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Elena Brožíková

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ON 3-BASIC QUASIGROUPS AND THEIR CONGRUENCES

ELENA BROŽÍKOVÁ, Praha

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Summary. A subgroup **G** of the full autotopy group of a given 3-basic quasigroup **Q** is said to be special if its component groups $\Gamma_1, \Gamma_2, \Gamma_3$ form a 3-basic quasigroup $(\Gamma_1, \Gamma_2, \Gamma_3; *)$, where $\alpha * \beta = \gamma \Leftrightarrow (\alpha, \beta, \gamma) \in \mathbf{G}$ for $\alpha \in \Gamma_1$, $\beta \in \Gamma_2$, $\gamma \in \Gamma_3$.

In this paper a one-to-one correspondence between special subgroups **G** and normal congruences ϱ of a given 3-basic quasigroup **Q** is proved.

Keywords: 3-basic quasigroup, autotopy, normal congruence, special autotopy group, $(n + 1)$-basic quasigroup.

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V. A. Beglarjan proved in [1] that every normal subgroup Γ of the associated group Q_τ of a given quasigroup $(Q, \cdot)$ induces a normal congruence R^Γ, and their corresponding decompositions fulfil $Q/R^\Gamma = Q/\Gamma$. Conversely, every normal congruence ϱ on a quasigroup $(Q, \cdot)$ induces a normal subgroup Γ^ϱ of the associated group Q_τ of $(Q, \cdot)$ such that the decomposition Q/Γ^ϱ is a refinement of the decomposition Q/ϱ. Further, every normal congruence ϱ on a quasigroup $(Q, \cdot)$ admits a refinement ϱ' such that $Q/\varrho' = Q/\Gamma^\varrho \leqq Q/\varrho$.

If we have a 3-basic quasigroup it is impossible to define an associated group. In the present paper we introduce as a certain compensation the connection between "special" subgroups of the full autotopy group of a given 3-basic quasigroup on one side and normal congruences of this quasigroup on the other side.

1. PRELIMINARIES

The quadruple $(Q_1, Q_2, Q_3; A)$, where Q_1, Q_2, Q_3 are non-void sets with the same cardinality and A is a map of $Q_1 \times Q_2$ onto Q_3 is called a *3-basic quasigroup* if in the equation $A(a_1, a_2) = a_3$ any two of the elements $a_1 \in Q_1$, $a_2 \in Q_2$, $a_3 \in Q_3$ uniquely determine the remaining one. If $Q_1 = Q_2 = Q_3$ we get a *usual quasigroup*. The triple of maps $\tau_i: Q_i \to Q_i'$, $i = 1, 2, 3$, is called *a homotopy* with components τ_1, τ_2, τ_3 of a 3-basic quasigroup $(Q_1, Q_2, Q_3; A)$ into a 3-basic quasigroup $(Q_1', Q_2', Q_3'; A')$ if $\tau_3 A(a_1, a_2) = A'(\tau_1 a_1, \tau_2 a_2)$ for all $a_1 \in Q_1$, $a_2 \in Q_2$. If in particular $Q_1 = Q_2 = Q_3$, $Q_1' = Q_2' = Q_3'$ and $\tau_1 = \tau_2 = \tau_3$ we obtain a quasigroup *homo-*

morphism. A homotopy with bijective components is called *an isotopy* and an isotopy of $(Q_1, Q_2, Q_3; A)$ onto itself is called *an autotopy*. The set of all autotopies $(\varphi_1, \varphi_2, \varphi_3)$ of a given 3-basic quasigroup forms a group under the composition $\circ$:

$$(\varphi_1, \varphi_2, \varphi_3) \circ (\varphi_1', \varphi_2' \varphi_3') = (\varphi_1 \circ \varphi_1', \varphi_2 \circ \varphi_2', \varphi_3 \circ \varphi_3') .$$

This group is called *a full autotopy group*.

Let $(Q_1, Q_2, Q_3; A_3)$ be a 3-basic quasigroup. Since any two of the elements a_1, a_2, a_3 in the equation $A_3(a_1, a_2) = a_3$ uniquely determine the remaining one, we can define operations

$$A_2(a_3, a_1) = a_2 , \quad A_1(a_2, a_3) = a_1$$

which are analogous to the left and right inverse operations of a usual quasigroup. Then

$$A_i(A_j(a_k, a_i), A_k(a_i, a_j)) = a_i$$

is fulfilled for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. Moreover, $(Q_2, Q_3, Q_1; A_1)$ and $(Q_3, Q_1, Q_2; A_2)$ are also 3-basic quasigroups which are called *cyclic parastrophes* of $(Q_1, Q_2, Q_3; A_3)$.

In the sequel we shall use symbols $\mathbf{Q}$, $\mathbf{Q}'$ as the notation for 3-basic quasigroups $(Q_1, Q_2, Q_3; A)$, $(Q_1', Q_2', Q_3'; A')$, respectively.

A congruence in a 3-basic quasigroup $\mathbf{Q}$ is a triple of equivalence relations ϱ_i of Q_i, $i = 1, 2, 3$, such that

(i) $x\varrho_1 y \Rightarrow A(x, z)\, \varrho_3\, A(y, z)$ for all $z \in Q_2$,

(ii) $x\varrho_2 y \Rightarrow A(z, x)\, \varrho_3\, A(z, y)$ for all $z \in Q_1$.

A congruence $(\varrho_1, \varrho_2, \varrho_3)$ of $\mathbf{Q}$ is said to be *normal* if

(iii) $A(x, z)\, \varrho_3\, A(y, z) \Rightarrow x\varrho_1 y$ for $x, y \in Q_1$ and $z \in Q_2$,

(iv) $A(z, x)\, \varrho_3\, A(z, y) \Rightarrow x\varrho_2 y$ for $x, y \in Q_2$ and $z \in Q_1$.

In the definition of normal congruence we can combine conditions (i) and (ii) to

(I) $x_1\varrho_1 y_1, x_2\varrho_2 y_2 \Rightarrow A(x_1, x_2)\, \varrho_3\, A(y_1, y_2)$ for $x_1, y_1 \in Q_1$ and $x_2, y_2 \in Q_2$,

and conditions (iii) and (iv) to

(II) if $A(x_1, x_2)\, \varrho_3\, A(y_1, y_2)$, then $x_1\varrho_1 y_1 \Leftrightarrow x_2\varrho_2 y_2$ for $x_1, y_1 \in Q_1$ and $x_2, y_2 \in Q_2$.

The connection between homotopies of a given 3-basic quasigroup and its normal congruences is well-known ([3]). Let (τ_1, τ_2, τ_3) be a homotopy of a 3-basic quasigroup $\mathbf{Q}$ onto a 3-basic quasigroup $\mathbf{Q}'$. Then we can define equivalence relations

$$R^{\tau_i} \subseteq Q_i \times Q_i \quad \text{by} \quad xR^{\tau_i}y \Leftrightarrow \tau_i x = \tau_i y , \quad i = 1, 2, 3 .$$

We shall show that $(R^{\tau_1}, R^{\tau_2}, R^{\tau_3})$ is a normal congruence on $\mathbf{Q}$.

(i) For $x, y \in Q_1$ let $xR^{\tau_1}y \Leftrightarrow \tau_1 x = \tau_1 y$, then $\tau_3 A(x, z) = A'(\tau_1 x, \tau_2 z) = A'(\tau_1 y, \tau_2 z) = \tau_3 A(y, z) \Rightarrow A(x, z)\, R^{\tau_3} A(y, z)$ for all $z \in Q_2$.

(ii) For $x, y \in Q_2$ let $xR^{\tau_2}y \Leftrightarrow \tau_2 x = \tau_2 y$, then $\tau_3 A(z, x) = A'(\tau_1 z, \tau_2 x) = A'(\tau_1 z, \tau_2 y) = \tau_3 A(z, y) \Rightarrow A(z, x)\, R^{\tau_3}\, A(z, y)$ for all $z \in Q_1$.

(iii) Let $A(x, z) R^{\tau_3} A(y, z) \Leftrightarrow \tau_3 A(x, z) = \tau_3 A(y, z)$, then $A'(\tau_1 x, \tau_2 z) = A'(\tau_1 y, \tau_2 z) \Rightarrow \tau_1 x = \tau_1 y \Rightarrow x R^{\tau_1} y$ for all $z \in Q_2$ and $x, y \in Q_1$.

(iv) Let $A(z, x) R^{\tau_3} A(z, y) \Leftrightarrow \tau_3 A(z, x) = \tau_3 A(z, y)$, then $A'(\tau_1 z, \tau_2 x) = A'(\tau_1 z, \tau_2 y) \Rightarrow \tau_2 x = \tau_2 y \Rightarrow x R^{\tau_2} y$ for all $z \in Q_1$ and $x, y \in Q_2$.

Conversely, every normal congruence $(\varrho_1, \varrho_2, \varrho_3)$ on a 3-basic quasigroup $\mathbf{Q}$ determines a homotopy of $\mathbf{Q}$ onto a convenient 3-basic quasigroup $\mathbf{Q}'$. Let $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ be a congruence on $\mathbf{Q} = (Q_1, Q_2, Q_3; \cdot)$ and let

$$C_a^{\varrho_i} = \{x \in Q_i;\ x \varrho_i a\}$$

be an element of the decomposition Q_i/ϱ_i for $a \in Q_i$, $i = 1, 2, 3$. Clearly $b \in C_a^{\varrho_i} \Rightarrow C_a^{\varrho_i} = C_b^{\varrho_i}$ and $b \notin C_a^{\varrho_i} \Rightarrow C_a^{\varrho_i} \cap C_b^{\varrho_i} = \emptyset$. Define a map $\odot: (Q_1/\varrho_1) \times (Q_2/\varrho_2) \to (Q_3/\varrho_3)$ by

$$(1) \qquad C_x^{\varrho_1} \odot C_y^{\varrho_2} = C_{x.y}^{\varrho_3} \quad \text{for all} \quad x \in Q_1,\ y \in Q_2 .$$

This map is independent of the choice of x, y because if $C_x^{\varrho_1} = C_{x'}^{\varrho_1}$ and $C_y^{\varrho_2} = C_{y'}^{\varrho_2}$, then $C_{x.y}^{\varrho_3} = C_{x'.y'}^{\varrho_3}$ for all $x, x' \in Q_1$ and $y, y' \in Q_2$. If $(\varrho_1, \varrho_2, \varrho_3)$ is a normal congruence, then $(Q_1/\varrho_1, Q_2/\varrho_2, Q_3/\varrho_3; \odot)$ is a 3-basic quasigroup. We need to verify that every equation

$$(2) \qquad C_a^{\varrho_1} \odot C_y^{\varrho_2} = C_c^{\varrho_3}, \quad a \in Q_1,\ y \in Q_2,\ c \in Q_3$$

and every equation

$$(2') \qquad C_x^{\varrho_1} \odot C_b^{\varrho_2} = C_c^{\varrho_3}, \quad x \in Q_1,\ b \in Q_2,\ c \in Q_3$$

are uniquely solvable by $C_y^{\varrho_2} \in Q_2/\varrho_2$ and $C_x^{\varrho_1} \in Q_1/\varrho_1$, respectively.

We have $C_a^{\varrho_1} \odot C_y^{\varrho_2} = C_{a.y}^{\varrho_3} = C_c^{\varrho_3}$ and consequently $(a \,.\, y)\, \varrho_3 c$. Let $y = b$ be the unique solution of the equation $a \,.\, y = c$ and let $y = b'$ be a solution of the relation $(a \,.\, y)\, \varrho_3 c$. Then $(a \,.\, b)\, \varrho_3 c$, $(a \,.\, b')\, \varrho_3 c$ and consequently $(a \,.\, b)\, \varrho_3 (a \,.\, b') \Rightarrow b \varrho_2 b' \Rightarrow C_b^{\varrho_2} = C_{b'}^{\varrho_2}$. (Here we have used the fact that $(\varrho_1, \varrho_2, \varrho_3)$ is a normal congruence.) The equation (2′) can be discussed similarly.

The quasigroup $\mathbf{Q}/\varrho = (Q_1/\varrho_1, Q_2/\varrho_2, Q_3/\varrho_3; \odot)$ is called the *factor-quasigroup of* $\mathbf{Q}$ *under* ϱ. The maps $\tau_i: Q_i \to Q_i/\varrho_i$ defined by $\tau_i a = C_a^{\varrho_i}$, $i = 1, 2, 3$, satisfy

$$\tau_3(x \,.\, y) = C_{x.y}^{\varrho_3} = C_x^{\varrho_1} \odot C_y^{\varrho_2} = (\tau_1 x) \odot (\tau_2 y) .$$

Consequently, (τ_1, τ_2, τ_3) is a homotopy of $\mathbf{Q}$ onto $\mathbf{Q}/\varrho$. ■

We shall still prove that

$$(3) \qquad x \,.\, C_y^{\varrho_2} = C_x^{\varrho_1} \,.\, y = C_{x.y}^{\varrho_3} \quad \text{for all} \quad x \in Q_1,\ y \in Q_2 .$$

Let us take an arbitrary element $z \in x \,.\, C_y^{\varrho_2}$ and let $b \in Q_2$ be another element satisfying the equation $z = x \,.\, b$. Then

$$\tau_3 z = \tau_3(x \,.\, b) = (\tau_1 x) \odot (\tau_2 b) = (\tau_1 x) \odot (\tau_2 y) = \tau_3(x \,.\, y) \Rightarrow$$

$z \varrho_3 (x \,.\, y)$ and thus $z \in C_{x.y}^{\varrho_3}$. Similarly, choose $z \in C_{x.y}^{\varrho_3}$, then $z \varrho_3 (x \,.\, y)$ and $\tau_3 z =$

$= \tau_3(x . y) = (\tau_1 x) \odot (\tau_2 y) = (\tau_1 x) \odot (\tau_2 b) = \tau_3(x . b)$ and $z\varrho_3(x . b)$ for all $b \in C_y^{\varrho_2}$, thus $z \in x . C_y^{\varrho_2}$ and $x . C_y^{\varrho_2} = C_{x.y}^{\varrho_3}$.

It can be verified analogously that $C_x^{\varrho_1} . y = C_{x.y}^{\varrho_3}$ for all $x \in Q_1$, $y \in Q_2$.

2. AUTOTOPIES

Let $\mathbf{Q} = (Q_1, Q_2, Q_3; \cdot)$ be a 3-basic quasigroup, Π_i the full permutation group of Q_i, $i = 1, 2, 3$, and $\mathscr{A}(\mathbf{Q})$ the full autotopy group of $\mathbf{Q}$. Starting from subgroups Γ_1 of Π_1 and Γ_2 of Π_2 we introduce Γ_3 by

(4) $$\Gamma_3 = \{\varphi_3 \in \Pi_3;\ \varphi_1 x . \varphi_2 y = \varphi_3(x . y) \text{ for all } x \in Q_1,\ y \in Q_2,\ \varphi_1 \in \Gamma_1,\ \varphi_2 \in \Gamma_2\}.$$

Clearly $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}(\mathbf{Q})$.

Lemma 1. Γ_3 *defined by* (4) *together with the map composition* $\circ$ *is a subgroup of* Π_3.

Proof. Clearly $e_1 = id_{Q_1} \in \Gamma_1$, $e_2 = id_{Q_2} \in \Gamma$ and by (4), $e_3 = id_{Q_3} \in \Gamma_3$. Now let $(\varphi_1, \varphi_2, \varphi_3), (\varphi'_1, \varphi'_2, \varphi'_3) \in \mathscr{A}(\mathbf{Q})$; $\varphi_1, \varphi'_1 \in \Gamma_1$; $\varphi_2, \varphi'_2 \in \Gamma_2$, then $\varphi_1(\varphi'_1 x) . \varphi_2(\varphi'_2 y) = \varphi_3(\varphi'_1 x . \varphi'_2 y) = \varphi_3(\varphi'_3(x . y))$ for all $x \in Q_1$, $y \in Q_2$. Since $\varphi_1 \circ \varphi'_1 \in \Gamma_1$, $\varphi_2 \circ \varphi'_2 \in \Gamma_2$ we have also $\varphi_3 \circ \varphi'_3 \in \Gamma_3$. If $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}(\mathbf{Q})$, $\varphi_1 \in \Gamma_1$, $\varphi_2 \in \Gamma_2$, then there exist $\varphi_1^{-1} \in \Gamma_1$, $\varphi_2^{-1} \in \Gamma_2$ (as Γ_1, Γ_2 are groups) and $\varphi'_3 \in \Gamma_3$ with $(\varphi_1^{-1}, \varphi_2^{-1}, \varphi'_3) \in \mathscr{A}(\mathbf{Q})$. Thus $\varphi_1^{-1}(\varphi_1 x) . \varphi_2^{-1}(\varphi_2 y) = \varphi'_3(\varphi_3(x . y)) \Rightarrow x . y = \varphi'_3(\varphi_3(x . y)) \Rightarrow \varphi'_3 \circ \varphi_3 = e_3 \Rightarrow \varphi'_3 = \varphi_3^{-1}$. ■

Lemma 2. (obvious). *All autotopies* $(\varphi_1, \varphi_2, \varphi_3)$ (*just obtained by* (4) *and said to be admissible*) *of* $\mathbf{Q}$ *form a subgroup* $G_{1,2}(\Gamma_1, \Gamma_2)$ *of* $\mathscr{A}(\mathbf{Q})$ *under the componentwise composition.*

Lemma 3. *Let* $(\varphi_1, \varphi_2, \varphi_3) \in \mathscr{A}(\mathbf{Q})$ *be admissible. Then arbitrary two of the components* $\varphi_1, \varphi_2, \varphi_3$ *determine uniquely the remaining one.*

Proof. Let us choose φ_1, φ_3 and suppose that there exist φ_2 and φ'_2 such that $\varphi_1 x . \varphi_2 y = \varphi_3(x . y)$ and $\varphi_1 x . \varphi'_2 y = \varphi_3(x . y)$. Then $\varphi_1 x . \varphi_2 y = \varphi_1 x . \varphi'_2 y$ and $\varphi_2 y = \varphi'_2 y$ for all $y \in Q_2$, $x \in Q_1 \Rightarrow \varphi_2 = \varphi'_2$.

Similarly, if we choose φ_2, φ_3, then we get a unique φ_1. ■

Thus we can choose Γ_1, Γ_2 arbitrary and obtain the unique corresponding Γ_3.

Using the permutations $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$ where $(i, j, k) = (1, 2, 3), (3, 1, 2), (2, 3, 1)$, and the corresponding cyclic parastrophes $(Q_i, Q_j, Q_k; A_k)$, we can start from groups Γ_i, Γ_j and introduce Γ_k by

$$\Gamma_k = \{\varphi_k \in \Pi_k;\ A_k(\varphi_i x, \varphi_j y) = \varphi_k A_k(x, y) \quad \text{for all} \quad x \in Q_i,\ y \in Q_j, \\ \varphi_i \in \Gamma_i,\quad \varphi_j \in \Gamma_j\}. \tag{5}$$

This permits us to choose any two groups of $\Gamma_1, \Gamma_2, \Gamma_3$ arbitrarily, the remaining one being then uniquely determined by (5). Thus we obtain a subgroup $G_{i,j}(\Gamma_i, \Gamma_j)$ of $\mathscr{A}(\mathbf{Q})$.

Remark. Passing from Γ_1, Γ_2 to Γ_3 by (4) and similarly from $\Gamma'_2 = \Gamma_2, \Gamma'_3 = \Gamma_3$ to Γ'_1 by (5), we get in general $\Gamma_1 \neq \Gamma'_1$, thus $G_{1,2}(\Gamma_1, \Gamma_2) \neq G_{2,3}(\Gamma_2, \Gamma_3)$.

Now we present several examples.

Example 1. Let $\Gamma_1 = \{e_1, \alpha\}$, $\Gamma_2 = \{e_2, \beta\}$, where $\alpha^2 = e_1$, $\beta^2 = e_2$. Then by (4), $\Gamma_3 = \{e_3, \gamma_1, \gamma_2, \gamma_3\}$ with the multiplication table

	γ_1	γ_2	γ_3
γ_1	e_3	γ_3	γ_2
γ_2	γ_3	e_3	γ_1
γ_3	γ_2	γ_1	e_3

The admissible autotopies are (e_1, e_2, e_3), (α, e_2, γ_1), (e_1, β, γ_2) and $(\alpha, \beta, \gamma_3)$. If $\gamma_3 = e_3$, then $\gamma_1 = \gamma_2$ and we obtain

Example 2. $\Gamma_1 = \{e_1, \alpha\}$, $\Gamma_2 = \{e_2, \beta\}$, $\Gamma_3 = \{e_3, \gamma\}$ with $\alpha^2 = e_1$, $\beta^2 = e_2$, $\gamma^2 = e_3$ and with the admissible autotopies (e_1, e_2, e_3), (α, e_2, γ), (e_1, β, γ), (α, β, e_3).

Example 3. Let $\Gamma_1 = \{e_1, \alpha\}$ and $\Gamma_2 = \{e_2, \beta_1, \beta_2\}$, where $\alpha^2 = e_1$ and

	β_1	β_2
β_1	β_2	e_2
β_2	e_2	β_1

Then, by (4), Γ_3 consists of 6 elements $e_3, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ with the multiplication table.

	γ_1	γ_2	γ_3	γ_4	γ_5
γ_1	γ_2	e_3	γ_4	γ_5	γ_3
γ_2	e_3	γ_1	γ_5	γ_3	γ_4
γ_3	γ_4	γ_5	e_3	γ_1	γ_2
γ_4	γ_5	γ_3	γ_1	γ_2	e_3
γ_5	γ_3	γ_4	γ_2	e_3	γ_1

The admissible autotopies are (e_1, e_2, e_3), (e_1, β_1, γ_1), (e_1, β_2, γ_2), (α, e_2, γ_3), $(\alpha, \beta_1, \gamma_4)$, $(\alpha, \beta_2, \gamma_5)$.

We can observe that in Examples 1 and 3 $G_{1,2}(\Gamma_1, \Gamma_2) \neq G_{2,3}(\Gamma_2, \Gamma_3)$, whereas in Example 2

$$G_{1,2}(\Gamma_1, \Gamma_2) = G_{2,3}(\Gamma_2, \Gamma_3) = G_{3,1}(\Gamma_3, \Gamma_1) \tag{6}$$

holds.

Now we restrict ourselves to the case when (6) is satisfied.

Lemma 4. *Let* $\mathbf{G}$ *be a subgroup of* $\mathscr{A}(\mathbf{Q})$ *such that*

$$\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2) = G_{2,3}(\Gamma_2, \Gamma_3) = G_{3,1}(\Gamma_3, \Gamma_1)\,. \tag{7}$$

Define a map $*: \Gamma_1 \times \Gamma_2 \to \Gamma_3$ *by*

$$\alpha * \beta = \gamma \Leftrightarrow (\alpha, \beta, \gamma) \in \mathbf{G} \quad \textit{for all} \quad \alpha \in \Gamma_1,\ \beta \in \Gamma_2\,. \tag{8}$$

Then $(\Gamma_1, \Gamma_2, \Gamma_3; *)$ *is a* 3*-basic quasigroup.*

Proof. If we choose any two elements of $\alpha \in \Gamma_1$, $\beta \in \Gamma_2$, $\gamma \in \Gamma_3$, then by (5) and (7) there exists a third element such that $(\alpha, \beta, \gamma) \in \mathbf{G}$, and by Lemma 3 this element is unique. ■

We say that the subgroup $\mathbf{G}$ of $\mathscr{A}(\mathbf{Q})$ is *special* if its component groups $\Gamma_1, \Gamma_2, \Gamma_3$ with the binary operation $*: \Gamma_1 \times \Gamma_2 \to \Gamma_3$ defined by (8) form a 3-basic quasigroup $(\Gamma_1, \Gamma_2, \Gamma_3; *)$.

3. CONGRUENCES

Let $\mathbf{Q} = (Q_1, Q_2, Q_3; \cdot)$ be a 3-basic quasigroup, $\mathbf{G}$ a subgroup of $\mathscr{A}(\mathbf{Q})$ and $\Gamma_1, \Gamma_2, \Gamma_3$ component groups of $\mathbf{G}$.

Lemma 5. $\{\Gamma_i(x);\ x \in Q_i\}$ *is a decomposition of* Q_i, $i = 1, 2, 3$.

Proof. For all $x \in Q_i$ we trivially have $x \in \Gamma_i(x)$, because $e_i = id_{Q_i} \in \Gamma_i$, $e_i x = x$. We need to prove that $\Gamma_i(x) \cap \Gamma_i(y) \neq \emptyset$ implies $\Gamma_i(x) = \Gamma_i(y)$, $x, y \in Q_i$. If $z \in \Gamma_i(x) \cap \Gamma_i(y)$, then there exist $\alpha, \beta \in \Gamma_i$ such that $z = \alpha x$, $z = \beta y$ and therefore $\Gamma_i(z) \subseteq \Gamma_i(x)$, $\Gamma_i(z) \subseteq \Gamma_i(y)$; at the same time there exist $\alpha^{-1}, \beta^{-1} \in \Gamma_i$ such that $x = \alpha^{-1} z$, $y = \beta^{-1} z$, thus $\Gamma_i(x) \subseteq \Gamma_i(z)$, $\Gamma_i(y) \subseteq \Gamma_i(z)$. This yields $\Gamma_i(x) = \Gamma_i(z) = \Gamma_i(y)$. ■

Now we can define for every $i = 1, 2, 3$ an equivalence relation R^{Γ_i} on Q_i by

$$x R^{\Gamma_i} y \Leftrightarrow \Gamma_i(x) = \Gamma_i(y) \quad \text{for} \quad x, y \in Q_i\,. \tag{9}$$

Theorem 1. $(R^{\Gamma_1}, R^{\Gamma_2}, R^{\Gamma_3})$ *defined by* (9) *is a congruence on* $\mathbf{Q}$ *if* $\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2)$.

Proof. We must prove

$$x_1 R^{\Gamma_1} y_1,\ x_2 R^{\Gamma_2} y_2 \Rightarrow (x_1 \,.\, x_2)\, R^{\Gamma_3} (y_1 \,.\, y_2)\,.$$

When $x_i R^{\Gamma_i} y_i$, then $\Gamma_i(x_i) = \Gamma_i(y_i)$ and there exists $\varphi_i \in \Gamma_i$ such that $y_i = \varphi_i x_i$ $(i = 1, 2)$ and

$$y_1 \,.\, y_2 = \varphi_1 x_1 \,.\, \varphi_2 x_2 \overset{(4)}{=} \varphi_3(x_1 \,.\, x_2) \Rightarrow y_1 \,.\, y_2 \in \Gamma_3(x_1 \,.\, x_2)\,.$$

By Lemma 5 we get $\Gamma_3(x_1 \,.\, x_2) = \Gamma_3(y_1 \,.\, y_2) \Rightarrow (x_1 \,.\, x_2)\, R^{\Gamma_3}(y_1 \,.\, y_2)$. ∎

Theorem 2. *Every special autotopy group* **G** *of a 3-basic quasigroup* **Q** *uniquely determines a normal congruence on* **Q**.

Proof. By the definition of a special autotopy group $\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2) = G_{3,1}(\Gamma_3, \Gamma_1) = G_{2,3}(\Gamma_2, \Gamma_3)$ and by Theorem 1, the triple $(R^{\Gamma_1}, R^{\Gamma_2}, R^{\Gamma_3})$ defined by (9) is a congruence. It remains to prove that this congruence is normal.

a) If $(x \,.\, z)\, R^{\Gamma_3}(y \,.\, z)$ for $x, y \in Q_1$, $z \in Q_2$, then $\Gamma_3(x \,.\, z) = \Gamma_3(y \,.\, z)$ and there exists $\varphi_3 \in \Gamma_3$ such that $x \,.\, z = \varphi_3(y \,.\, z)$. When we choose $\varphi_2 = id_{Q_2}$, then there exists a unique $\varphi_1 \in \Gamma_1$ (**G** is special) such that $x \,.\, z = \varphi_3(y \,.\, z) = \varphi_1 y \,.\, z$. Thus $x = \varphi_1 y$ and $x R^{\Gamma_1} y$.

b) If $(z \,.\, x)\, R^{\Gamma_3}(z \,.\, y)$ for $z \in Q_1$, $x, y \in Q_2$, then $\Gamma_3(z \,.\, x) = \Gamma_3(z \,.\, y)$ and $z \,.\, x = \varphi_3(z \,.\, y)$ for $\varphi_3 \in \Gamma_3$. If we choose $\varphi_1 = id_{Q_1}$, then there exists a unique $\varphi_2 \in \Gamma_2$ such that $z \,.\, x = \varphi_3(z \,.\, y) = z \,.\, \varphi_2 y$. Thus $x = \varphi_2 y$ and $x R^{\Gamma_2} y$. ∎

Now we shall prove the converse theorem.

Theorem 3. *Let* $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ *be a congruence on a 3-basic quasigroup* $\mathbf{Q} = (Q_1, Q_2, Q_3)$. *Then for every* $i = 1, 2, 3$,

$$\Gamma_i = \{\varphi \in \Pi_i;\; C^{\varrho_i}_{\varphi x} = C^{\varrho_i}_x \text{ for all } x \in Q_i\} \tag{10}$$

forms a subgroup of Π_i *and* $\Gamma_1, \Gamma_2, \Gamma_3$ *are components of an autotopy group* $\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2)$. *If* $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ *is a normal congruence on* **Q**, *then* $\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2)$ *is a special autotopy group on* **Q**.

Proof. a) It follows from (10) that $\Gamma_i(x) = C^{\varrho_i}_x$. Consequently, Γ_i is transitive on $C^{\varrho_i}_x$. It is clear that $\mathrm{id}_{Q_i} \in \Gamma_i$. If $\varphi, \varphi' \in \Gamma_i$, then $C^{\varrho_i}_{\varphi x} = C^{\varrho_i}_x = C^{\varrho_i}_{\varphi' x}$ and $C^{\varrho_i}_{\varphi'(\varphi x)} = C^{\varrho_i}_{\varphi x} = C^{\varrho_i}_x \Rightarrow \varphi' \circ \varphi \in \Gamma_i$. If $\varphi \in \Gamma_i$, then $C^{\varrho_i}_x = C^{\varrho_i}_{\varphi x}$ for $x \in Q_i$ and there exists $y \in Q_i$ such that $y \varrho_i x$ and $\varphi x = y$. Since φ is a permutation there is $\varphi^{-1} \in \Pi_i$ such that $x = \varphi^{-1} y$ and $C^{\varrho_i}_x = C^{\varrho_i}_{\varphi^{-1} y}$, $C^{\varrho_i}_y = C^{\varrho_i}_x = C^{\varrho_i}_{\varphi^{-1} y} \Rightarrow \varphi^{-1} \in \Gamma_i$. Thus we have proved that each Γ_i forms a subgroup of Π_i, $i = 1, 2, 3$.

b) Now we shall prove that $\Gamma_1, \Gamma_2, \Gamma_3$ are components of an autotopy group $\mathbf{G} = G_{1,2}(\Gamma_1, \Gamma_2)$. We need to prove that every two elements $\varphi_1 \in \Gamma_1$, $\varphi_2 \in \Gamma_2$ uniquely determine $\varphi_3 \in \Gamma_3$ such that $\varphi_1 x \,.\, \varphi_2 y = \varphi_3(x \,.\, y)$ for all $x \in Q_1$, $y \in Q_2$. Let $\varphi_1 \in \Gamma_1$. $\varphi_2 \in \Gamma_2$, then for any $x \in Q_1$ and $y \in Q_2$ we have $C^{\varrho_3}_{x.y} = C^{\varrho_1}_x \odot C^{\varrho_2}_y = C^{\varrho_1}_{\varphi_1 x} \odot C^{\varrho_2}_{\varphi_2 y} = C^{\varrho_3}_{\varphi_1 x . \varphi_2 y}$ and $(x \,.\, y)\, \varrho_3(\varphi_1 x \,.\, \varphi_2 y)$. We know that every congruence relation is always reflexive and therefore for some $z \in Q_3$ we get $\varphi_1 x \,.\, \varphi_2 y = z$ and $C^{\varrho_3}_z = C^{\varrho_3}_{x.y}$. The transitivity of Γ_3 on $C^{\varrho_3}_{x.y}$ implies that there exists $\varphi_3 \in \Gamma_3$ with $z = \varphi_3(x \,.\, y) \Rightarrow \varphi_1 x \,.\, \varphi_2 y = \varphi_3(x \,.\, y) \Rightarrow (\varphi_1, \varphi_2, \varphi_3)$ is an autotopy on **Q**.

c) Let $(\varrho_1, \varrho_2, \varrho_3)$ be a normal congruence on $\mathbf{Q}$. We must prove that every two elements of $\varphi_1 \in \Gamma_1$, $\varphi_2 \in \Gamma_2$, $\varphi_3 \in \Gamma_3$ uniquely determine the remaining one such that $\varphi_1 x \,.\, \varphi_2 y = \varphi_3(x \,.\, y)$ for all $x \in Q_1$, $y \in Q_2$. If $\varphi_1 \in \Gamma_1$, $\varphi_3 \in \Gamma_3$, then for $x \in Q_1$, $y \in Q_2$ we have $C_x^{\varrho_1} = C_{\varphi_1 x}^{\varrho_1}$, $C_{x.y}^{\varrho_3} = C_{\varphi_3(x.y)}^{\varrho_3}$, $C_{x.y}^{\varrho_3} = C_x^{\varrho_1} \odot C_y^{\varrho_2} = C_{\varphi_1 x}^{\varrho_1} \odot C_y^{\varrho_2} = C_{\varphi_1 x.y}^{\varrho_3} = C_{\varphi_3(x.y)}^{\varrho_3}$ and $(\varphi_1 x \,.\, y)\, \varrho_3 \varphi_2(x \,.\, y)$. The reflexivity of ϱ_3 implies that there exists an element $y' \in Q_2$ such that that $\varphi_1 x \,.\, y' = \varphi_3(x \,.\, y)$ and $C_{\varphi_3(x.y)}^{\varrho_3} = C_{\varphi_1 x}^{\varrho_1} \odot C_{y'}^{\varrho_2}$. Since simultaneously $C_{\varphi_3(x.y)}^{\varrho_3} = C_{\varphi_1 x}^{\varrho_1} \odot C_y^{\varrho_2}$, we obtain $C_y^{\varrho_2} = C_{y'}^{\varrho_2}$. Here we have used the fact that $(\varrho_1, \varrho_2, \varrho_3)$ is a normal congruence. Now the transitivity of Γ_2 on $C_y^{\varrho_2}$ yields the existence of $\varphi_2 \in \Gamma_2$ with $y' = \varphi_2 y$. Thus $\varphi_3(x \,.\, y) = \varphi_1 x \,.\, \varphi_2 y$.

Similarly, if $\varphi_2 \in \Gamma_2$, $\varphi_3 \in \Gamma_3$, then for $x \in Q_1$, $y \in Q_2$ we have $C_y^{\varrho_2} = C_{\varphi_2 y}^{\varrho_2}$, $C_{x.y}^{\varrho_3} = C_{\varphi_3(x.y)}^{\varrho_3}$ and $C_{x.y}^{\varrho_3} = C_x^{\varrho_1} \odot C_y^{\varrho_2} = C_x^{\varrho_1} \odot C_{\varphi_2 y}^{\varrho_2} = C_{x.\varphi_2 y}^{\varrho_3} = C_{\varphi_3(x.y)}^{\varrho_3}$ so that $(x \,.\, \varphi_2 y)\, \varrho_3 \varphi_3(x \,.\, y)$. The reflexivity of ϱ_3 yields the existence of an element $x' \in Q_1$ such that $x' \,.\, \varphi_2 y = \varphi_3(x \,.\, y)$ and $C_{\varphi_3(x.y)}^{\varrho_3} = C_{x'}^{\varrho_1} \odot C_{\varphi_2 y}^{\varrho_2}$ so that $C_x^{\varrho_1} = C_{x'}^{\varrho_1}$. Using the transitivity of Γ_1 on $C_x^{\varrho_1}$ we get $\varphi_1 \in \Gamma_1$ such that $x' = \varphi_1 x$ and $\varphi_3(x \,.\, y) = \varphi_1 x \,.\, \varphi_2 y$. ■

Theorems 2 and 3 yield a 1-1-correspondence between the special autotopy groups and the normal congruences of a given 3-basic quasigroup $\mathbf{Q}$. On the other hand, we know that there exists a 1-1-correspondence between the normal congruences of $\mathbf{Q}$ and the homotopies of $\mathbf{Q}$ onto $\mathbf{Q}'$. So we have also a 1-1-correspondence between the special autotopy groups $\mathbf{G}$ and the homotopies (τ_1, τ_2, τ_3) of $\mathbf{Q}$. This correspondence $\mathbf{G} \Leftrightarrow (\tau_1, \tau_2, \tau_3)$ is given directly by

$$\Gamma_i = \{\varphi \in \Pi_i;\ \tau_i(\varphi x) = \tau_i(x),\ x \in Q_i\}$$

and $\tau_i(x) = \tau_i(y) \Leftrightarrow y \in \Gamma_i(x)$, where $x, y \in Q_i$, $i = 1, 2, 3$.

Now let $\mathbf{G}$ and $\mathbf{G}'$ be special autotopy groups. If Γ_i' is a subgroup of Γ_i for every $i = 1, 2, 3$, then $\mathbf{G}'$ is a subgroup of $\mathbf{G}$ and $R^{\Gamma_i'}$ is a refinement of R^{Γ_i} for $i = 1, 2, 3$.

If $\Gamma_i = \Pi_i$ for every $i = 1, 2, 3$, then we get the maximal normal congruence $(R^{\Pi_1}, R^{\Pi_2}, R^{\Pi_3})$, i.e., $xR^{\Pi_i}y$ for all $x, y \in Q_i$ and $C_x^{R^{\Pi_i}} = Q_i = \Gamma_i(x)$ for every $x \in Q_i$.

If $\Gamma_i = \{e_i\}$, $i = 1, 2, 3$, then we get the minimal normal congruence $(R^{e_1}, R^{e_2}, R^{e_3})$ so that $xR^{e_i}y \Leftrightarrow x = y$ and $C_x^{R^{e_i}} = \Gamma_i(x) = \{x\}$ for every $x \in Q_i$ and $e_i = id_{Q_i}$.

Now we pass to a usual quasigroup $(Q, Q, Q; \cdot)$ and take a special autotopy group $\mathbf{G}$ with components $\Gamma_1, \Gamma_2, \Gamma_3$. Using Theorem 2 with $Q_1 = Q_2 = Q_3 = Q$ we get a normal congruence $(R^{\Gamma_1}, R^{\Gamma_2}, R^{\Gamma_3})$ with decomposition classes $C_x^{R^{\Gamma_i}} = \Gamma_i(x)$, $x \in Q$. So we have three (in general, mutually distinct) decompositions forming a 3-basic quasigroup $(Q/R^{\Gamma_1}, Q/R^{\Gamma_2}, Q/R^{\Gamma_3}; \odot)^p$with $C_x^{R^{\Gamma_1}} \odot C_y^{R^{\Gamma_2}} = C_{x.y}^{R^{\Gamma_3}}$, where $x, y \in Q$.

All these results can be trivially generalized to $(n + 1)$-basic quasigroups. We shall mention some primary notions. $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{n+1}; A)$ is said to be an $(n + 1)$-*basic quasigroup* if $Q_1, Q_2, \ldots, Q_{n+1}$ are sets with the same cardinality, A is an n-ary operation with

(11) $$A(a_1, \ldots, a_n) = a_{n+1} \quad \text{for} \quad a_i \in Q_i\,, \quad i = 1, \ldots, n + 1\,,$$

and in (11) any n elements of $a_i \in Q_i$, $i = 1, \ldots, n + 1$, uniquely determine the remaining one. Under *a homotopy* of $\mathbf{Q}$ onto $\mathbf{Q}'$ we mean an ordered $(n + 1)$-tuple $(\tau_1, \ldots, \tau_{n+1})$ of maps $\tau_i : Q_i \to Q'_i$, $\tau_i(Q_i) = Q'_i$, $i = 1, \ldots, n + 1$, such that $\tau_{n+1} A(a_1, \ldots, a_n) = A'(\tau_1 a_1, \ldots, \tau_n a_n)$ for all $a_i \in Q_i$, $i = 1, \ldots, n$. By analogy we can define *an isotopy* and *an autotopy*. The $(n + 1)$-tuple $(\varrho_1, \ldots, \varrho_{n+1})$ of equivalence relations ϱ_i of Q_i, $i = 1, \ldots, n + 1$, is called *a normal congruence* on $\mathbf{Q}$ if $a \varrho_i b$, $a, b \in Q_i \Leftrightarrow A(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n)\, \varrho_{n+1} A(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$ for all $i = 1, \ldots, n$ and all $x_j \in Q_j$, $j = 1, \ldots, i - 1, i + 1, \ldots, n$. A *subgroup* $\mathbf{G} = (\Gamma_1, \ldots, \Gamma_{n+1})$ of the full autotopy group of is said to be *special* if $(\Gamma_1, \ldots, \Gamma_{n+1}; \Phi)$ is an $(n + 1)$-basic quasigroup, where

$$\Phi(\varphi_1, \ldots, \varphi_n) = \varphi_{n+1} \Leftrightarrow (\varphi_1, \ldots, \varphi_n, \varphi_{n+1}) \in \mathbf{G}\,, \quad \varphi_i \in \Gamma_i\,, \quad i = 1, \ldots, n + 1\,.$$

Similarly as in the case $n = 2$, we can prove that there exists a 1-1-correspondence between the normal congruences on $\mathbf{Q}$ and the special autotopy groups $\mathbf{G}$ on $\mathbf{Q}$.

If an $(n + 1)$-basic quasigroup $(Q_1, \ldots, Q_{n+1}; A)$ satisfies $Q_1 = \ldots = Q_{n+1}$ then we get the n-quasigroup $(Q; A) = (Q, \ldots, Q; A)$. R. F. Kramareva ([2]) proved that every homotopy of an n-quasigroup $(Q; A)$ onto an n-quasigroup $(Q'; A')$ determines a normal congruence $(\varrho_1, \ldots, \varrho_{n+1})$, and that $(Q/\varrho_1, \ldots, Q/\varrho_{n+1}; \tilde{A})$ with

$$\tilde{A}(C^{\varrho_1}_{a_1}, \ldots, C^{\varrho_n}_{a_n}) = C^{\varrho_{n+1}}_{A(a_1, \ldots, a_n)}\; C^{\varrho_i}_{a_i} \in Q/\varrho_i\,, \quad i = 1, \ldots, n + 1$$

forms a partial n-quasigroup. This is exactly our $(n + 1)$-basic quasigroup.

References

[1] *V. A. Beglarjan:* On normality of subquasigroups (in Russian), Issledovanija po teorii kvazigrupp i lup, Štiinca, Kišinev 1973, 18–32.

[2] *R. F. Kramareva:* About homotopies of n-quasigroups (in Russian), Voprosy teorii kvazigrupp i lup, Štiinca, Kišinev 1970, 76–91.

[3] *E. I. Tebyrce:* Some questions of the homotopy theory for quasigroups (in Russian), Iss'edovanija po obščej algebre, Štiinca, Kišinev 1968, 74–81.

Souhrn

O 3-BÁZOVÝCH KVAZIGRUPÁCH A JEJICH KONGRUENCÍCH

ELENA BROŽÍKOVÁ

Podgrupa $\mathbf{G}$ úplné grupy autotopií dané 3-bázové kvazigrupy $\mathbf{Q}$ se nazývá speciální, jestliže její grupy komponent $\Gamma_1, \Gamma_2, \Gamma_3$ tvoří 3-bázovou kvazigrupu $(\Gamma_1, \Gamma_2, \Gamma_3; *)$, kde

$$\alpha * \beta = \gamma \Leftrightarrow (\alpha, \beta, \gamma) \in \mathbf{G} \quad \text{pro} \quad \alpha \in \Gamma_1,\ \beta \in \Gamma_2,\ \gamma \in \Gamma_3\,.$$

V této práci je dokázána vzájemně jednoznačná korespondence mezi speciálními podgrupami $\mathbf{G}$ a normálními kongruencemi ϱ dané 3-bázové kvazigrupy $\mathbf{Q}$.

Резюме

О 3-БАЗОВЫХ КВАЗИГРУППАХ И ИХ КОНГРУЭНЦИЯХ

ELENA BROŽÍKOVÁ

Подгруппа $\mathbf{G}$ полной группы автотопий данной 3-базовой квазигруппы $\mathbf{Q}$ называется специальной, если ее группы компонент $\Gamma_1, \Gamma_2, \Gamma_3$ образуют 3-базовую квазигруппу $(\Gamma_1, \Gamma_2, \Gamma_3; *)$, где

$$\alpha * \beta = \gamma \Leftrightarrow (\alpha, \beta, \gamma) \in \mathbf{G} \text{ для } \alpha \in \Gamma_1, \beta \in \Gamma_2, \gamma \in \Gamma_3.$$

В работе показано, что существует взаимно однозначное соответствие между специальными подгруппами $\mathbf{G}$ и нормальными конгруэнциями ϱ данной 3-базовой квазигруппы $\mathbf{Q}$.

Author's address: Strojní fakulta ČVUT, K201, Karlovo nám. 13, 121 35 Praha 2.