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DEFORMATIONS OF SUBMANIFOLDS OF HOMOGENEOUS SPACES

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Among the topics of the classical differential geometry there is the theory of the deformations of submanifolds of homogeneous spaces (first order deformation of surfaces in Euclidean 3-space, second order deformation of surfaces and line congruences in projective 3-space, etc.). It is interesting that there is no general definition of the deformation, the paper [1] being not very precise. The purpose of this paper is to present such a definition and to produce a process solving the question whether two given submanifolds are in a deformation. The deformation of high order being an equivalence, the theory of deformations may lead to the solution of the equivalence problem for submanifolds in homogeneous spaces. I present this process explicitly for the second order deformation of curves; the generalisation to submanifolds is quite trivial. The deformations of higher order lead to very complicated calculations, and I have no general formulas.

The main problem in the theory of deformations is as follows: Be given a homogeneous space G/H, a manifold M with $\dim M < \dim G/H$ and a natural number n; we have to find out all couples $V, W : M \to G/H$ which are in the deformation of order n without being equivalent. I think that there are no general theorems covering the known results.

1. Be given a Lie group G and its closed subgroup H; let us consider the homogeneous space G/H. For the sake of simplicity, suppose that G is a linear group, i.e. a subgroup of the full linear group $GL(\mu, R)$. Further, let us suppose that the normalizer of H coincides with H, i.e.,

$$gHg^{-1} \subset H \Rightarrow g \in H\,, \tag{1.1}$$

$$[v, \mathfrak{h}] \subset \mathfrak{h} \Rightarrow v \in \mathfrak{h}\,. \tag{1.2}$$

In the Lie algebra $\mathfrak{g}$, we have

$$[A, B] = AB - BA\,, \quad \text{ad}(g) A = gAg^{-1}\,; \quad A, B \in \mathfrak{g},\ g \in G\,. \tag{1.3}$$

Let us recall the well known formula

$$\text{ad}(g)[A, B] = [\text{ad}(g) A, \text{ad}(g) B]\,. \tag{1.4}$$

Let M be a differentiable manifold, $\dim M \leqq \dim G/H$. Consider the embedding $V: M \to G/H$. The *lift* of V is a map $v : M \to G$ such that the diagram

(1.5)
$$\begin{array}{ccc} & \overset{v}{\nearrow} & G \\ M & & \downarrow \pi \\ & \underset{V}{\searrow} & G/H \end{array}$$

is commutative; of course, $\pi : G \to G/H$ is the natural projection. If $v_1, v_2 : M \to G$ are two lifts of the embedding $V: M \to G/H$, there is the map $h : M \to H$ such that

(1.6)
$$v_2(m) = v_1(m)\, h(m) \quad \text{for} \quad m \in M\,.$$

To each $v : M \to G$, there is associated the $\mathfrak{g}$-valued 1-form ω_v on M defined by

(1.7)
$$\omega_v = v^{-1}\, \mathrm{d}v\,.$$

If the lifts $v_1, v_2 : M \to G$ satisfy (1.6), we have

(1.8)
$$\omega_{v_1} = \mathrm{ad}\,(h^{-1})\, \omega_{v_2} + h^{-1}\, \mathrm{d}h\,.$$

If $v : M \to G$ is a lift of the embedding $V: M \to G/H$,

(1.9)
$$\omega_v(T_m(M)) \cap \mathfrak{h} = 0 \quad \text{for each} \quad m \in M\,.$$

Let us present the fundamental existence theorem.

Theorem 1.1. *Be given a Lie group G, a manifold M and a $\mathfrak{g}$-valued 1-form ω_v on M such that*

(1.10)
$$\mathrm{d}\omega_v = -\omega_v \wedge \omega_v\,.$$

Further, be given points $m_0 \in M$ and $g_0 \in G$. Then there is a neighborhood $U \subset M$ of the point m_0 and a unique map $v : U \to G$ satisfying (1.7) and $v(m_0) = g_0$. If $\dim M = 1$ and M is an interval, we may set $U = M$; the condition (1.10) is always satisfied.

The group G acts transitively on G/H to the left; to the element $g \in G$ there is associated the map $A_g : G/H \to G/H$ given by $A_g(g_1 H) = (g g_1)\, H$.

Definition 1.1. The embeddings V, $W: M \to G/H$ are equivalent if there is an element $g \in G$ such that the diagram

(1.11)
$$\begin{array}{ccc} & \overset{V}{\nearrow} & G/H \\ M & & \downarrow A_g \\ & \underset{W}{\searrow} & G/H \end{array}$$

is commutative.

In the differential geometry, the following problem is of fundamental importance: Be given a homogeneous space G/H and a manifold M, $\dim M \leqq \dim G/H$. On M, be given $\mathfrak{g}$-valued 1-forms ω_1, ω_2 such that $(i = 1, 2)$

$$\text{(1.12)} \qquad \mathrm{d}\omega_i = -\omega_i \wedge \omega_i\,, \quad \omega_i(T_m(M)) \cap \mathfrak{h} = 0\,.$$

Suppose that there are mappings (according to Theorem 1.1) $v_1, v_2 : M \to G$ such that $\omega_i = v_i^{-1}\,\mathrm{d}v_i$. Define the embeddings $V_1, V_2 : M \to G/H$ by $V_i = \pi v_i$. We have to decide whether V_1 and V_2 are equivalent.

From (1.8), we get

Theorem 1.2. *The embeddings $V_1, V_2 : M \to G/H$ are equivalent if and only if there is a map $h : M \to H$ such that*

$$\text{(1.13)} \qquad \omega_2 = \mathrm{ad}\,(h^{-1})\,\omega_1 + h^{-1}\,\mathrm{d}h\,.$$

The condition (1.13) is, of course, a differential equation for h, and it is not very convenient for our purposes. In what follows, we wish to replace it by a sequence of algebraic conditions.

Denote by $\mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ the Grassman manifold of all subspaces $K \subset \mathfrak{g}$ such that $\dim K = \dim \mathfrak{h}$. Further, denote by $\mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ the Stiefel manifold of all ordered sets $(b_1, \ldots, b_\tau)$ of linearly independent vectors $b_1, \ldots, b_\tau \in \mathfrak{g}$, $\tau = \dim \mathfrak{h}$. Let $\pi : \mathrm{St}\,(\mathfrak{g}/\mathfrak{h}) \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ be the natural projection. The group G acts on $\mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ to the left; if $\mathscr{B} = (b_1, \ldots, b_\tau) \in \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ we set

$$\text{(1.14)} \qquad \mathrm{ad}\,(g)\,\mathscr{B} = (\mathrm{ad}\,(g)\,b_1, \ldots, \mathrm{ad}\,(g)\,b_\tau) \in \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})\,.$$

If $K \in \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ and $\mathscr{B}_1, \mathscr{B}_2 \in \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ are such that $\pi(\mathscr{B}_1) = \pi(\mathscr{B}_2) = K$, we have $\pi(\mathrm{ad}\,(g)\,\mathscr{B}_1) = \pi(\mathrm{ad}\,(g)\,\mathscr{B}_2)$, and the group G acts on $\mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ to the left; we denote its action by ad, and we have

$$\text{(1.15)} \qquad \pi(\mathrm{ad}\,(g)\,\mathscr{B}) = \mathrm{ad}\,(g)\,\pi(\mathscr{B}) \quad \text{for} \quad \mathscr{B} \in \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})\,.$$

The full linear group $GL(\dim \mathfrak{h}, \mathbf{R})$ acts on the Stiefel manifold $\mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ to the right according to the rule

$$\text{(1.16)} \qquad \mathscr{B}S = (b_1, \ldots, b_\tau)\,(s_i^j) = \Big(\sum_{i=1}^{\tau} b_i s_1^i, \ldots, \sum_{i=1}^{\tau} b_i s_\tau^i\Big)\,.$$

Be given an embedding $V : M \to G/H$ and let $v : M \to G$ be its arbitrary lift. Define the mapping $V^* : M \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ by

$$\text{(1.17)} \qquad V^*(m) = \mathrm{ad}\,(v(m))\,\mathfrak{h}\,.$$

The mapping V^* is obviously independent on the choice of the lift $v : M \to G$. Be given another embedding $W : M \to G/H$ and its associated mapping $W^* : M \to$

$\to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$. Suppose that V and W are equivalent; hence, there is an element $g \in G$ such that the diagram (1.11) is commutative. If $v : M \to G$ is a lift of V, gv is a lift of W, and we have

$$W^*(m) = \mathrm{ad}\,(g)\, V^*(m) \quad \text{for each} \quad m \in M\,. \tag{1.18}$$

Now, suppose the existence of an element $g \in G$ such that we have (1.18), and let us choose lifts $v, w : M \to G$ of the maps $V, W : M \to G/H$. Then the equation (1.18) may be written as $\mathrm{ad}\,(w(m))\,\mathfrak{h} = \mathrm{ad}\,(g)\,\mathrm{ad}\,(v(m))\,\mathfrak{h}$, i.e. $\mathrm{ad}\,(v(m)^{-1}\, g^{-1}\, w(m))\,\mathfrak{h} = \mathfrak{h}$. There is a mapping $h : M \to H$ such that

$$v(m)^{-1}\, g^{-1}\, w(m) = h(m)\,, \quad \text{i.e.} \quad w(m) = g\, v(m)\, h(m)\,.$$

The embedding $V : M \to G/H$ has a lift $v'(m) = v(m)\, h(m)$ such that $w(m) = g\, v'(m)$, and the embeddings V and W are equivalent. We have just proved

Theorem 1.3. *The embeddings $V, W : M \to G/H$ are equivalent if and only if there is an element $g \in G$ such that we have* (1.18).

Be given the embeddings $V, W : M \to G/H$ and the associated mappings V^*, $W^* : M \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$. Introduce the following

Definition 1.2. Let Z be a vector space, $r, s : M \to Z$ mappings and $m \in M$ a fixed point. Let ζ_A be a basis of the space Z and u^α; $\alpha = 1, \ldots, k$; be local coordinates in a neighborhood $U \subset M$ of m; let $u^\alpha(m) = u_0^\alpha$. The restrictions of the mappings r and s to U are given by the functions $z^A = r^A(u^\alpha)$ and $z^A = s^A(u^\alpha)$ in such a manner that we have $r(u^\alpha) = r^A(u^\alpha)\,\zeta_A$, $s(u^\alpha) = s^A(u^\alpha)\,\zeta_A$. We write $j_m^t(r) = j_m^t(s)$ if

$$\frac{\partial^\varrho r_A(m)}{(\partial u^{\alpha_1})^{\varrho_1}, \ldots, (\partial u^{\alpha_k})^{\varrho_k}} = \frac{\partial^\varrho s^A(m)}{(\partial u^{\alpha_1})^{\varrho_1}, \ldots, (\partial u^{\alpha_k})^{\varrho_k}} \quad \text{for} \quad 0 \leqq \varrho \leqq t,\ \varrho_1 + \ldots + \varrho_k = \varrho\,. \tag{1.19}$$

It is well known that this is a good definition. The spaces $\mathrm{St}\,(\mathfrak{g}/\mathfrak{h}) \subset X^\tau\mathfrak{g}$ and $X^\tau\mathfrak{g}$ being vector spaces, the relation $j_m^t(\mathscr{V}) = j_m^t(\mathscr{W})$, $\mathscr{V}$ and $\mathscr{W} : M \to \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ being given, is well defined. *Be given mappings $V^*, W^* : M \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$. We write $j_m^t(V^*) = j_m^t(W^*)$ if and only if there are lifts $\mathscr{V}, \mathscr{W} : M \to \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ of the mappings V^*, W^* such that we have $j_m^t(\mathscr{V}) = j_m^t(\mathscr{W})$.*

Definition 1.3. Be given the embeddings $V, W : M \to G/H$. We say that the embeddings V and W are in the deformation of order t if there is a mapping $g : M \to G$ such that for each $m_0 \in M$ we have

$$j_{m_0}^t(\mathrm{ad}\,(g(m_0))\, V^*) = j_{m_0}^t(W^*)\,, \tag{1.20}$$

$V^*, W^* : M \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$ being the associated mappings (1.17).

2. Let G/H be a homogeneous space, $M = (t_1, t_2) \subset \mathbf{R}$ an interval and $V: M \to G/H$ an embedding. Let $v : M \to G$ be an arbitrary lift of V, and let $A : M \to \mathfrak{g}$ be defined by

$$A(t) = v(t)^{-1} \frac{\mathrm{d}v(t)}{\mathrm{d}t}. \tag{2.1}$$

Analogously, be given an embedding $W: M \to G/H$, its lift $w : M \to G$ and the associated mapping

$$B(t) = w(t)^{-1} \frac{\mathrm{d}w(t)}{\mathrm{d}t}. \tag{2.2}$$

The mappings $A, B : M \to \mathfrak{g}$ being given, we have to decide whether the embeddings $V, W: M \to G/H$ are in the deformation of the given order.

The associated mappings $V^*, W^* : M \to \mathrm{Gr}(\mathfrak{g}/\mathfrak{h})$ are

$$V^*(t) = \mathrm{ad}\,(v(t))\,\mathfrak{h}\,, \quad W^* = \mathrm{ad}\,(w(t))\,\mathfrak{h}\,. \tag{2.3}$$

Let $\mathscr{B} \in \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ be a fixed basis of the space $\mathfrak{h}$. The mappings $\mathscr{V}^*, \mathscr{W}^* : M \to \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ given by

$$\mathscr{V}^*(t) = \mathrm{ad}\,(v(t))\,\mathscr{B}\,, \quad \mathscr{W}^*(t) = \mathrm{ad}\,(w(t))\,\mathscr{B} \tag{2.4}$$

are lifts of the mappings $V^*, W^* : M \to \mathrm{Gr}\,(\mathfrak{g}/\mathfrak{h})$. If $S : M \to GL(\dim \mathfrak{h}, \mathbf{R})$ is an arbitrary mapping, the mapping $\mathscr{W}^*_S : M \to \mathrm{St}\,(\mathfrak{g}/\mathfrak{h})$ given by

$$\mathscr{W}^*_S(t) = \mathrm{ad}\,(w(t))\,\mathscr{B}\,S(t) \tag{2.5}$$

is certainly a lift of W^*; we get all the lifts of W^* by means of this procedure. Obviously we have

Theorem 2.1. *The embeddings $V, W: M \to G/H$ are in the deformation of order k at the point t_0 if and only if there is $g \in G$ and $S : M \to GL(\dim \mathfrak{h}, \mathbf{R})$ such that*

$$\mathrm{ad}\,(g) \frac{\mathrm{d}^\varkappa}{\mathrm{d}t^\varkappa} \mathscr{V}^*(t_0) = \frac{\mathrm{d}^\varkappa}{\mathrm{d}t^\varkappa} \mathscr{W}^*_S(t_0)\,; \quad 0 \leqq \varkappa \leqq k\,. \tag{2.6}$$

Let us study the deformations of low orders; first of all, let us consider the deformation of order 0. For the sake of simplicity, let us write $v(t_0) = v_0$, etc. We have

$$\mathscr{V}^*_0 = \mathrm{ad}\,(v_0)\,\mathscr{B}\,, \quad \mathscr{W}^*_{S0} = \mathrm{ad}\,(w_0)\,\mathscr{B}S_0\,, \tag{2.7}$$

and the condition (2.6) reduces to the existence of $g \in G$ and $S_0 \in GL\,(\dim \mathfrak{h}, \mathbf{R})$ such that $\mathrm{ad}\,(gv_0)\,\mathscr{B} = \mathrm{ad}\,(w_0)\,\mathscr{B}S_0$, i.e.

$$\mathrm{ad}\,(w_0^{-1}gv_0)\,\mathscr{B} = \mathscr{B}S_0\,. \tag{2.8}$$

$\mathscr{B}S_0$ being a basis of the space $\mathfrak{h}$ we have $w_0^{-1}gv_0 \in H$. The general solution g and S_0 of (2.7) is obtained as follows: choose $h \in H$, set $g = \omega_0 h v_0^{-1}$, and determine S_0 from

$$\text{(2.9)} \qquad \operatorname{ad}(h)\,\mathscr{B} = \mathscr{B}S_0 .$$

Every two curves are thus in the deformation of order 0; of course, this is obvious, the group G acting on G/H transitively.

Let us now consider the deformation of order $k = 1$. From (2.4), we get

$$\mathscr{V}^*(t)\, v(t) = v(t)\,\mathscr{B},$$

and we have

$$\frac{\mathrm{d}\mathscr{V}^*(t)}{\mathrm{d}t} v(t) + v(t)\,\mathscr{B}\,A(t) = v(t)\,A(t)\,\mathscr{B},$$

i.e.

$$\text{(2.10)} \qquad \frac{\mathrm{d}\mathscr{V}^*(t)}{\mathrm{d}t} = \operatorname{ad}(v(t))\,[A(t), \mathscr{B}] ;$$

here, we use the obvious notation $[A, (b_1, \ldots, b_\tau)] = ([A, b_1], \ldots, [A, b_\tau])$. Analogously, we get

$$\text{(2.11)} \qquad \frac{\mathrm{d}\mathscr{W}^*(t)}{\mathrm{d}t} = \operatorname{ad}(w(t))\,[B(t), \mathscr{B}] .$$

From (2.5), we get

$$\frac{\mathrm{d}\mathscr{W}^*_S(t)}{\mathrm{d}t} = \operatorname{ad}(w(t)) \left\{ \mathscr{B}\,\frac{\mathrm{d}S(t)}{\mathrm{d}t} + [B(t), \mathscr{B}\,S(t)] \right\}.$$

The condition (2.6) for $\varkappa = 0, 1$ and $g = w_0 h v_0^{-1}$ yields (2.9) and

$$\operatorname{ad}(w_0 h)\,[A_0, \mathscr{B}] = \operatorname{ad}(w_0)\,(\mathscr{B}S_1 + [B_0, \operatorname{ad}(h)\,\mathscr{B}]) ; \quad \dot{S}_1 = \frac{\mathrm{d}S(t_0)}{\mathrm{d}t} .$$

Applying $\operatorname{ad}(h^{-1}w_0^{-1})$, we get

$$\text{(2.12)} \qquad [A_0 - \operatorname{ad}(h^{-1})\,B_0, \mathscr{B}] = \operatorname{ad}(h^{-1})\,\mathscr{B}S_1 .$$

The curves V, $W: M \to G/H$ are in the deformation of order 1 at the point $t = t_0$ if and only if there is an element $h \in H$ and a $(\dim \mathfrak{h} \times \dim \mathfrak{h})$-matrix S_1 – possibly singular – such that we have (2.12). From (2.12), we get $[A_0 - \operatorname{ad}(h^{-1})\,B_0, \mathfrak{h}] \subset \mathfrak{h}$ and $A_0 - \operatorname{ad}(h^{-1})\,B_0 \in \mathfrak{h}$. We have proved

Theorem 2.2. *Be given curves V, $W: M \to G/H$, lifts v, $w: M \to G$ and the associated mappings $A, B: M \to \mathfrak{g}$* (2.1) *and* (2.2) *resp. The curves V and W are in the deformation of order* 1 *if and only if there is a mapping $h: M \to H$ such that*

$$\text{(2.13)} \qquad A(t) - \operatorname{ad}(h(t)^{-1})\,B(t) \in \mathfrak{h} \quad \textit{for each} \quad t \in M .$$

Suppose that the curves V, $W: M \to G/H$ are in the deformation of order 1. For each $t \in M$, choose $h(t) \in H$ satisfying (2.13), and replace the lift $w: M \to G$ by the lift $w': M \to G$ defined by $w'(t) = w(t)\, h(t)$. For the associated mapping

$$B'(t) = w'(t)^{-1} \frac{\mathrm{d}w'(t)}{\mathrm{d}t},$$

we have

$$B'(t) = \mathrm{ad}\,(h(t)^{-1})\, B(t) + h(t)^{-1} \frac{\mathrm{d}h(t)}{\mathrm{d}t}$$

according to (1.8), and the relation (2.13) is equivalent to $A(t) - B'(t) \in \mathfrak{h}$. Thus we have

Theorem 2.3. *The curves V, $W: M \to G/H$ are in the deformation of order 1 if and only if there are lifts $v, w: M \to G$ such that we have*

(2.14) $$A(t) - B(t) \in \mathfrak{h} \quad \textit{for each} \quad t \in M$$

for the associated mappings $A, B: M \to \mathfrak{g}$.

Finally, let us consider the deformation of order 2. From (2.10) and (2.11), we get

(2.15) $$\frac{\mathrm{d}^2 \mathscr{V}^*(t)}{\mathrm{d}t^2} = \mathrm{ad}\,(v(t)) \left\{ \left[\frac{\mathrm{d}A(t)}{\mathrm{d}t}, \mathscr{B} \right] + [A(t), [A(t), \mathscr{B}]] \right\},$$

(2.16) $$\frac{\mathrm{d}^2 \mathscr{W}^*(t)}{\mathrm{d}t^2} = \mathrm{ad}\,(w(t)) \left\{ \left[\frac{\mathrm{d}B(t)}{\mathrm{d}t}, \mathscr{B} \right] + [B(t), [B(t), \mathscr{B}]] \right\}.$$

Further,

$$\frac{\mathrm{d}^2 \mathscr{W}_S^*(t_0)}{\mathrm{d}t^2} = \mathrm{ad}\,(w_0) \left\{ \left[\frac{\mathrm{d}B(t)}{\mathrm{d}t}, \mathrm{ad}\,(h)\, \mathscr{B} \right] + [B_0, \mathrm{ad}\,(h)\, \mathscr{B}]] + \right.$$
$$\left. + 2[B_0, [\mathrm{ad}\,(h)\, A_0 - B_0, \mathrm{ad}\,(h)\, \mathscr{B}]] + \mathscr{B}S_2 \right\}.$$

The condition (2.6) $\varkappa = 2$ yields

(2.17) $$\left[\frac{\mathrm{d}A(t_0)}{\mathrm{d}t} - \mathrm{ad}\,(h^{-1}) \frac{\mathrm{d}B(t_0)}{\mathrm{d}t} + [A_0, \mathrm{ad}\,(h^{-1})\, B_0], \mathscr{B} \right] =$$
$$= \mathrm{ad}\,(h^{-1})\, \mathscr{B}S_2 - [A_0 - \mathrm{ad}\,(h^{-1})\, B_0, [A_0 - \mathrm{ad}\,(h^{-1})\, B_0, \mathscr{B}]],$$

and we have

Theorem 2.4. *Be given curves V, $W: M \to G/H$, the lifts $v, w: M \to G$ and the associated mapping $A, B: M \to \mathfrak{g}$ given by (2.1) and (2.2) resp. The curves V and W are in the deformation of order 2 if and only if there is a mapping $h: M \to H$*

such that we have (2.13) *and*

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} - \operatorname{ad}\left(h(t)^{-1}\right) \frac{\mathrm{d}B(t)}{\mathrm{d}t} + \left[A(t), \operatorname{ad}\left(h(t)^{-1}\right) B(t)\right] \in \mathfrak{h} \tag{2.18}$$

$$\text{for each} \quad t \in M .$$

Let there be a mapping $h : M \to H$ such that we have (2.13), i.e.

$$A(t) - \operatorname{ad}\left(h(t)^{-1}\right) B(t) = \varphi(t)\,, \quad \varphi(t) \in \mathfrak{h}\,.$$

Then

$$h(t)\, A(t) - B(t)\, h(t) = h(t)\, \varphi(t)\,,$$

and we get

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} - \operatorname{ad}\left(h(t)^{-1}\right) \frac{\mathrm{d}B(t)}{\mathrm{d}t} + \left[h(t)^{-1} \frac{\mathrm{d}h(t)}{\mathrm{d}t} A(t)\right] \in \mathfrak{h}\,.$$

The relation (2.18) is equivalent to

$$\left[A(t), \operatorname{ad}\left(h(t)^{-1}\right) B(t) + h(t)^{-1} \frac{\mathrm{d}h(t)}{\mathrm{d}t}\right] \in \mathfrak{h} \quad \text{for each} \quad t \in M\,. \tag{2.19}$$

Replacing w by the lift $w'(t) = w(t)\, h(t)$, we get

Theorem 2.5. *The curves V, $W : M \to G/H$ are in the deformation of order* 2 *if and only if there are lifts v, $w : M \to G$ such that we have* (2.14) *and*

$$[A(t), B(t)] \in \mathfrak{h} \quad \textit{for each} \quad t \in M \tag{2.20}$$

for the associated mapping A, $B : M \to \mathfrak{g}$.

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