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SOME INEQUALITIES CONCERNING Π-ISOMORPHISMS

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In this article two problems of S. M. Ulam are solved.

In his book [1] (page 18 of the Russian translation) S. M. ULAM defines the Π-isomorphism in a given Cartesian power E^m, where $m \geqq 2$, as a mapping by which to an element of E^m with coordinates $[x_1, x_2, \ldots, x_m]$ an element with coordinates $[f(x_1), f(x_2), \ldots, f(x_m)]$ is assigned, where f is a one-to-one mapping of E onto E. Using this concept, the Π-automorphism is defined in the usual manner. Now in [1] one asks the questions to find suitable inequalities for the cardinality of the class of subsets of E^m which are Π-isomorphic to a given subset and of the set of Π-automorphisms of a given set, supposing that the cardinality e of the set E is finite. At first we shall solve the second problem.

Let a set $A \subset E^m$ be given and $\tilde{A}$ be the set of coordinates of elements of A, i.e. such a subset of E, that each element of $\tilde{A}$ is a coordinate at least of one element of A and $\tilde{A}$ contains all such elements. Let $\tilde{a}$ be the cardinality of the set $\tilde{A}$; it is evidently a finite number. Let us denote $J(A)$ the set of Π-automorphisms of the set A (we do not consider their values outside A). Then the following theorem is true.

Theorem 1. *Given $\tilde{a}$, for the cardinality of the set $J(A)$ we have the following inequality*:

$$1 \leqq \text{card } J(A) \leqq \tilde{a}!$$

This inequality cannot be improved.

Proof. The proof of the inequality itself is simple. In the set A there exists always an identical Π-automorphism, so that card $J(A) \geqq 1$. Each Π-automorphism of the set A is induced by some one-to-one mapping (permutation) of $\tilde{A}$ onto $\tilde{A}$; such mappings and Π-automorphisms induced by them are assigned one to another in one-to-one manner, so that card $J(A) \leqq \tilde{a}!$, because $\tilde{a}!$ is the number of permutations of the set $\tilde{A}$. Next, we shall prove that the cases card $J(A) = 1$ and card $J(A) = \tilde{a}!$ can occur. At first we take the first case with $\tilde{a} \geqq 2$ (for $\tilde{a} = 1$ the proof is trivial). Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\tilde{a}}$ be the elements of the set $\tilde{A}$. Let A be the set of elements p_i for $i = 1, \ldots, \tilde{a} - 1$ such that the first coordinate of the element p_i is $\tilde{p}_i$ and all other coordinates of the element p_i are equal to $\tilde{p}_{i+1}$. The set constructed in such a manner has only the identical Π-automorphism. Each of the elements $\tilde{p}_1$ and $\tilde{p}_{\tilde{a}}$ is a coordinate of only one element of A and each other element is a coordinate of exactly two elements of A. Let φ be an arbitrary Π-automorphism of the set A induced by a permutation $\tilde{\varphi}$ of the set $\tilde{A}$. As $\tilde{p}_1$ is a coordinate of exactly one element of A and is its first coordinate, $\tilde{\varphi}(\tilde{p}_1)$ must be also a coordinate of exactly one element of A, and must be its first coordinate. But such an element is only $\tilde{p}_1$ and consequently, $\tilde{\varphi}(\tilde{p}_1) = \tilde{p}_1$. But then $\varphi(p_1) = p_1$ and therefore $\tilde{\varphi}(\tilde{p}_2) = \tilde{p}_2$. From this it follows that $\varphi(p_2) = p_2$, as p_2 is the only element of A with the first coordinate $\tilde{p}_2$; from this again it follows that $\tilde{\varphi}(\tilde{p}_3) = \tilde{p}_3$. In this manner we shall prove after a finite number of steps that φ is an identical Π-automorphism. As we have chosen φ arbitrarily, we have proved that in A only an identical Π-automorphism exists. In the second case let again $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\tilde{a}}$ be the elements of the set $\tilde{A}$ and let now p_i for $i = 1, \ldots, \tilde{a}$ be the elements of the set A such that all coordinates of the element p_i are equal to $\tilde{p}_i$. Easily we can verify that each permutation of the set $\tilde{A}$ induces a Π-automorphism of the set A and therefore card $J(A) = \tilde{a}!$

Using Theorem 1 we shall prove a new theorem concerning the first problem. For simplifying the considerations we shall consider the Π-isomorphism as a mapping of the set A into E, so the matter will be with the contracting of the Π-isomorphism onto the set A.

Theorem 2. *For the cardinality of the set* $\mathbf{A}$ *of the sets* Π*-isomorphic with the set* A *the following inequality is true:*

$$\binom{e}{\tilde{a}} \leqq \text{card } \mathbf{A} \leqq \tilde{a}! \binom{e}{\tilde{a}}$$

This inequality cannot be improved.

Proof. Every one-to-one mapping of $\tilde{A}$ into E induces some Π-isomorphism of the set A onto some subset of E^m. The number of those mappings is the same as the number of variations with $\tilde{a}$ elements of e elements, i.e. $\tilde{a}! \binom{e}{\tilde{a}}$; also, each of those Π-isomorphisms is induced by some of those mappings. Now, if the Π-isomorphism φ maps the set A onto some set $B \subset E^m$ and ψ is some Π-automorphism of the set A, then the composed Π-isomorphism $\varphi\psi$ also maps the set A onto B and each Π-isomorphism of A onto B can evidently be expressed so. Therefore, if B is Π-isomorphic with A, then the number of Π-isomorphisms mapping A onto B is equal to card $J(A)$. The cardinality of the class $\mathbf{A}$ is therefore equal to $\tilde{a}!\binom{e}{\tilde{a}}/\text{card } J(A$

Using the inequality of Theorem 1, we get the inequality

$$\binom{e}{\tilde{a}} \leqq \text{card } \mathbf{A} \leqq \tilde{a}! \binom{e}{\tilde{a}} .$$

As the inequality of Theorem 1 cannot be improved, also this inequality cannot be improved.

Corollary. *For the cardinality of the set* $\mathbf{A}$ *the following inequality is true*:

$$1 \leqq \text{card } \mathbf{A} \leqq e!$$

This inequality cannot be improved in general case. (Both the bounds are attained for $\tilde{a} = e$.)

References

[1] *S. M. Ulam:* A Collection of Mathematical Problems. The Russian translation: Нерешенные математические задачи, Москва 1964.

Výtah

NĚKTERÉ NEROVNOSTI TÝKAJÍCÍ SE Π-ISOMORFISMŮ

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V článku jsou dokázány nerovnosti pro mohutnost třídy podmnožin E^m Π-isomorfních dané podmnožině a pro mohutnost množiny Π-automorfismů dané množiny za předpokladu, že mohutnost množiny E je konečná. Je to řešení problémů z [1].

Резюме

НЕКОТОРЫЕ НЕРАВЕНСТВА КАСАЮЩИЕСЯ Π-ИЗОМОРФИЗМОВ

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В статье доказаны неравенства для мощности класса подмножеств E^m Π-изоморфных данному подмножеству и для мощности множества Π-автоморфизмов данного множества с предположением, что мощность множества E конечна. Это решение задач из [1].