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THE SPECTRUM OF THE 6-LAPLACIAN ON KÄHLER MANIFOLDS

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Summary. Given two Kähler manifolds whose spectra of Laplacian acting on the 6-forms coincide, it is shown that one of them is of real constant holomorphic sectional curvature h if and only if the other is, provided their complex dimension n satisfies $n = 4, 5$ or $7 \leqq n \leqq 12$ or $18 \leqq n \leqq 264$. A similar result is established for Kähler-Einstein manifolds.
Keywords: Kähler manifold, Kähler-Einstein manifold, constant holomorphic sectional curvature, spectrum of the 6-Laplacian.

1. INTRODUCTION

Let (M, J, g) be an n-dimensional Kähler manifold (all manifolds are assumed to be compact, connected and of complex dimension $n > 1$) with complex structure J and Kähler metric g. By Δ^p we denote the Laplacian acting on p-forms on M. Then we have the spectrum for each p:

$$\mathrm{Spec}^p(M, g) = \{0 \leqq \lambda_{1,p} \leqq \lambda_{2,p} \leqq \ldots \to +\infty\},$$

where each eigenvalue is repeated as many times as its multiplicity indicates. It is well known that $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^{2n-p}(M, g)$ and, immediately from Hodge's theory. $0 \in \mathrm{Spec}^p(M, g)$ if and only if $\beta_p(M) \neq 0$ (and 0 has multiplicity $\beta_p \neq 0$).

One of the most interesting problems on spectrum is the following: "Let (M, J, g) and (M', J', g') be compact Kähler manifolds with $\mathrm{Spec}^p(M, g) = \mathrm{Spec}^p(M', g')$ for a fixed but arbitrary p. Is it true that (M, J, g) is of real constant holomorphic sectional curvature h if and only if (M', J', g') is of real constant holomorphic sectional curvature h' and $h = h'$?".

The answer to the problem is affirmative for $p = 0$, $n \leqq 6$, [4]; $p = 1$, $8 \leqq n \leqq 51$, [5]; $p = 2$, $n = 3, 4, 7$ or $9 \leqq n \leqq 94$, [6]; $p = 3$, $11 \leqq n \leqq 136$, [3]; $p = 4$, $5 \leqq n \leqq 9$ or $12 \leqq n \leqq 179$, [3] and $p = 5$, $n = 4$, [2].

In this paper we study the effect of $\mathrm{Spec}^6(M, g) = \mathrm{Spec}^6(M', g')$. To this aim we apply Patodi's results [1] to the coefficients of the Minakshisundaram-Pleijel-Gaffney asymptotic expansion.

2. PRELIMINARIES

Let M be a Kähler manifold of complex dimension n, If $(\theta^1, \ldots, \theta^n)$ form a local field of unitary coframes, then the Kähler metric g and the fundamental 2-form ϕ are given respectively by

$$g = \tfrac{1}{2}\sum(\theta^i \otimes \bar\theta^i + \bar\theta^i \otimes \theta^i)\,,$$

$$\phi = \frac{\sqrt{-1}}{2}\sum \theta^i \wedge \bar\theta^i\,.$$

Let $\Omega^i_j = \sum R^i_{jkl}\,\theta^k \wedge \bar\theta^l$ be the curvature form of M. Then the curvature tensor R is the tensor field with local components R^i_{jkl}. The Ricci tensor E and the scalar curvature τ are given respectively by

$$E = \tfrac{1}{2}\sum(R_{ij}\theta^i \wedge \bar\theta^j + \bar R_{ij}\bar\theta^i \wedge \theta^j)\,,$$

$$\tau = 2\sum R_{ii}\,,$$

where $R_{ij} = 2\sum R^k_{ikj}$. We denote by $|R|$ and $|E|$ the lengths of R and E, respectively.

The Minakshisundaram-Pleijel-Gaffney formula for $\mathrm{Spec}^6(M, g)$ reads

$$\sum e^{\lambda_{k,6}t} \underset{t\to 0}{\sim} (4\pi t)^{-n}\sum_{i=0}^{\infty} a_{i,6}t^i\,, \tag{2.1}$$

where

$$a_{0,6} = \binom{2n}{6}\int_M \mathrm{d}M\,, \tag{2.2}$$

$$a_{1,6} = \left\{\frac{1}{6}\binom{2n}{6} - \binom{2n-2}{5}\right\}\int_M \tau\,\mathrm{d}M\,, \tag{2.3}$$

$$a_{2,6} = \frac{1}{360}\int_M \{a\tau^2 + b|E|^2 + c|R|^2\}\,\mathrm{d}M \tag{2.4}$$

with

$$\left\{\begin{aligned} a &= 5\binom{2n}{6} - 60\binom{2n-2}{5} + 180\binom{2n-4}{4},\\ b &= -2\binom{2n}{6} + 180\binom{2n-2}{5} - 720\binom{2n-4}{4},\\ c &= 2\binom{2n}{6} - 30\binom{2n-2}{5} + 180\binom{2n-4}{4}. \end{aligned}\right. \tag{2.5}$$

3. MAIN RESULTS

3.1. Theorem. *Let (M, J, g) and (M', J', g') be two compact Kähler manifolds with* $\mathrm{Spec}^6(M, g) = \mathrm{Spec}^6(M', g')$. *If n is the complex dimension of M, then for*

$n = 4, 5$ or $7 \leqq n \leqq 12$ or $18 \leqq n \leqq 264$, (M, J, g) is of real constant holomorphic sectional curvature h if and only if (M', J', g') is of real constant holomorphic sectional curvature h' and $h' = h$.

Proof. Let C, G, B be the Weyl conformal curvature tensor field, the Einstein tensor and the Bochner curvature tensor field, respectively, on (M, g). The components $(C_{ijkl}), (G_{ij}), (B_{ijkl})$ of C, G and B, respectively, are given by

$$(3.1) \qquad C_{ijkl} = R_{ijkl} - \frac{1}{n-1}(E_{jk}g_{il} - E_{jl}g_{ik} - g_{jk}E_{il} - g_{il}E_{ik}) + \\ + \frac{1}{(n-1)(1-2)}(g_{jk}g_{il} - g_{jl}g_{ik})\tau ;$$

$$(3.2) \qquad G_{ij} = E_{ij} - \frac{1}{n} g_{ij}\tau ;$$

$$(3.3) \qquad B_{ijkl} = R_{ijkl} + \frac{1}{n-1}(E_{jk}g_{il} - E_{jl}g_{ik} + E_{il}g_{jk} - \\ - E_{ik}g_{jl} + E_{jr}J^r_k J_{il} - E_{jr}J^r_l J_{ik} - E_{jr}J^r_l J_{ik} + \\ + J_{jk}E_{ir}J^r_l - J_{jl}E_{ir}J^r_k - 2E_{kr}J^r_l J_{ij} - 2E_{ir}J^r_j J_{kl}) + \\ + \frac{1}{(n+2)(n+4)}[g_{jk}g_{il} - g_{jl}g_{ik} - J_{jk}J_{il} - J_{jl}J_{il} - J_{jl}J_{ik} - 2J_{kl}J_{ij}]\tau .$$

Then we have

$$(3.4) \qquad |C|^2 = |R|^2 - \frac{4}{n-2}|E|^2 + \frac{2}{(n-1)(n-2)}\tau^2 ;$$

$$(3.5) \qquad |G|^2 = |E|^2 - \frac{1}{n}\tau^2 ;$$

$$(3.6) \qquad |B|^2 = |R|^2 - \frac{16}{n+4}|E|^2 + \frac{8}{(n+2)(n+4)}\tau^2 .$$

By means of (3.4), (3.5), (3.6) we arrange formula (2.4) to the form

$$(3.7) \qquad a_{2,6} = \frac{\binom{2n-4}{4}}{37800(n-3)(2n-7)}\int_M \left\{\alpha|B|^2 + \frac{\beta}{n+2}|G|^2 + \right. \\ \left. + \frac{\gamma}{2n(n+1)}\tau^2\right\} dM ,$$

where

$$\alpha = 8n^4 - 384n^3 + 7402n^2 - 38346n + 58320,$$

$$\beta = -8n^5 + 2232n^4 - 32206n^3 + 152058n^2 - 223596n - 6480,$$

$$\gamma = 40n^6 - 1528n^5 + 1947n^4 - 97214n^3 + 192642n^2 - 110178n - 3240.$$

If n is the complex dimension of M, then for $n = 4, 5$ or $7 \leqq n \leqq 12$ or $18 \leqq n \leqq \leqq 264$, we have

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0. \tag{3.8}$$

If we assume that the Kähler manifold (M', J', g') has constant holomorphic sectional curvature h', then relation (3.7) takes the form

$$a'_{2,6} = \frac{\binom{2n-4}{4}}{37800(n-3)(2n-7)} \frac{1}{2n(n+1)} \int_{M'} \gamma \, . \, \tau'^2 \, \mathrm{d}M' .$$

From the hypothesis $a'_{1,6} = a_{1,6}$ hence

$$\int_M \tau \, \mathrm{d}M = \int_{M'} \tau' \, \mathrm{d}M', \tag{3.9}$$

while $a'_{2,6} = a_{2,6}$ implies

$$\int_M \left\{ \alpha |B|^2 + \frac{\beta}{n+2} |G|^2 + \frac{\gamma}{2n(n+1)} \tau^2 \right\} \mathrm{d}M = \tag{3.10}$$

$$= \int_{M'} \frac{\gamma}{2n(2n+1)} \tau'^2 \, \mathrm{d}M' .$$

Since the holomorphic sectional curvature h' of (M', J', g') is constant, (3.9) implies

$$\int_M \tau^2 \, \mathrm{d}M \geqq \int_{M'} \tau'^2 \, \mathrm{d}M' . \tag{3.11}$$

Relation (3.10) by virtue of (3.8) and (3.11) gives $|B|^2 = 0$ and $|G|^2 = 0$ which imply $B = 0$ and $G = 0$. That is, (M, J, g) has constant holomorphic sectional curvature h. From (3.9) we obtain that $h = h'$. Q.e.d.

As an immediate consequence of the above theorem we have the following corollary:

3.2. Corollary. *The complex projective space $(P^n(\mathbb{C}), J_0, g_0)$ with the Fubini-Study metric g_0 and complex dimension n such that $n = 4, 5$ or $7 \leqq n \leqq 12$ or $18 \leqq n \leqq 264$, is completely characterized by the spectrum of the Laplacian on the exterior 6-forms.*

3.3. Theorem. *Let (M, J, g) and (M', J', g') be two compact Kähler-Einstein manifolds with* $\mathrm{Spec}^6(M, g) = \mathrm{Spec}^6(M', g')$ *(which implies that the complex* *dimension of M coincides with the complex dimension of M' and is equal to n). If $2 \leqq n \leqq 12$ or $n \geqq 18$ then (M, J, g) has real constant holomorphic sectional curvature h if and only if (M', J', g') has real constant holomorphic sectional curvature h' and $h' = h$.*

Proof. If the manifold (M, J, g) is an Einstein one, then we have $G = 0$ and the formula (3.7) takes the form

$$a_{2,6} = \frac{\binom{2n-4}{4}}{37800(n-3)(2n-7)} \int_M \left\{\alpha|B|^2 + \frac{\gamma}{2n(n+1)} \tau^2\right\} \mathrm{d}M \,. \tag{3.12}$$

If $2 \leqq n \leqq 12$ or $n \geqq 18$ then we have

$$\alpha > 0 \,, \quad \gamma > 0 \,. \tag{3.13}$$

From the hypothesis and the formulas (3.12), (3.13) we obtain that $|B|^2 = 0$ which implies $B = 0$. Hence the Kähler-Einstein manifold (M, J, g) has real constant holomorphic sectional curvature h. The relation (3.9) implies that $h = h'$. Q.e.d.

As a consequence of the above theorem we obtain

3.4. Corollary. *Let (M, J, g) be a compact Kähler-Einstein manifold whose complex dimenston is n. If $2 \leqq n \leqq 12$, or $n \geqq 18$ and* $\mathrm{Spec}^6 (M, g) = \mathrm{Spec}^6 (P^n(\mathbb{C}), g_0)$, *then (M, J, g) is holomorphically isometric to $(P^n(\mathbb{C}), J_0, g_0)$.*

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Souhrn

SPEKTRUM 6-LAPLACIÁNU NA KÄHLEROVÝCH VARIETÁCH

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Nechť jsou dány dvě Kählerovy variety se stejnými spektry Laplaciánu na 6-formách. Za předpokladu, že pro jejich komplexní dimenzi platí $n = 4, 5$ nebo $7 \leqq n \leqq 12$ nebo $18 \leqq n \leqq 264$,

je dokázáno, že první varieta má reálnou konstantní holomorfní řezovou křivost h právě tehdy, když totéž platí pro druhou varietu. Obdobný výsledek se dokazuje pro Kählerovy-Einsteinovy variety.

Резюме

СПЕКТР 6-ЛАПЛАСИАНА НА КЭЛЕРОВЫХ МНОГООБРАЗИЯХ

MIRCEA PUTA, ANDREI TÖRÖK

Пусть заданы два кэлеровых многообразия с одинаковыми спектрами оператора Лапласа на 6-формах. В статье доказано, что если для их комплексной размерности n выполнено одно из условий $n = 4, 5$ или $7 \leqq n \leqq 12$ или $18 \leqq n < 264$, то первое многообразие имеет действительную постоянную голоморфную секционную кривизну h тогда и только тогда, когда это же верно для второго многообразия. Аналогичный результат доказан для многообразий Кэлера-Эйнштейна.

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