-1} \cdot \frac{1 - b_n}{1 - b_{n+1}}$$

for all $n \in N - \{1\}$. Hence it remains to prove that $F(b) < \infty$ implies $G(b) < \infty$. However,

$$G(b) \leqq \lim_{i \to \infty} G_i(b) = G_1(b) + \sum_{i=1}^{\infty} (G_{i+1}(b) - G_i(b)) \leqq$$

$$\leqq 2 \,.\, F_1(b) + \sum_{i=1}^{\infty} 2 \,.\, (F_{i+1}(b) - F_i(b)) = 2 \,.\, F(b) < \infty \,, \quad \text{q.e.d.}$$

2.3. Lemma. *Let $k > 1$, let $(a_1, a_2, a_3, \ldots)$ be a decreasing sequence of positive reals, let*

$$\sum_{n=1}^{\infty} a_n < \infty$$

and let for all $n \in N$ $a_{n+1} - a_{n+2} \leqq k \,.\, (a_n - a_{n+1})$. Then

$$\sum_{n=1}^{\infty} a_{n+1} \,.\, \ln \frac{k \,.\, (a_n - a_{n+1})}{a_{n+1} - a_{n+2}} < \infty \,.$$

Proof. Denote $b_n = a_n - a_{n+1}$ for all $n \in N$. Then obviously $a_n = b_n + b_{n+1} + b_{n+2} + \ldots$ for all $n \in N$. Without loss of generality we may assume $b_1 = 1$ and $b_{n+1} < 1$ for all $n \in N$. Then we have

$$\sum_{n=1}^{\infty} a_{n+1} \,.\, \ln \frac{k \,.\, (a_n - a_{n+1})}{a_{n+1} - a_{n+2}} = \sum_{n=1}^{\infty} a_{n+1} \,.\, \ln \frac{k \,.\, b_n}{b_{n+1}} =$$

$$= (b_2 + b_3 + b_4 + \ldots) \,.\, \ln \frac{k \,.\, b_1}{b_2} + (b_3 + b_4 + \ldots) \,.\, \ln \frac{k \,.\, b_2}{b_3} +$$

$$+ (b_4 + b_5 + \ldots) \,.\, \ln \frac{k \,.\, b_3}{b_4} + \ldots =$$

$$= b_2 \,.\, \ln \frac{k \,.\, b_1}{b_2} + b_3 \,.\, \left(\ln \frac{k \,.\, b_1}{b_2} + \ln \frac{k \,.\, b_2}{b_3}\right) +$$

$$+ b_4 \,.\, \left(\ln \frac{k \,.\, b_1}{b_2}\right) + \ln \frac{k \,.\, b_2}{b_3} + \ln \frac{k \,.\, b_3}{b_4}\right) + \ldots =$$

$$= \sum_{n=1}^{\infty} b_{n+1} \,.\, \ln (k^n / b_{n+1}) = \sum_{n=1}^{\infty} b_{n+1} \,.\, n \,.\, \ln k + \sum_{n=1}^{\infty} b_{n+1} \,.\, |\ln b_{n+1}| \leqq$$

$$\leqq \ln k \,.\, \sum_{n=1}^{\infty} a_{n+1} + \sum_{n=1}^{\infty} n \,:\, (b_{n+1} + e^{-n}) =$$

$$= (1 + \ln k) \,.\, \sum_{n=1}^{\infty} a_{n+1} + \sum_{n=1}^{\infty} n \,.\, e^{-n} < \infty \,.$$

We have used the inequality

$$b \,.\, |\ln b| \leqq n \,.\, (b + e^{-n}) \quad \text{for all} \quad b \in (0, 1) \quad \text{and} \quad n \in N \,.$$

To verify it, let us distinguish two cases. If $\ln b \geqq -n$ then $b \,.\, |\ln b| \leqq n \,.\, b$. If $\ln b \leqq -n$ then we have $b \,.\, |\ln b| = |\ln b| \,.\, e^{-|\ln b|} \leqq n \,.\, e^{-n}$, since the function $x \,.\, e^{-x}$ is decreasing on $[1, \infty)$, q.e.d.

3. OTHER LEMMAS

3.1. Lemma. *Let* $a, b, c \in (0, T)$, $a < b < c$, *let* $w(x) < x$ *for all* $x \in [b, c]$, $w(c) = b$, $w(b) = a$ *and* $k > 1$.

a) *If* $w(y) - w(x) \leqq k \,.\, (y - x)$ *for all* $x, y \in [b, c]$, $x < y$ *then*

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \geqq \frac{b}{k} \,.$$

b) *If* $w(y) - w(x) \geqq -k \,.\, (y - x)$ *for all* $x, y \in [b, c]$, $x < y$ *then*

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \geqq \frac{b}{k + 1} \,.\, \ln \frac{(b - a) + (k + 1) \,.\, (c - b)}{b - a} \,.$$

Proof. a) Let $u(x) = b + k \,.\, (x - c)$. Then $w(x) \geqq u(x)$ for all $x \in [b, c]$ and hence

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \geqq \int_b^c \frac{x \,.\, dx}{x - u(x)} = \int_b^c \frac{dx}{1 - u(x)/x} \geqq \int_b^c \frac{dx}{1 - u(b)/b} = \frac{b}{k} \,.$$

b) Let $u(x) = a - k \,.\, (x - b)$. Then $w(x) \geqq u(x)$ for all $x \in [b, c]$ and hence

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \geqq \int_b^c \frac{x \,.\, dx}{x - u(x)} = \int_b^c \frac{x \,.\, dx}{(k + 1) \,.\, x - k \,.\, b - a} \,;$$

by an easy computation we obtain the required expression, q.e.d.

3.2. Lemma. *Let* $a, b, c \in (0, T)$, $a < b < c$, *let* $w(x) < x$ *for all* $x \in [b, c]$, *let* $w(c) = b$, $w(b) = a$ *and let* k *be a real.*

a) *If* $k \in (0, 1)$ *and* $w(y) - w(x) \geqq k \,.\, (y - x)$ *for all* $x, y \in [b, c]$, $x < y$ *then*

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \leqq \frac{c}{k} \,.$$

b) *If* $k > 1$ *and* $0 \leqq w(y) - w(x) \leqq k \,.\, (y - x)$ *for all* $x, y \in [b. c]$, $x < y$ *then*

$$\int_b^c \frac{x \,.\, dx}{x - w(x)} \leqq c + b \,.\, \ln \frac{(c - b) \,.\, k}{b - a} \,.$$

c) *If the function $w(x)/x$ is non-increasing on $[b, c]$ then*

$$\int_b^c \frac{x \, . \, dx}{x - w(x)} \leqq b \, . \, \frac{c - b}{b - a} \, .$$

Proof. a) Let $u(x) = b + k \, . \, (x - c)$. Then $w(x) \leqq u(x)$ for all $x \in [b, c]$ and since $u(x)/x$ is monotone on $[b, c]$ we have

$$\int_b^c \frac{x \, . \, dx}{x - w(x)} \leqq \int_b^c \frac{x \, . \, dx}{x - u(x)} \leqq \max \left(\frac{b \, . \, (c - b)}{b - u(b)} \, , \; \frac{c \, . \, (c - b)}{c - u(c)} \right) =$$

$$= \max \left(\frac{b}{k} \, , \; c \right) \leqq \frac{c}{k} \, .$$

b) Let $d = b + (b - a)/k$ and $u(x) = b + k \, . \, (x - d)$ for $x \in [b, d]$, $u(x) = b$ for $x \in (d, c]$. Then $w(x) \leqq u(x)$ for all $x \in [b, c]$ and therefore

$$\int_b^c \frac{x \, . \, dx}{x - w(x)} \leqq \int_b^c \frac{x \, . \, dx}{x - u(x)} = \int_b^d \frac{x \, . \, dx}{x - u(x)} + \int_d^a \frac{x \, . \, dx}{x - u(x)} \leqq$$

$$\leqq d + \left(c - d + b \, . \, \ln \frac{c - b}{b - a} \right) = c + b \, . \, \ln \frac{k \, . \, (c - b)}{b - a} \, .$$

c) We have

$$\int_b^c \frac{x \, . \, dx}{x - w(x)} = \int_b^c \frac{dx}{1 - w(x)/x} = \int_b^c \frac{dx}{1 - a/b} = b \, . \, \frac{c - b}{b - a} \, , \quad \text{q.e.d.}$$

3.3. Lemma. *Let $w(x) < x$ for all $x \in (0, T)$, let $k > 1$ and $w(y) - w(x) \leqq \leqq k \, . \, (y - x)$ for all $x, y \in (0, T)$, $x < y$ or $w(y) - w(x) \geqq -k \, . \, (y - x)$ for all $x, y \in (0, T)$, $x < y$. Let there be $b \in (0, T)$ such that the point $[b, b]$ is a limit point of the graph of $w(x)$. Then there are $a, c \in (0, T)$, $a < c$ such that*

$$\int_a^c \frac{x \, . \, dx}{x - w(x)} = \infty \, .$$

Proof. Let e.g. $w(y) - w(x) \leqq k \, . \, (y - x)$ for $x < y$ and let the point $[b, b]$ be a limit point of the graph of $w(x)$. It can be easily shown that $b - w(x) \leqq \leqq k \, . \, (b - x)$ for all $x \in (0, b)$. Take $c = b$ and $a \in (0, c)$. Then

$$\int_a^c \frac{x \, . \, dx}{x - w(x)} \geqq \int_a^b \frac{a \, . \, dx}{(k - 1) \, . \, (b - x)} = \infty \, .$$

If $w(y) - w(x) \geqq -k \, . \, (y - x)$ for $x < y$, the proof is similar. We choose $a = b$ and $c \in (b, T)$. Q.e.d.

3.4. Lemma. *Let $w(x) < x$ for all $x \in (0, T)$, let $w(x)$ be Lebesgue measurable on $(0, T)$ nad let no point $[b, b]$, $b \in (0, T)$ be a limit point of the graph of $w(x)$. Then for all $a, c \in (0, T)$, $a < c$,*

$$\int_a^c \frac{x \,.\, dx}{x - w(x)} < \infty \,.$$

Proof. Take $a, c \in (0, T)$, $a < c$. Then there exists a positive number ε such that $w(x) \leqq x - \varepsilon$ for all $x \in [a, c]$ and hence

$$\int_a^c \frac{x \,.\, dx}{x - w(x)} \leqq \int_a^c \frac{c \,.\, dx}{\varepsilon} < \infty \,. \quad \text{Q.e.d.}$$

3.5. Lemma. *Let at least one of the functions $u(x)$, $v(x)$ be non-decreasing on $(0, t)$, let $u(x) \leqq v(x)$ for every $x \in (0, t)$. Let $a, b \in (0, T)$, $V(b) < \infty$ and $\lim_{n \to \infty} u^n(a) = 0$. Then $U(a) < \infty$.*

Proof. Without loss of generality we may assume $a \leqq b < t$. Now we can prove $u^n(a) \leqq v^n(b)$ by induction. If $n = 0$ then obviously $u^n(a) = a \leqq b = v^n(b)$. For $n \in N$ we have

$$u^n(a) = u(u^{n-1}(a)) \leqq X \leqq v(v^{n-1}(b)) = v^n(b)$$

where $X = u(v^{n-1}(a))$ if $u(x)$ is non-decreasing and $X = v(u^{n-1}(a))$ if $v(x)$ is non-decreasing. Comparing $U(a)$, $V(b)$ term by term we obtain $U(a) \leqq V(b) < \infty$, q.e.d.

4. CRITERIA OF SMALLNESS

Let t be a fixed element of $(0, T)$; it is suitable to imagine it small. An obvious necessary condition for a function $w(x)$ to be small is

$$\text{(4.0)} \qquad \lim_{n \to \infty} w^n(x) = 0 \quad \text{for every} \quad x \in (0, T)\,;$$

this condition will be called the zero-condition for the function $w(x)$. It is easy to see that if a function $w(x)$ satisfies the zero-condition and is small on $(0, t)$ then it is small (i.e. small on the whole $(0, T)$). The problem whether a function $w(x)$ is small is usually much more difficult than the problem whether $w(x)$ satisfies the zero-condition. Therefore it is usually reasonable first to verify (4.0) and only if it holds to find out whether $w(x)$ is small. Hence it is suitable to investigate smallness on $(0, t)$.

We shall also assume

$$\text{(4.1)} \qquad w(x) < x \quad \text{for all} \quad x \in (0, t]$$

in most theorems. The condition (4.1) is obviously very natural even if it is not necessary for $w(x)$ to be small (see Example 6.5). Our basic result is the following theorem.

4.1. Theorem. *Let $w(x)$ satisfy (4.1) and (4.0), let there be a real k such that*

$$w(y) - w(x) \leqq k . (y - x) \quad \textit{for all} \quad x, y \in (0, t], \quad x < y \tag{4.2}$$

or

$$w(y) - w(x) \geqq k . (y - x) \quad \textit{for all} \quad x, y \in (0, t], \quad x < y, \tag{4.3}$$

and let

$$\int_0^t \frac{x . dx}{x - w(x)} < \infty . \tag{4.4}$$

Then the function $w(x)$ is small.

Proof. We may obviously assume $|k| > 1$ and prove only $W(a) < \infty$ for all $a \in (0, t)$. Denote $a_n = w^n(a)$ for all $n \in N \cup \{0\}$. The sequence $(a_0, a_1, a_2, \ldots)$ is decreasing and its limit is 0. Now let us distinguish two cases.

If (4.2) holds then using Lemma 3.1a for $b = a_n$, $c = a_{n-1}$ we obtain

$$W(a) = a_0 + \sum_{n=1}^{\infty} a_n \leqq a_0 + \sum_{n=1}^{\infty} k . \int_{a_n}^{a_{n-1}} \frac{x . dx}{x - w(x)} =$$

$$= a_0 + k . \int_0^a \frac{x . dx}{x - w(x)} \leqq a_0 + k . \int_0^t \frac{x . dx}{x - w(x)} < \infty .$$

Let (4.3) hold. Using Lemma 3.1b we obtain

$$\infty > \int_0^a \frac{x . dx}{x - w(x)} = \sum_{n=1}^{\infty} \int_{a_n}^{a_{n-1}} \frac{x . dx}{x - w(x)} \geqq$$

$$\geqq \frac{a_{n+1}}{-k + 1} . \ln \frac{(a_n - a_{n+1}) + (-k + 1) . (a_{n-1} - a_n)}{a_n - a_{n+1}} .$$

Now we can use Lemma 2.1. It implies that $a_1 + a_2 + a_3 + \ldots < \infty$ and hence $W(a) = a_0 + a_1 + a_2 + \ldots < \infty$, q.e.d.

Theorem 4.1 shows that (4.4) is a sufficient smallness condition for a rather large class of functions $w(x)$. Generally speaking, it is not a necessary condition (see Example 6.7). However, (4.4) can turn out to be a necessary and sufficient smallness condition if we restrict the class of functions $w(x)$ considered. Some convenient restrictions are given in the next three theorems. In their proofs the necessity of (4.4) is verified only, the sufficiency being obvious consequence of Theorem 4.1.

4.2. Theorem. *Let $w(x)$ satisfy (4.0) and (4.1), let there be a positive real k such that (4.3) holds. Then the function $w(x)$ is small if and only if (4.4) holds.*

Proof. We may obviously assume $k < 1$. Let $w(x)$ be small. Denote $a_n = w^{n-1}(t)$ for all $n \in N$. By Lemma 3.2 we have

$$\int_0^t \frac{x \,.\, dx}{x - w(x)} = \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \frac{x \,.\, dx}{x - w(x)} \leqq \sum_{n=1}^{\infty} \frac{a_n}{k} = \frac{W(t)}{k} < \infty \,. \quad \text{Q.e.d.}$$

4.3. Theorem. *Let $w(x)$ satisfy (4.0) and (4.1) and let the function $w(x)/x$ be non-increasing on $(0, t]$. Then the function $w(x)$ is small if and only if (4.4) holds.*

Proof. Let $w(x)$ be small. Denote $a_n = w^{n-1}(t)$ for all $n \in N$. Then

$$\int_0^t \frac{x \,.\, dx}{x - w(x)} = \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \frac{x \,.\, dx}{x - w(x)} \leqq \sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} \,.\, a_{n+1} < \infty \,;$$

the first inequality follows from Lemma 3.2c and the other from Lemma 2.2, q.e.d.

4.4. Theorem. *Let $w(x)$ satisfy (4.0) and (4.1), let $w(x)$ be non-decreasing and let there be a real k such that (4.2) holds. Then $w(x)$ is small if and only if (4.4) holds.*

Proof. We may obviously assume $k > 1$. Let $w(x)$ be small. Denote $a_n = w^{n-1}(t)$ for all $n \in N$. It holds

$$\int_0^t \frac{x \,.\, dx}{x - w(x)} = \sum_{n=1}^{\infty} \int_{a_{n+1}}^{a_n} \frac{x \,.\, dx}{x - w(x)} \leqq \sum_{n=1}^{\infty} \left(a_n + a_{n+1} \,.\, \ln \frac{(a_n - a_{n+1}) \,.\, k}{a_{n+1} - a_{n+2}} \right) =$$

$$= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} a_{n+1} \,.\, \ln \frac{(a_n - a_{n+1}) \,.\, k}{a_{n+1} - a_{n+2}} < \infty \,.$$

The first inequality follows from Lemma 3.2b, the second from $W(t) < \infty$ and Lemma 2.3. Q.e.d.

Up to now we have tried to find smallness conditions which were as general as possible. Now we are going to give some more easily applicable conditions. We begin with a simple theorem for verifying (4.0).

4.5. Theorem. *A continuous function $w(x)$ satisfies the zero-condition if and only if $w(x) < x$ for all $x \in (0, T)$.*

Proof. Let $w(x) < x$ for all $x \in (0, T)$, and $a \in (0, T)$. Denote $a_n = w^n(a)$ for all $n \in N \cup \{0\}$. The decreasing sequence $(a_0, a_1, a_2, \ldots)$ has a limit $b \geqq 0$. If b is positive then $w(b) = b$, which contradicts the assumption. Therefore $b = 0$.

Conversely, let $w(a) \geqq a$ for some $a \in (0, T)$. We have to find b such that $W(b) = \infty$. If $W(a) = \infty$ take $b = a$. Otherwise there is $n \in N$ such that $w^n(a) \geqq w^{n+1}(a)$, $w^n(a) \leqq a$. Denote $c = w^n(a)$. It holds $w(c) \leqq c$, $w(a) \geqq a$, and therefore there is $b \in [c, a]$ such that $w(b) = b$. Then obviously $W(b) = \infty$, q.e.d.

Now we shall reformulate Theorems 4.1–4.4 for the functions $w(x)$ which have the first derivative on $(0, t)$. The proofs of both reformulated theorems are very easy and we shall omit them.

4.6. Theorem. *Let $w(x)$ satisfy (4.0) and (4.1), let $w'(x)$ exist for all $x \in (0, t]$, let there be a real k such that $w'(x) \leqq k$ for all $x \in (0, t]$ or $w'(x) \geqq k$ for all $x \in (0, t]$ and let (4.4) hold. Then the function $w(x)$ is small.*

4.7. Theorem. *Let k be a positive real, let $w(x)$ satisfy (4.0) and (4.1), let $w'(x)$ exist for all $x \in (0, t]$ and let at least one of the following conditions hold:*

(i) $w'(x) \geqq k$ *for all* $x \in (0, t]$;

(ii) $w'(x) \leqq w(x)/x$ *for all* $x \in (0, t]$;

(iii) $0 \leqq w'(x) \leqq k$ *for all* $x \in (0, t]$.

Then the function $w(x)$ is small if and only if (4.4) holds.

4.8. Corollary. *Let a function $w(x)$ satisfy (4.0) and (4.1), let r be a real, $t < e^{-e}$ and let for all $x \in (0, t)$ either*

$$w(x) = x - x^{2-r}$$

or

$$w(x) = x - x^2 \,.\, |\ln x|^{1+r}$$

or

$$w(x) = x - x^2 \,.\, |\ln x| \,.\, (\ln |\ln x|)^{1+r} \,.$$

Then $w(x)$ is small if and only if $r > 0$.

Proof. Let e.g. $w(x) = x - x^2 \,.\, |\ln x|^{1+r}$. Then

$$\int_0^t \frac{x \,.\, dx}{x - w(x)} = \int_0^t \frac{dx}{x \,.\, |\ln x|^{1+r}} = \int_{|\ln t|}^{\infty} \frac{dy}{y^{1+r}} \,.$$

The last integral converges if and only if $r > 0$. Now it suffices to use Theorem 4.7. The other two cases for $w(x)$ are similar. It is also clear how to continue the sequence of formulae for $w(x)$; then the number e^{-e} must be replaced by a smaller number depending on the considered formula. Q.e.d.

4.9. Corollary. *Let a function $w(x)$ satisfy (4.0) and (4.1) and let*

$$w(x) = c_1 \,.\, x + c_2 \,.\, x^2 + c_3 \,.\, x^3 + \ldots$$

for all $x \in (0, t]$ where c_i are real constants. Then $w(x)$ is small if and only if $c_1 < 1$.

The corollary is an immediate consequence of Theorem 4.7. Notice that if $w(x) = c_0 + c_1 \,.\, x + c_2 \,.\, x^2 + \ldots$ then $w(x)$ can satisfy (4.1) only if $c_0 = 0$.

5. COMPARATIVE CRITERIA

In the preceding section we have given a list of small functions. Now we give some theorems which enable us to conclude that a given function is small if it is related in a certain way to some other small functions. As in Section 4, t denotes a fixed element of $(0, T)$.

5.1. Theorem. *Let $u(x)$ satisfy the zero-condition, let at least one of the functions $u(x)$, $v(x)$ be non-decreasing on $(0, t)$, let $u(x) \leqq v(x)$ for every $x \in (0, t)$ and let the function $v(x)$ be small on $(0, t)$. Then the function $u(x)$ is small.*

Proof. Take an arbitrary $x \in (0, T)$. The zero-condition implies that there is $n \in N$ such that $a = u^n(x) \in (0, t)$. Obviously $U(x) < \infty$ if and only if $U(a) < \infty$. Since the function $v(x)$ is small on $(0, t)$ it holds $V(a) < \infty$. Then by Lemma 3.5 (used for $b = a$) we have $U(a) < \infty$, and hence $U(x) < \infty$, q.e.d.

Another corollary of Lemma 3.5 follows by taking $u(x) = v(x) = w(x)$.

5.2. Theorem. *Let $w(x)$ satisfy the zero-condition and be non-decreasing on $(0, t]$. Then $w(x)$ is small if and only if $W(t) < \infty$.*

If we want to use Theorem 5.1 it is sometimes useful to extend the list of small functions by the theorem below.

5.3. Theorem. *Let $w(x)$ satisfy the assumptions of Theorem 4.7, $r \in (0, 1)$, let $u(x) = r \,.\, x + (1 - r) \,.\, w(x)$ for all $x \in (0, t]$ and let $u(x)$ satisfy the zero-condition. Then $u(x)$ is small if and only if $w(x)$ is small.*

Proof. The function $u(x)$ also satisfies the assumptions of Theorem 4.7 and

$$\int_0^t \frac{x \,.\, dx}{x - u(x)} = \frac{1}{1 - r} \,.\, \int_0^t \frac{x \,.\, dx}{x - w(x)} \,.$$

The integral on the left converges if and only if the integral on the right converges. Now it is sufficient to use Theorem 4.7. Q.e.d.

We give one example how to use Theorem 5.3.

5.4. Corollary. *If $w(x)$ satisfies the zero-condition, $r > 0$ and*

$$\liminf_{x \to 0+} \frac{x - w(x)}{x^2 \,.\, |\ln x| \,.\, (\ln |\ln x|)^{1-r}} > 0$$

then the function $w(x)$ is small.

The assumption that at least one of $u(x)$, $v(x)$ is non-decreasing cannot be omitted in Theorem 5.1. (See Examples 6.3, 6.4.) However, it can be replaced by the continuity of $v(x)$. We shall see that from the following theorem.

5.5. Theorem. (J. Smítal). *Let a continuous function $w(x)$ be small and let for all $x \in (0, T)$*

$$u(x) = \sup \{w(y);\ y \in (0, x]\} .$$

Then the function $u(x)$ is small.

Proof. Let $b \in (0, T)$. There is the least real a_0 satisfying $w(a_0) = w(b)$. Denote $a_n = u^n(a_0)$ for all $n \in N$. Since $w(x)$ is continuous and small, we have $w(x) < x$ for all $x \in (0, T)$. Then $u(x) < x$ for all $x \in (0, T)$, $u(x)$ being continuous. Therefore $u(x)$ satisfies the zero-condition.

Denote by G_0 the set of all $x \in (0, T)$ such that $u(x)$ is constant in a (sufficiently small) neighbourhood of x. Further, denote for all $n \in N$

$$G_n = \{x \in (0, T);\ u^n(x) \in G_0\} ,$$

$$A_0 = [a_1, a_0] - G_0 , \quad A_n = A_{n-1} - G_n .$$

All sets $A_0, A_1, A_2, \ldots$ are closed and $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$. We shall show $A_n \neq \emptyset$ for all $n \in N \cup \{0\}$. If we denote $u^n(X) = \{u^n(x);\ x \in X\}$ for every $X \subseteq (0, T)$ then it holds

$$u^{n+1}(A_n) = [a_{n+2}, a_{n+1}] , \tag{5.1}$$

$$u^n(A_n) \cap G_0 = \emptyset . \tag{5.2}$$

For $n = 0$, (5.1) and (5.2) obviously hold. Let they be true for some n; we prove them for $n + 1$. It holds $u^{n+1}(A_{n+1}) = u^{n+1}(A_n - G_{n+1}) \subseteq [a_{n+2}, a_{n+1}] - G_0$, hence $u^{n+1}(A_{n+1}) \cap G_0 = \emptyset$. Further, we have $u^{n+2}(A_{n+1}) = u^{n+2}(A_n - G_{n+1}) = u(u^{n+1}(A_n - G_{n+1})) \supseteq u(u^{n+1}(A_n) - u^{n+1}(G_{n+1})) = u([a_{n+2}, a_{n+1}] - G_0) = [a_{n+3}, a_{n+2}]$. The converse inclusion is obvious: $u^{n+2}(A_{n+1}) \subseteq u^{n+2}([a_1, a_0]) = [a_{n+3}, a_{n+2}]$.

We have proved (5.1) and (5.2). (5.1) implies $A_n \neq \emptyset$ for all $n \in N$ and since $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are closed sets we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Take $c \in \bigcap_{n=1}^{\infty} A_n$. It holds $w^n(c) = u^n(c)$ for all $n \in N \cup \{0\}$. Therefore $U(c) = W(c) < \infty$. Now we use Theorem 5.2. Since $u(x)$ is obviously non-decreasing and satisfies (4.1), it is small, q.e.d.

5.6. Corollary. *A continuous function $w(x)$ is small if and only if $w(x) < x$ for all $x \in (0, T)$ and $W(t) < \infty$.*

5.7. Corollary. *Let $v(x)$ be a continuous small function and let $u(x) \leqq v(x)$ for all $x \in (0, T)$. Then the function $u(x)$ is small.*

6.1. Example. A function $w(x)$ such that $w(x) < x$ for all $x \in (0, T)$ which does not satisfy the zero-condition.

Let $w(x) = x/2$ for $x \in (0, t]$, $w(x) = (x + t)/2$ for $x \in (t, T)$. Then obviously $w(x) < x$ for all $x \in (0, T)$. However, for every $a \in (t, T)$ it holds $\lim_{n\to\infty} w^n(a) = t > 0$, hence $w(x)$ does not satisfy the zero-condition.

Remark. The just constructed function $w(x)$ is continuous from the left on $(0, T)$. From Theorem 4.5 we know that it cannot be continuous. It is easy to see that it cannot be even continuous from the right on $(0, T)$.

6.2. Example. A function $w(x)$ satisfying the zero-condition, $w(x) < x$ for all $x \in (0, T)$ and $W(t) < \infty$ which is not small on $(0, t)$.

Choose $r \in (0, t)$ such that r/t is irrational (e.g. $r = t/\sqrt{2}$) and for all $x \in (0, T)$ define $w(x) = r/(r/x + 1)$ if $r/x \in N$, $w(x) = x/2$ otherwise. The function $w(x)$ obviously satisfies (4.0) and $w(x) < x$. It holds also $W(t) = t + t/2 + t/4 + t/8 + \dots$ $\dots < \infty$. However, $W(r) = r + r/2 + r/3 + r/4 + \dots = \infty$, hence $w(x)$ is not small on $(0, t)$.

6.3. Example. Functions $u(x), v(x)$ satisfying the zero-condition and $u(x) <$ $< v(x) < x$ for all $x \in (0, T)$, such that $v(x)$ is small and $U(x) = \infty$ for all $x \in (0, T)$.

Let for all $n \in N$

$$u(t/(2n)) = t/(4n + 1)\,, \qquad v(t/(2n)) = t/(4n)\,,$$
$$u(t/(2n - 1)) = t/(2n + 1)\,, \quad v(t/(2n - 1)) = t/(2n)$$

and for all $x \in (0, T)$ such that $t/x \notin N$ let

$$u(x) = t/(2n + 3)\,, \quad v(x) = t/(2n + 2)\,,$$

where n is the integer part of t/x. Then for every $x \in (0, T)$ there is $m \in N$ such that $u(x) = t/(2m + 1)$, $v(x) = t/(2m)$ and we have

$$U(x) = x + U(u(x)) = x + t/(2m + 1) + t/(2m + 3) + t/(2m + 5) + \dots = \infty\,,$$
$$V(x) = x + V(v(x)) = x + t/(2m) + t/(4m) + t/(8m) + \dots < \infty\,.$$

Remark. Neither $u(x)$ nor $v(x)$ can be non-decreasing, and $v(x)$ cannot be continuous.

6.4. Example. A small function $v(x)$ and a continuous but not small function $u(x)$ satisfying the zero-condition and $u(x) \leqq v(x)$ for all $x \in (0, T)$.

Let for all $n \in N$

$$u(t/(2n - 1)) = t/(2n + 3)\,, \quad u(t/(2n)) = t/(4n + 2)\,,$$

let $u(x) = t/5$ for all $x \in (t, T)$ and let $u(x)$ be defined by linear interpolation on each interval $(t/(n + 1), t/n)$, $n \in N$. Let for all $n \in N$

$$v(t/(2n)) = t/(4n),$$

$$v(x) = t/(2n + 2) \quad \text{for} \quad x \in (t/(2n + 1), t/(2n - 1)] - \{t/2n\}$$

and let $v(x) = t/2$ for $x \in (t, T)$. Then obviously $u(x)$ is continuous and $u(x) \leqq v(x) < x$ for all $x \in (0, T)$. It is easy to verify that $v(x)$ is small. However, $u(x)$ is not small because $U(t) = t + t/5 + t/9 + t/13 + \ldots = \infty$.

Remark. If we replace the linear interpolation by a finer construction we can reach e.g. that $u(x)$ has all derivatives on $(0, T)$.

6.5. Example. A small function $w(x)$ such that $w(x) > x$ for all but countably many $x \in (0, T)$.

Let $(t_0, t_1, t_2, \ldots)$ be an increasing sequence, $t_0 = t$ and $\lim_{n\to\infty} t_n = T$. Denote $t_{-n} = t/2^n$ for all $n \in N$, $Z = \bigcup_{n=0}^{\infty} \{t_{-n}, t_n\}$ and for all integers n and all $x \in (0, T)$

$$w(x) = t_{n+1} \quad \text{if} \quad x \in (t_{n-1}, t_n), \qquad w(x) = t_{n-1} \quad \text{if} \quad x = t_n.$$

Then for all $x \in (0, T) - Z$ we have $w(x) > x$. In spite of that the function $w(x)$ is small: For each x, $W(x)$ converges if and only if $t_{-1} + t_{-2} + t_{-3} + \ldots < \infty$, which obviously holds.

6.6. Example. A function $w(x)$ satisfying (4.0), (4.1) and (4.4) which is not small. Let for all $x \in (0, T)$

$$w(x) = x/2 \quad \text{if} \quad t/x \notin N, \qquad w(x) = t/(t/x + 1) \quad \text{if} \quad t/x \in N.$$

Then $w(x)$ satisfies (4.0) and (4.1). It also satisfies (4.4) because $w(x)$ can be replaced by $x/2$ in the integral. However, $w(x)$ is not small since $W(t) = t + t/2 + t/3 + \ldots \ldots = \infty$.

Remark. The function $w(x)$ just constructed is not continuous. However, a continuous function with all the mentioned properties can be found. It could be constructed as an "approximation" of the function $w(x)$. Therefore in Theorem 4.1 the conditions (4.2) or (4.3) cannot be replaced by continuity of $w(x)$.

6.7. Example. A non-decreasing small function $w(x)$ satisfying (4.0) and (4.1) which does not satisfy (4.4).

Denote $a_n = t/2^n$ for all $n \in N$, and define

$$w(x) = a_{n+1} \quad \text{for all} \quad x \in (a_{n+1}, a_n], \qquad w(x) = a_1 \quad \text{for all} \quad x > a_1.$$

Then obviously $w(x)$ satisfies (4.1) and is non-decreasing. Further, $w(x)$ is small since $W(x) \leqq x + a_1 + a_2 + a_3 + \ldots < \infty$, and hence it satisfies (4.0), too. However, (4.4) does not hold. Moreover, for every positive k we have

$$\int_0^k \frac{x \,.\, dx}{x - w(x)} = \infty$$

because the graph of $w(x)$ has a limit point $[a_n, a_n]$ for some $a_n \in (0, k)$.

Remark. There is also an increasing continuous small function $w(x)$ satisfying (4.0) and (4.1) and not satisfying (4.4). It can be constructed as an "approximation" of the function from Example 6.7. Hence we cannot replace (4.3) with a positive k by the assumption that $w(x)$ is increasing in Theorem 4.2. Analogously we cannot replace the assumption (i) in Theorem 4.7 by the assumption $w'(x) > 0$.

6.8. Example. A decreasing sequence $(a_1, a_2, a_3, \ldots)$ of positive reals such that $a_1 + a_2 + a_3 + \ldots < \infty$, $a_{n+1} - a_{n+2} \leqq a_n - a_{n+1}$ for all $n \in N$ and

$$\sum_{n=1}^{\infty} a_n \,.\, \ln \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} = \infty \,.$$

(Compare with Lemma 2.3.)

Let $c_1 = 1$, $c_{n+1} = c_n \,.\, e^{c_n}$, $b_n = 1/c_n$, $a_n = b_n + b_{n+1} + b_{n+2} + \ldots$ for all $n \in N$. Then

$$\sum_{n=1}^{\infty} a_n \,.\, \ln \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} \geqq \sum_{n=1}^{\infty} \frac{1}{c_n} \,.\, \ln \frac{c_{n+1}}{c_n} = \sum_{n=1}^{\infty} 1 = \infty \,.$$

The other conditions can be easily verified.

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