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INHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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In the foreword to the paper [1], we stated that the well-posedness of the Duhamel problem is sufficient for the corresponding nonhomogeneous problem to be solvable. However, more is true and proved in the present paper, viz. that an appropriate well-posedness of the nonhomogeneous problem, which will be introduced and studied in the sequel, is equivalent to the well-posedness of an initial value problem, i.e. to the correctness in the sense of [1] (see Theorems 27 nad 28).

The main technical tool in the realization of the above program is an abstract version of the classical Duhamel integral.

In the text, we use the notation and definitions introduced in [1]. In particular, it is necessary to get acquainted with the points 1.10, 5.1 – 5.3, 7.1, 7.4 and 7.7 of [1]. Moreover, we use some results of [1] and [2] which will be quoted when necessary.

1. We denote by $L_{\mathrm{loc}}(R^+, E)$ the space of all functions $f \in R^+ \to E$, integrable over every finite subinterval of R^+ equipped with the following system of seminorms:

$$|f|_T = \int_0^T \|f(\tau)\| \, \mathrm{d}\tau \,, \quad T > 0 \,.$$

2. Proposition. *The space $L_{\mathrm{loc}}(R^+, E)$ is a Fréchet space.*

3. Let $a < b$ and $f \in (a, b) \to E$. The function f will be called disintegrable (in (a, b)) if there exists a function $g \in (a, b) \to E$ such that for every $a < \alpha < \beta < b$, g is integrable over (α, β) and $f(\beta) - f(\alpha) = \int_\alpha^\beta g(\tau) \, \mathrm{d}\tau$.

The function g will be called the disintegral of the function f and denoted by f' or $\mathrm{d}f(t)/\mathrm{d}t$.

By induction, we define the disintegrability of the r-th order and denote by $f^{(r)}$ or $\mathrm{d}^r f(t)/\mathrm{d}t^r$ the r-th disintegral of f.

4. Lemma. *Let $f \in R^+ \to E$ and $r \in \{0, 1, \ldots\}$. If the function f is $(r + 1)$-times disintegrable in R^+, then*

(a) *f is r-times differentiable on R^+,*
(b) *the functions $f, f', \ldots, f^{(r)}$ are continuous on R^+.*
If moreover $f^{(r+1)} \in L_{\mathrm{loc}}(R^+, E)$, then
(c) *$f(0_+), f(0_+), \ldots, f^{(r)}(0_+)$ exist.*

5. Lemma. *For every $k \in \{1, 2, \ldots\}$, the set of all k-times disintegrable functions f on R^+ such that*
(α) *$f^{(k)} \in L_{\mathrm{loc}}(R^+, E)$,*
(β) *$f(0_+) = f'(0_+) = \ldots = f^{(k-1)}(0_+) = 0$*
is dense in $L_{\mathrm{loc}}(R^+, E)$.

6. Lemma. *Let $f \in R^+ \to E$. If $f \in L_{\mathrm{loc}}(R^+, E)$, then*
(a) *$\int_0^t (t-\tau)^l f(\tau)\,\mathrm{d}\tau$ exists for every $t \in R^+$ and $l \in \{0, 1, \ldots\}$,*
(b) *the function $\int_0^t (t-\tau)^l f(\tau)\,\mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$ for every $l \in \{0, 1, \ldots\}$,*
(c) *the function $\int_0^t (t-\tau)^l f(\tau)\,\mathrm{d}\tau$ is $(l+1)$-times disintegrable in R^+ for every $l \in \{0, 1, \ldots\}$,*

(d)
$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{1}{l!}\int_0^t (t-\tau)^l f(\tau)\,\mathrm{d}\tau\right) = \frac{1}{(l-j)!}\int_0^t (t-\tau)^{l-j} f(\tau)\,\mathrm{d}\tau$$

for every $t \in R^+$ and $l \in \{0, 1, \ldots\}$, $j \in \{0, 1, \ldots, l\}$,

(e)
$$\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\left(\frac{1}{l!}\int_0^t (t-\tau)^l f(\tau)\,\mathrm{d}\tau\right) = f(t) \quad \text{for almost all} \quad t \in R^+$$

and every $l \in \{0, 1, \ldots\}$,

(f)
$$\frac{1}{l_1!}\int_0^t (t-\tau)^{l_1}\left(\frac{1}{l_2!}\int_0^\tau (\tau-\sigma)^{l_2} f(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau =$$
$$= \frac{1}{(l_1+l_2+1)!}\int_0^t (t-\tau)^{l_1+l_2+1} f(\tau)\,\mathrm{d}\tau \quad \text{for every} \quad t \in R^+$$

and $l_1, l_2 \in \{0, 1, \ldots\}$,

(g)
$$\frac{1}{l_1!}\int_0^t (t-\tau)^{l_1}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{l_2!}\int_0^\tau (\tau-\sigma)^{l_2} f(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau =$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(l_1+l_2+1)!}\int_0^t (t-\tau)^{l_1+l_2+1} f(\tau)\,\mathrm{d}\tau \quad \text{for almost all} \quad t \in R^+$$

and every $l_1, l_2 \in \{0, 1, \ldots\}$.

7. Lemma. *Let $f \in R^+ \to E$ and $r \in \{0, 1, \ldots\}$. If*

(α) *the function f is $(r + 1)$-times disintegrable in R^+,*

(β) $f(0_+) = f(0_+) = \ldots = f^{(r)}(0_+) = 0$,

(γ) $f^{(r+1)} \in \boldsymbol{L}_{\text{loc}}(R^+, E)$,

then

(a) $$\frac{1}{(l + r - j + 1)!} \int_0^t (t - \tau)^{l+r-j+1} f^{(r+1)}(\tau) \, \mathrm{d}\tau = \frac{1}{l!} \int_0^t (t - \tau)^l f^{(j)}(\tau) \, \mathrm{d}\tau$$

for every $t \in R^+$, $j \in \{0, 1, \ldots, r\}$ and $l \in \{0, 1, \ldots\}$,

(b) $$\frac{1}{r!} \int_0^t (t - \tau)^r f^{(r+1)}(\tau) \, \mathrm{d}\tau = f(t) \text{ for almost every } t \in R^+.$$

8. Lemma. *Let Λ be an open interval, $f \in \Lambda \to E$ and let K be a nonnegative constant. If*

(α) *the function f is integrable over Λ,*

(β) *there exists a dense subset Φ in the space of Lebesgue integrable real functions on Λ such that*

$$\left\| \int_\Lambda \varphi(\tau) f(\tau) \, \mathrm{d}\tau \right\| \leqq K \quad \text{for every} \quad \varphi \in \Phi,$$

then $\|f(t)\| \leqq K$ for almost all $t \in R^+$.

9. Lemma. *Let $q \in R^+ \to R$ and $Q \in R^+ \to E$. If*

(α) $q \in \boldsymbol{L}_{\text{loc}}(R^+, R)$,

(β) *the function Q is continuous on R^+ and bounded on $(0, 1)$,*

then

(a) $\int_0^t Q(t - \tau) \, q(\tau) \, \mathrm{d}\tau$ *exists for every $t \in R^+$,*

(b) *the function $\int_0^t Q(t - \tau) \, q(\tau) \, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$,*

(c) $$\int_0^t (t - \tau)^l \left(\int_0^\tau Q(\tau - \sigma) \, q(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}\tau = \int_0^t \int_0^{t-\tau} (t - \tau - \sigma)^l \, Q(\sigma) \, q(\tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau =$$
$$= \int_0^t Q(t - \tau) \int_0^\tau (\tau - \sigma)^l \, q(\sigma) \, \mathrm{d}\sigma \, \mathrm{d}\tau \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad l \in \{0, 1, \ldots\}.$$

10. Lemma. *Let $q \in R^+ \to E$ and $Q \in R^+ \times E \to E$. If*

(α) $q \in \boldsymbol{L}_{\text{loc}}(R^+, E)$,

(β) *for every $x \in E$, the function $Q(., x)$ is continuous on R^+,*

(γ) *for every $t \in R^+$, the function $Q(t, .)$ is linear,*

(δ) *for every* $T \in R^+$, *there exists a nonnegative constant* K *so that* $\|Q(t, x)\| \leqq$ $\leqq K\|x\|$ *for every* $0 < t < T$ *and* $x \in E$,

then

(a) $\int_0^t Q(t - \tau, q(\tau))\, d\tau$ *exists for every* $t \in R^+$,

(b) *the function* $\int_0^t Q(t - \tau, q(\tau))\, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(c)
$$\int_0^t (t - \tau)^l \left(\int_0^\tau Q(\tau - \sigma, q(\sigma))\, d\sigma \right) d\tau =$$

$$= \int_0^t \left(\int_0^{t-\tau} (t - \tau - \sigma)^l\, Q(\sigma, q(\tau))\, d\sigma \right) d\tau = \int_0^t Q\left(t - \tau, \int_0^\tau (\tau - \sigma)^l\, q(\sigma)\, d\sigma \right) d\tau$$

$$\textit{for every} \quad t \in R^+ \quad \textit{and} \quad l \in \{0, 1, \ldots\}.$$

11. Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $u \in R^+ \to E$. The function u will be called a response for the operators $A_1, A_2, \ldots, A_n$ if

(1) u is n-times disintegrable in R^+,
(2) $u^{(n-i)}(t) \in D(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,
(3) the functions $A_i u^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,
(4) $u(0_+) = u'(0_+) = \ldots = u^{(n-1)}(0_+) = 0$.

12. Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $u \in R^+ \to E$. The function u will be called a Duhamel response for the operators $A_1, A_2, \ldots, A_n$ if u is a response for the operators $A_1, A_2, \ldots, A_n$ such that

$$u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E).$$

13. Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $u \in R^+ \to E$. The function u will be called a null response for the operators $A_1, A_2, \ldots, A_n$ if u is a response for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u = 0$.

14. Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in R^+ \to E$. *If the function* u *is a Duhamel response for the operators* $A_1, A_2, \ldots, A_n$, *then* $u^{(n)} \in$ $\in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$.

Proof. Immediate, from Definition 12 by means of 11(3).

15. Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in R^+ \to E$. *If the operators* $A_1, A_2, \ldots, A_n$ *are closed and the function* u *is a response for the operators* $A_1, A_2, \ldots, A_n$, *then*

(a) *the functions* $u^{(n-i)}$ *are continuous on* R^+ *and bounded on* $(0, 1)$ *for every* $i \in \in \{1, 2, \ldots, n\}$,

(b)
$$\int_0^t (t - \tau)^l u^{(n-i)}(\tau) \, d\tau \in D(A_i) \quad \textit{for every} \quad t \in R^+,$$
$$i \in \{1, 2, \ldots, n\} \quad \textit{and} \quad l \in \{0, 1, \ldots\}.$$

(c)
$$A_i \int_0^t (t - \tau)^l u^{(n-i)}(\tau) \, d\tau = \int_0^t (t - \tau)^l A_i u^{(n-i)}(\tau) \, d\tau$$
$$\textit{for every} \quad t \in R^+, \quad i \in \{1, 2, \ldots, n\} \quad \textit{and} \quad l \in \{0, 1, \ldots\}.$$

Proof. The statement (a) follows from 4 and 11 directly and the statements (b) and (c) by means of [1] 2.4.

16. Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in R^+ \to E$. *If the operators* $A_1, A_2, \ldots, A_n$ *are closed and the function* u *is a Duhamel response for the operators* $A_1, A_2, \ldots, A_n$, *then for every* $i \in \{1, 2, \ldots, n\}$ *and* $l \in \{0, 1, \ldots\}$,
(a) *the function* $\int_0^t (t - \tau)^{l+i} u^{(n)}(\tau) \, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(b)
$$\int_0^t (t - \tau)^{l+i} u^{(n)}(\tau) \, d\tau \in D(A_i) \quad \textit{for every} \quad t \in R^+,$$

(c)
$$A_i \frac{1}{(l + i)!} \int_0^t (t - \tau)^{l+i} u^{(n)}(\tau) \, d\tau = \frac{1}{l!} \int_0^t (t - \tau)^l A_i u^{(n-i)}(\tau) \, d\tau$$
$$\textit{for every} \quad t \in R^+.$$

Proof. The statement (a) follows from 6 (a), (b) and 14.
Since u is supposed to be a response for the operators $A_1, A_2, \ldots, A_n$ we can write by 7 and 14 for every $t \in R^+$

$$\frac{1}{(l + i)!} \int_0^t (t - \tau)^{l+i} u^{(n)}(\tau) \, d\tau = \frac{1}{l!} \int_0^t (t - \tau)^l u^{(n-i)}(\tau) \, d\tau.$$

Now it suffices to apply 15.

17. Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $u \in R^+ \to E$. *If the operators* $A_1, A_2, \ldots, A_n$ *are closed and the function* u *is a Duhamel response for the operators* $A_1, A_2, \ldots, A_n$, *then for every* $l \in \{0, 1, \ldots\}$

(a) *the function* $\int_0^t (t - \tau)^{l+j} u^{(n)}(\tau) \, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$ *for every* $j \in \{0, 1, \ldots, n\}$,

(b) $\int_0^t (t - \tau)^{l+i} u^{(n)}(\tau) \, d\tau \in D(A_i)$ *for every* $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$,

(c) *the function* $A_i \int_0^t (t-\tau)^{l+i} u^{(n)}(\tau)\,d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$ *for every* $i \in \{1, 2, \dots, n\}$,

(d)
$$\frac{1}{l!}\int_0^t (t-\tau)^l u^{(n)}(\tau)\,d\tau + A_1 \frac{1}{(l+1)!}\int_0^t (t-\tau)^{l+1} u^{(n)}(\tau)\,d\tau +$$
$$+ A_2 \frac{1}{(l+2)!}\int_0^t (t-\tau)^{l+2} u^{(n)}(\tau)\,d\tau + \dots + A_n \frac{1}{(l+n)!}\int_0^t (t-\tau)^{l+n} u^{(n)}(\tau)\,d\tau =$$
$$= \frac{1}{l!}\int_0^t (t-\tau)^l [u^{(n)}(\tau) + A_1 u^{(n-1)}(\tau) + \dots + A_n u(\tau)]\,d\tau \quad \textit{for every} \quad t \in R^+ .$$

Proof. The statement (a) follows from 6 and 14. The statements (b), (c) and (d) are easy consequences of 12 and 16.

18. Proposition. *Let* $A_1, A_2, \dots, A_n \in L^+(E)$, $n \in \{1, 2, \dots\}$, *and* $u \in R^+ \to E$. *The function* u *is a null response for the operators* $A_1, A_2, \dots, A_n$ *if and only if it is a null solution for these operators.*

19. Let $A_1, A_2, \dots, A_n \in L^+(E)$, $n \in \{1, 2, \dots\}$, and $m \in \{0, 1, \dots\}$. The system of operators $A_1, A_2, \dots, A_n$ will be called converse of class m if

(A) there exists a set J which is a dense subset of $R^+ \to D(A_1) \cap D(A_2) \cap \dots \cap D(A_n)$ in the space $L_{\mathrm{loc}}(R^+, E)$ so that for every $h \in J$ we can find a Duhamel response u for the operators $A_1, A_2, \dots, A_n$ for which $u^n + A_1 u^{(n-1)} + \dots + A_n u = h$,

(B) there exist two nonnegative constants M, ω such that for every Duhamel response u for the operators $A_1, A_2, \dots, A_n$, for every $t \in R^+$ and for every $i \in \{1, 2, \dots, n\}$
$$\left\| \frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,d\tau \right\| \leqq$$
$$\leqq M e^{\omega t} \int_0^t \| u^{(n)}(\tau) + A_1 u^{(n-1)}(\tau) + \dots + A_n u(\tau) \|\,d\tau .$$

20. Remark. The property 19(B) can be modified in a similar way as in [1] 7.5. These modifications are left to the reader.

21. Let $A_1, A_2, \dots, A_n \in L^+(E)$, $n \in \{1, 2, \dots\}$. The system of operators $A_1, A_2, \dots, A_n$ will be called converse if there exists an $m \in \{0, 1, \dots\}$ so that it is converse of class m.

22. Proposition. *Let* $A_1, A_2, \dots, A_n \in L^+(E)$, $n \in \{1, 2, \dots\}$. *If the system of operators* $A_1, A_2, \dots, A_n$ *is converse, then it is also definite.*

Proof. An immediate consequence of 18, 19 and 21.

23. Lemma. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ and let $N, \varkappa$ be two nonnegative constants. If the operators $A_1, A_2, \ldots, A_n$ are closed and if there exists a dense linear set $H \subseteq L_{\text{loc}}(R^+, E)$ such that for every $h \in H$, there exists a Duhamel response u for the operators $A_1, A_2, \ldots, A_n$ satisfying*

$$u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u = h\,,$$

$$\left\|\frac{1}{m!}\int_0^t (t-\tau)^m A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau\right\| \leqq N e^{\varkappa t}\int_0^t \|h(\tau)\|\,\mathrm{d}\tau$$

for every $t \in R^+$ and $i \in \{1. 2, \ldots, n\}$, then for every $g \in L_{\text{loc}}(R^+, E)$ there exists a function $v \in R^+ \to E$ such that

(a) *v is continuous on R^+ and bounded on $(0, 1)$,*

(b) *$\int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,*

(c) *the functions $A_i \int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,*

(d) $$v(t) + A_1\int_0^t v(\tau)\,\mathrm{d}\tau + \ldots + \frac{1}{(n-1)!}A_n\int_0^t (t-\tau)^{n-1} v(\tau)\,\mathrm{d}\tau = $$
$$= \frac{1}{m!}\int_0^t (t-\tau)^m g(\tau)\,\mathrm{d}\tau \quad \textit{for every} \quad t \in R^+\,,$$

(e) $$\left\|\frac{1}{(i-1)!}A_i\int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau\right\| \leqq N e^{\varkappa t}\int_0^t \|g(\tau)\|\,\mathrm{d}\tau$$
$$\textit{for every} \quad t \in R^+ \quad \textit{and} \quad i \in \{1, 2, \ldots, n\}\,.$$

Proof. Let us fix a linear dense subset $H \subseteq L_{\text{loc}}(R^+, E)$ for which our hypothesis holds.

Let $g \in L_{\text{loc}}(R^+, E)$ be arbitrary but fixed.

By our hypothesis we can find a sequence u_k, $k \in \{1, 2, \ldots\}$, of Duhamel responses for the operators $A_1, A_2, \ldots, A_n$ such that

(1) $$u_k^{(n)} + A_1 u_k^{(n-1)} + \ldots + A_n u_k \in H \quad \text{for every} \quad k \in \{1, 2, \ldots\}\,,$$

(2) $$u_k^{(n)} + A_1 u_k^{(n-1)} + \ldots + A_n u_k \xrightarrow[k\to\infty]{} g \quad \text{in} \quad L_{\text{loc}}(R^+, E)\,,$$

(3) $$\left\|\frac{1}{m!}\int_0^t (t-\tau)^m A_i\, u^{(n-i)}(\tau)\,\mathrm{d}\tau\right\| \leqq$$
$$\leqq N e^{\varkappa t}\int_0^t \left\|u_k^{(n)}(\tau) + A_1\, u_k^{(n-1)}(\tau) + \ldots + A_n\, u_k(\tau)\right\|\,\mathrm{d}\tau$$
$$\text{for every} \quad t \in R^+\,, \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad k \in \{1, 2, \ldots\}\,,$$

(4) $$\left\|\frac{1}{m!}\int_0^t (t-\tau)^m A_i u_{k_1}^{(n-i)}(\tau)\,d\tau - \frac{1}{m!}\int_0^t (t-\tau)^m A_i u_{k_2}^{(n-i)}(\tau)\,d\tau\right\| \leqq$$
$$\leqq N e^{\varkappa t}\int_0^t \| [u_{k_1}^{(n)}(\tau) + A_1 u_{k_1}^{(n-1)}(\tau) + \ldots + A_n u_{k_1}(\tau)] -$$
$$- [u_{k_2}^{(n)}(\tau) + A_1 u_{k_2}^{(n-1)}(\tau) + \ldots + A_n u_{k_2}(\tau)] \|\, d\tau$$
for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k_1, k_2 \in \{1, 2, \ldots\}$.

Let us now write for $t \in R^+$ and $k \in \{1, 2, \ldots\}$

(5) $$v_k(t) = \frac{1}{m!}\int_0^t (t-\tau)^m u_k^{(n)}(\tau)\,d\tau\,.$$

It is clear from (6) (f) that

(6) $$\frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1} v_k(\tau)\,d\tau = \frac{1}{(m+i)!}\int_0^t (t-\tau)^{m+i} u_k^{(n)}(\tau)\,d\tau$$
for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$.

Using 6 and 16 we obtain from (5) and (6) that

(7) the functions v_k are continuous on R^+ and bounded on $(0, 1)$ for every $k \in \in \{1, 2, \ldots\}$,

(8) $$\int_0^t (t-\tau)^i v_k(\tau)\,d\tau \in D(A_i)$$
for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$,

(9) the functions $A_i \int_0^t (t-\tau)^{i-1} v_k(\tau)\,d\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$,

(10) $$A_i \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1} v_k(\tau)\,d\tau = \frac{1}{m!}\int_0^t (t-\tau)^m A_i u_k^{(n-i)}(\tau)\,d\tau$$
for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$.

Moreover, using 17, we obtain from (5) and (6) that

(11) $$v_k(t) + A_1 \int_0^t v_k(\tau)\,d\tau + \ldots + A_n \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} v_k(\tau)\,d\tau =$$
$$= \frac{1}{m!}\int_0^t (t-\tau)^m [u_k^{(n)}(\tau) + A_1 u_k^{(n-1)}(\tau) + \ldots + A_n u_k(\tau)]\,d\tau$$
for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

We have by (3), (4) and (10)

$$(12)\qquad \left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} v_k(\tau)\, \mathrm{d}\tau \right\| \leqq$$

$$\leqq N e^{\varkappa t} \int_0^t \left\| u_k^{(n)}(\tau) + A_1 u_k^{(n-1)}(\tau) + \ldots + A_n u_k(\tau) \right\| \mathrm{d}\tau$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots\}$,

$$(13)\qquad \left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} v_{k_1}(\tau)\, \mathrm{d}\tau - A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} v_{k_2}(\tau)\, \mathrm{d}\tau \right\| \leqq$$

$$\leqq N e^{\varkappa t} \int_0^t \left\| \left[u_{k_1}^{(n)}(\tau) + A_1 u_{k_1}^{(n-1)}(\tau) + \ldots + A_n u_{k_1}(\tau) \right] - \right.$$

$$\left. - \left[u_{k_2}^{(n)}(\tau) + A_1 u_{k_2}^{(n-1)}(\tau) + \ldots + A_n u_{k_2}(\tau) \right] \right\| \mathrm{d}\tau$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $k_1, k_2 \in \{1, 2, \ldots\}$.

Moreover, we obtain from (11) and (13) that

$$(14)\qquad \left\| v_{k_1}(t) - v_{k_2}(t) \right\| \leqq$$

$$\leqq \left[n N e^{\varkappa t} + \frac{t^m}{m!} \right] \int_0^t \left\| \left[u_{k_1}^{(n)}(\tau) + A_1 u_{k_1}^{(n-1)}(\tau) + \ldots + A_n u_{k_1}(\tau) \right] - \right.$$

$$\left. - \left[u_{k_2}^{(n)}(\tau) + A_1 u_{k_2}^{(n-1)}(\tau) + \ldots + A_n u_{k_2}(\tau) \right] \right\| \mathrm{d}\tau$$

for every $t \in R^+$ and $k_1, k_2 \in \{1, 2, \ldots\}$.

In virtue of (2) and (14), there exists a function $v \in R^+ \to E$ such that

$$(15)\qquad v_k(t) \xrightarrow[k \to \infty]{} v(t) \quad \text{for every} \quad t \in R^+ .$$

It is clear from (2), (7), (14) and (15) that
(16) the function v is continuous on R^+ and bounded on $(0, 1)$,

$$(17)\qquad \int_0^t (t-\tau)^{i-1} v_k(\tau)\, \mathrm{d}\tau \xrightarrow[k \to \infty]{} \int_0^t (t-\tau)^{i-1} v(\tau)\, \mathrm{d}\tau$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Since the operators $A_1, A_2, \ldots, A_n$ are supposed to be closed we see easily from (2), (8), (13) and (17) that

$$(18)\qquad \int_0^t (t-\tau)^{i-1} v(\tau)\, \mathrm{d}\tau \in D(A_i) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\},$$

(19) $$A_i \int_0^t (t-\tau)^{i-1} v_k(\tau)\, d\tau \xrightarrow[k\to\infty]{} A_i \int_0^t (t-\tau)^{i-1} v(\tau)\, d\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \dots, n\}\,.$$

Further, we see from (2), (9), (10), (13) and (19) that

(20) the functions $A_i \int_0^t (t-\tau)^{i-1} v(\tau)\, d\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \dots, n\}$,

(21) $$v(t) + A_1 \int_0^t v(\tau)\, d\tau + \dots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau = \frac{1}{m!} \int_0^t (t-\tau)^m g(\tau)\, d\tau \quad \text{for every} \quad t \in R^+\,.$$

Finally, by (2), (12) and (19)

(22) $$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} v(\tau)\, d\tau \right\| \leqq N e^{\varkappa t} \int_0^t \|g(\tau)\|\, d\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \dots, n\}\,.$$

The statements of our lemma are contained in (16), (18) and (20)–(22).

24. Proposition. *Let* $A_1, A_2, \dots, A_n \in L^+(E)$, $n \in \{1, 2, \dots\}$, *and* $m \in \{0, 1, \dots\}$. *If*

(α) *the operators* $A_1, A_2, \dots, A_n$ *are closed,*

(β) *the set* $D_1(A_1, A_2, \dots, A_n)$ *is dense in* E,

(γ) *the system of operators* $A_1, A_2, \dots, A_n$ *is converse of class* m,

then there exists an $\mathscr{F} \in R^+ \times L_{loc}(R^+, E) \to E$ *such that*

(a) *for every* $h \in L_{loc}(R^+, E)$, *the function* $\mathscr{F}(.,\, h)$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(b) $\int_0^t (t-\tau)^{i-1} \mathscr{F}(\tau, h)\, d\tau \in D(A_i)$ *for every* $h \in L_{loc}(R^+, E)$, $t \in R^+$ *and* $i \in \{1, 2, \dots, n\}$,

(c) *for every* $h \in L_{loc}(R^+, E)$ *and* $i \in \{1, 2, \dots, n\}$, *the function* $A_i \int_0^t (t-\tau)^{i-1} \cdot \mathscr{F}(\tau, h)\, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(d) $$\mathscr{F}(t, h) + A_1 \int_0^t \mathscr{F}(\tau, h)\, d\tau + A_2 \int_0^t (t-\tau)\, \mathscr{F}(\tau, h)\, d\tau + \dots$$

$$\dots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{F}(\tau, h)\, d\tau = \frac{1}{m!} \int_0^t (t-\tau)^m h(\tau)\, d\tau$$

$$\textit{for every} \quad h \in L_{loc}(R^+, E) \quad \textit{and} \quad t \in R^+\,,$$

(e) *for every $t \in R^+$, the function $\mathscr{F}(t, .)$ is a linear mapping,*

(f) *there exist two nonnegative constants M, ω so that for every $h \in L_{loc}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots\}$*

$$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} \mathscr{F}(\tau, h)\, d\tau \right\| \leqq M e^{\omega t} \int_0^t \|h(\tau)\|\, d\tau .$$

Proof. It follows from (21) and (22) that

(1) the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Further, we can choose, by the assumption, a dense linear subset $H \subseteq L_{loc}(R^+, E)$ and two nonnegative constants M, ω so that

(2) for every $h \in H$, there exists a response u for the operators $A_1, A_2, \ldots, A_n$ so that $u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u = h$,

(3) for every response u for the operators $A_1, A_2, \ldots, A_n$, for every $t \in R^+$ and every $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\, d\tau \right\| \leqq$$

$$\leqq M e^{\omega t} \int_0^t \|u^{(n)}(\tau) + A_1 u^{(n-1)}(\tau) + \ldots + A_n u(\tau)\|\, d\tau .$$

Now we see easily from the assumptions and from (1)–(3) that the hypotheses of 23 and [1] 7.10 are fulfilled and the assertion of our proposition then follows from here.

25. Proposition. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ and $\mathscr{F} \in R^+ \times L_{loc}(R^+, E) \to E$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *the properties 24 (a)–(d) are fulfilled,*

then for every $l \in \{0, 1, \ldots\}$

(a) *for every $h \in L_{loc}(R^+, E)$, the function $\int_0^t (t-\tau)^l \mathscr{F}(\tau, h)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,*

(b) $\int_0^t (t-\tau)^{i+l} \mathscr{F}(\tau, h)\, d\tau \in D(A_i)$ *for every $h \in L_{loc}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,*

(c) *for every $h \in L_{loc}(R^+ E)$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t-\tau)^{i+l} . \mathscr{F}(\tau, h)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,*

(d) $$\frac{1}{l!} \int_0^t (t-\tau)^l \mathscr{F}(\tau, h) + A_1 \frac{1}{(l+1)!} \int_0^t (t-\tau)^{l+1} \mathscr{F}(\tau, h)\, d\tau +$$

$$+ A_2 \frac{1}{(l+2)!} \int_0^t (t-\tau)^{l+2} \mathscr{F}(\tau, h)\, d\tau + \ldots + A_n \frac{1}{(l+n)!} \int_0^t (t-\tau)^{l+n} \mathscr{F}(\tau, h)\, d\tau =$$

$$= \frac{1}{(l+m+1)!} \int_0^t (t-\tau)^{l+m+1} h(\tau)\, d\tau$$

$$\textit{for every} \quad h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E) \quad \textit{and} \quad t \in R^+ ,$$

(e) *for every* $t \in R^+$, *the function* $\int_0^t (t-\tau)^l \mathscr{F}(\tau, .)\, d\tau$ *is a linear mapping,*

(f) *there exist two nonnegative constants* M, ω *so that for every* $h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$, *and* $i \in \{1, 2, \ldots, n\}$

$$\left\| A_i \frac{1}{(i+l)!} \int_0^t (t-\tau)^{i+l} \mathscr{F}(\tau, h)\, d\tau \right\| \leqq M e^{\omega t} \frac{t^{l+1}}{(l+1)!} \int_0^t \|h(\tau)\|\, d\tau .$$

Proof. An easy consequence of 24 by virtue of [1] 1.8, [1] 2.4, [1] 2.7 and [1] 2.9.

26. Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ *and* $\mathscr{F} \in R^+ \times \boldsymbol{L}_{\mathrm{loc}}(R^+, E) \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite,*

(γ) *the conditions* 24 (a)–(d) *are fulfilled,*
then for every $h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$ *such that* $A_1 h, A_2 h, \ldots, A_n h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$ *and for every* $t \in R^+$

$$\mathscr{F}(t, h) + \int_0^t \mathscr{F}(\tau, A_1 h)\, d\tau + \int_0^t (t-\tau) \mathscr{F}(\tau, A_2 h)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{F}(\tau, A_n h)\, d\tau = \frac{1}{m!} \int_0^t (t-\tau)^m h(\tau)\, d\tau .$$

Proof. Let us fix an $h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$ such that $A_1 h, A_2 h, \ldots, A_n h$ belong also to $\boldsymbol{L}_{\mathrm{loc}}(R^+, E)$ and let us put for $t \in R^+$

$$w(t) = \mathscr{F}(t, h) + \int_0^t \mathscr{F}(\tau, A_1 h) + \int_0^t (t-\tau) \mathscr{F}(\tau, A_2 h)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{F}(\tau, A_n h)\, d\tau - \frac{1}{m!} \int_0^t (t-\tau)^m h(\tau)\, d\tau .$$

A simple calculation using the properties 24 (a)–(d) and 25 (a)–(d) shows that the function w has the properties [1] 7.10 (1)–(4). Hence by lemma [1] 7.10, $w(t) = 0$ for every $t \in R^+$ and this proves our proposition.

27. Theorem. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $m \in \{0, 1, \ldots\}$. *If*
(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*
(β) *the set* $D_1(A_1, A_2, \ldots, A_n)$ *is dense in* E,
(γ) *the system of operators* $A_1, A_2, \ldots, A_n$ *is correct of class* m,
then this system is also converse of class m.

Proof. We see easily that, by our assumptions, [2] 2.13 is applicable and consequently we can find a function $\mathscr{W} \in R^+ \times E \to E$ so that
(1) the conditions [2] 2.13 (a)–(f) are fulfilled.
Since, by (1), the assumptions of 10 are fulfilled, we have
(2) $\int_0^t \mathscr{W}(t - \tau, h(\tau))\, d\tau$ exists for every $h \in \boldsymbol{L}_{loc}(R^+, E)$ and $t \in R^+$,
(3) the function $\int_0^t \mathscr{W}(t - \tau, h(\tau))\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(4)
$$\frac{1}{l!}\int_0^t (t-\tau)^l \int_0^t \mathscr{W}(\tau - \sigma, h(\sigma))\, d\sigma\, d\tau =$$
$$= \int_0^t \frac{1}{l!}\int_0^{t-\tau} (t - \tau - \sigma)^l\, \mathscr{W}(\sigma, h(\tau))\, d\sigma\, d\tau =$$
$$= \int_0^t \mathscr{W}\Big(t-\tau, \frac{1}{l!}\int_0^t (\tau - \sigma)^l\, h(\sigma)\Big)\, d\sigma\, d\tau$$

for every $h \in \boldsymbol{L}_{loc}(R^+, E)$, $t \in R^+$ and $l \in \{0, 1, \ldots\}$,

(5)
$$\int_0^t A_i \frac{1}{(i-1)!}\int_0^{t-\tau} (t - \tau - \sigma)^{i-1}\, \mathscr{W}(\sigma, h(\tau))\, d\sigma\, d\tau$$

exists for every $h \in \boldsymbol{L}_{loc}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(6) the function
$$\int_0^t A_i \frac{1}{(i-1)!}\int_0^{t-\tau} (t - \tau - \sigma)^{i-1}\, \mathscr{W}(\sigma, h(\tau))\, d\sigma\, d\tau$$

is countinuous on R^+ and bounded on $(0,1)$ for every $h \in \boldsymbol{L}_{loc}(R^+, E)$ and $i \in \{1, 2, \ldots, n\}$.

Using [1] 2.4, we obtain from (2)–(6) that

(7)
$$\frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\int_0^t \mathscr{W}(\tau - \sigma, h(\sigma))\, d\sigma\, d\tau \in D(A_i)$$

for every $h \in \boldsymbol{L}_{loc}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(8)
$$A_i \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\int_0^t \mathscr{W}(\tau - \sigma, h(\sigma))\, d\sigma\, d\tau =$$

$$= \int_0^t A_i \frac{1}{(i-1)!} \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}(\sigma, h(\tau)) \, d\sigma \, d\tau$$

for every $h \in L_{loc}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Moreover, let us fix according to (1) [[2] 2.13 (f)] two nonnegative constants M, ω so that

(9) $$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x) \, d\tau \right\| \leqq M e^{\omega t} \|x\|$$

for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Let us further denote by $\mathbf{C}^{(m)}(R^+, E)$ the set of all $(m + 1)$-times desintegrable functions f in R^+ such that $f^{(m+1)} \in L_{loc}(R^+, E)$ and $f(0_+) = f'(0_+) = \ldots = f^{(m)}(0_+) = 0$.

It follows from 5 that

(10) the set $\mathbf{C}^{(m)}(R^+, E)$ is dense in $L_{loc}(R^+, E)$.

Since the system of operators $A_1, A_2, \ldots, A_n$ is definite by [2] 2.10 and [2] 2.12, it is easy to see from (10) that the assertion of our theorem will follow if we prove that

(11) for every $h \in \mathbf{C}^{(m)}(R^+, E)$, there exists a Duhamel response u for the operators $A_1, A_2, \ldots, A_n$ such that

$$u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u = h$$

and for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t - \tau)^m A_i u^{(n-1)}(\tau) \, d\tau \right\| \leqq M e^{\omega t} \int_0^t \|h(\tau)\| \, d\tau .$$

To this aim, let us choose for the rest of the proof an arbitrary $h \in \mathbf{C}^{(m)}(R^+, E)$. With regard to 4, it has the following properties:

(12) h is $(m + 1)$-times disintegrable in R^+,

(13) $h, h', \ldots, h^m$ are continuous on R^+ and bounded on $(0, 1)$ and $h^{(m+1)} \in L_{loc}(R^+, E)$,

(14) $h(0_+) = h'(0_+) = \ldots = h^{(m)}(0_+) = 0$.

Now we define for $t \in R^+$

(15) $$u(t) = \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} \left(\int_0^t \mathscr{W}(\tau - \sigma, h^{(m+1)}(\sigma)) \, d\sigma \right) d\tau .$$

Using [1] 2.8 we obtain from (2), (3), (12), (13) and (15) that

(16) the function u is a n-times disintegrable in R^+,

(17) $$u^{(n-i)}(t) = \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \left(\int_0^t \mathscr{W}(\tau - \sigma, h^{(m+1)}(\sigma)) \, d\sigma \right) d\tau$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

$$(18) \qquad u^{(n)}(t) = \int_0^t \mathscr{W}(t - \tau, h^{(m+1)}(\tau))\, d\tau \quad \text{for every} \quad t \in R^+ .$$

Now we obtain from (7), (8), (12), (13) and (17) that

$$(19) \qquad u^{(n-i)}(t) \in D(A_i) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\} ,$$

$$(20) \quad A_i u^{(n-i)}(t) = \int_0^t \left(A_i \frac{1}{(i-1)!} \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}(\sigma, h^{(m+1)}(\tau))\, d\sigma \right) d\tau$$
$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\} .$$

It follows easily from (6), (12), (13) and (20) that
(21) the functions $A_i u^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$.
Further, by means of 8, we get from (12)–(14) that

$$(22) \qquad \frac{1}{m!} \int_0^t (t - \tau)^m h^{(m+1)}(\tau)\, d\tau = h(t) \quad \text{for every} \quad t \in R^+ .$$

Now it follows from (1), (12), (13), (14), (18), (19), (20) and (22) that

$$(23) \quad u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = \frac{1}{m!} \int_0^t (t - \tau)^m h^{(m+1)}(\tau)\, d\tau = h(t)$$
$$\text{for every} \quad t \in R^+ .$$

We conclude from (16), (19), (21) and (23) that
(24) the function u is a Duhamel response for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n)} + A_1 u^{(n-1)} + \ldots + A_n u = h$.
Using [1] 2.9, we obtain from (4), (17) and (22) that

$$(25) \qquad \frac{1}{m!} \int_0^t (t - \tau)^m u^{(n-i)}(\tau)\, d\tau =$$
$$= \frac{1}{m!} \int_0^t (t - \tau)^m \left[\frac{1}{(i-1)!} \int_0^\tau (\tau - \sigma)^{i-1} \left(\int_0^\sigma \mathscr{W}(\sigma - \varrho, h^{(m+1)}(\varrho))\, d\varrho \right) d\sigma \right] d\tau =$$
$$= \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \left[\frac{1}{m!} \int_0^\tau (\tau - \sigma)^m \left(\int_0^\sigma \mathscr{W}(\sigma - \varrho, h^{(m+1)}(\varrho))\, d\varrho \right) d\sigma \right] d\tau =$$
$$= \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \left[\int_0^\tau \mathscr{W}\left(\tau - \sigma, \frac{1}{m!} \int_0^\sigma (\sigma - \varrho)^m h^{(m+1)}(\varrho)\, d\varrho \right) d\sigma \right] d\tau =$$
$$= \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \left[\int_0^\tau \mathscr{W}(\tau - \sigma, h(\sigma))\, d\sigma \right] d\tau$$
$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\} .$$

On the other hand, we get from (16), (19) and (21) by means of [1] 2.4 that

$$(26)\quad \frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-i)}(\tau) \in \mathsf{D}(A_i) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\},$$

$$(27)\quad A_i \frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-i)}(\tau)\,\mathrm{d}\tau = \frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,\mathrm{d}\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Using [1] 2.9, we obtain from (2)–(4), (17) and (22) that

$$(28)\quad \frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-i)}(\tau)\,\mathrm{d}\tau =$$

$$= \frac{1}{m!}\int_0^t (t-\tau)^m \left[\frac{1}{(i-1)!}\int_0^\tau (\tau-\sigma)^{i-1}\left(\int_0^\sigma \mathscr{W}(\sigma-\varrho, h^{(m+1)}(\varrho))\,\mathrm{d}\varrho\right)\mathrm{d}\sigma\right]\mathrm{d}\tau =$$

$$= \frac{1}{(m+i)!}\int_0^t (t-\tau)^{m+i}\int_0^\tau \mathscr{W}(\tau-\sigma, h^{(m+1)}(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau =$$

$$= \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\left[\frac{1}{m!}\int_0^\tau (\tau-\sigma)^m\left(\int_0^\sigma \mathscr{W}(\sigma-\varrho, h^{(m+1)}(\varrho)\,\mathrm{d}\varrho\right)\mathrm{d}\sigma\right]\mathrm{d}\tau =$$

$$= \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\int_0^\tau \mathscr{W}(\tau-\sigma, h(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Now it follows from (7), (8) and (26)–(28) that

$$(29)\quad \frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,\mathrm{d}\tau = A_i \frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-i)}(\tau)\,\mathrm{d}\tau =$$

$$= A_i \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\int_0^\tau \mathscr{W}(\tau-\sigma, h(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau =$$

$$= \int_0^t A_i \frac{1}{(i-1)!}\int_0^{t-\tau} (t-\tau-\sigma)^{i-1}\,\mathscr{W}(\sigma, h(\tau))\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Finally, by (9) and (29),

$$(30)\quad \left\|\frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,\mathrm{d}\tau\right\| \leqq$$

$$\leqq \int_0^t \left\|A_i \frac{1}{(i-1)!}\int_0^{t-\tau} (t-\tau-\sigma)^{i-1}\,\mathscr{W}(\sigma, h(\tau))\,\mathrm{d}\sigma\right\|\mathrm{d}\tau \leqq$$

$$\leqq \int_0^t M e^{\omega(t-\tau)} \|h(\tau)\| \, \mathrm{d}\tau \leqq M e^{\omega t} \int_0^t \|h(\tau)\| \, \mathrm{d}\tau$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Since h was an arbitrary element from $\mathsf{C}^{(m)}(R^+, E)$ as follows from (12)–(14) by means of 4, it is obvious that the desired assertion (11) was proved in (24) and (30). The proof is complete.

28. Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. If*
(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*
(β) *the set $\mathsf{D}_{n+m+1}(A_1, A_2, \ldots, A_n)$ is dense in E,*
(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is converse of class m,*
then this system is also correct of class m.

Proof. We see easily that, by our assumptions, 24 is applicable and consequently we can find a function $\mathscr{F} \in R^+ \times \boldsymbol{L}_{\mathrm{loc}}(R^+, E) \to E$ so that
(1) the conditions 24 (a)–(f) are fulfilled.
According to (1) we can fix nonnegative constants M, ω so that

(2) $$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} \mathscr{F}(\tau, h) \, \mathrm{d}\tau \right\| \leqq M e^{\omega t} \int_0^t \|h(\tau)\| \, \mathrm{d}\tau$$

for every $h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

For $x \in E$ denote by
(3) $\mathsf{c}(x)$ the constant function from $R^+ \to E$ identically equal to x.
Put for $x \in E$ and $t \in R^+$

(4) $$\mathscr{V}(t, x) = \mathscr{F}(t, \mathsf{c}(x)).$$

From (1)–(4), we infer easily that
(5) for every $x \in E$, the function $\mathscr{V}(., x)$ is continuous on R^+ and bounded on $(0, 1)$,
(6) $\int_0^t (t-\tau)^{i-1} \mathscr{V}(\tau, x) \in \mathsf{D}(A_i)$ for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,
(7) for every $x \in E$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t-\tau)^{i-1} \mathscr{V}(\tau, x) \, \mathrm{d}\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(8) $$\mathscr{V}(t, x) + A_1 \int_0^t \mathscr{V}(\tau, x) \, \mathrm{d}\tau + A_2 \int_0^t (t-\tau) \mathscr{V}(\tau, x) \, \mathrm{d}\tau + \ldots$$
$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{V}(\tau, x) \, \mathrm{d}\tau = \frac{t^{m+1}}{(m+1)!} x$$

for every $x \in E$ and $t \in R^+$,

(9) for every $t \in R^+$, the function $\mathscr{V}(t, .)$ is a linear mapping,

(10)
$$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} \mathscr{V}(\tau, x)\, d\tau \right\| \leqq M e^{\omega t}\, t \|x\|$$
for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

On the other hand, using 25, we obtain from (1), (3), (9) and (10) that
(11) for every $x \in E$ and $l \in \{0, 1, \ldots\}$, the function
$\int_0^t (t-\tau)^l \mathscr{V}(\tau, x)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(12) $\int_0^t (t-\tau)^{i+l} \mathscr{V}(\tau, x)\, d\tau \in D(A_i)$ for every $x \in E$, $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$,

(13) for every $x \in E$, $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$, the function $A_i \int_0^t (t-\tau)^{i+l}$. $\mathscr{V}(\tau, x)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(14)
$$\frac{1}{l!} \int_0^t (t-\tau)^l \mathscr{V}(\tau, x)\, d\tau + A_1 \frac{1}{(l+1)!} \int_0^t (t-\tau)^{l+1} \mathscr{V}(\tau, x)\, d\tau +$$
$$+ A_2 \frac{1}{(l+2)!} \int_0^t (t-\tau)^{l+2} \mathscr{V}(\tau, x)\, d\tau + \ldots$$
$$\ldots + A_n \frac{1}{(l+n)!} \int_0^t (t-\tau)^{l+n} \mathscr{V}(\tau, x)\, d\tau = \frac{t^{l+m+2}}{(l+m+2)!} x$$
for every $x \in E$, $t \in R^+$ and $l \in \{0, 1, \ldots\}$.

Similarly, by 26

(15)
$$\mathscr{V}(t, x) = -\int_0^t \mathscr{V}(\tau, A_1 x)\, d\tau - \int_0^t (t-\tau)\, \mathscr{V}(\tau, A_2 x)\, d\tau - \ldots$$
$$\ldots - \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{V}(\tau, A_n x) + \frac{t^{m+1}}{(m+1)!} x$$
for every $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$.

Using 10, we obtain from (5)–(7)
(16) $\int_0^t \mathscr{V}(t-\tau, h(\tau))\, d\tau$ exists for every $h \in \boldsymbol{L}_{loc}(R^+, E)$ and $t \in R^+$,

(17) for every $h \in \boldsymbol{L}_{loc}(R^+, E)$, the function $\int_0^t \mathscr{V}(t-\tau, h(\tau))\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(18) $$\int_0^t (t-\tau)^{i-1} \int_0^\tau \mathscr{V}(\tau-\sigma, h(\sigma))\, d\sigma\, d\tau = \int_0^t \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau$$

for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(19) $$\int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau$$

exists for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(20) for every $h \in L_{\text{loc}}(R^+, E)$ and $i \in \{1, 2, \ldots, n\}$, the function

$\int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau$ is continuous on R^+

and bounded on $(0, 1)$.

By means of [1] 2.4, we get from (16)–(20) that

(21) $$\int_0^t \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau \in D(A_i)$$

for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(22) $$A_i \int_0^t \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau =$$
$$= \int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau$$

for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

By (18), (21) and (22)

(23) $$\int_0^t (t-\tau)^{i-1} \mathscr{V}(\tau-\sigma, h(\sigma))\, d\sigma\, d\tau \in D(A_i)$$

for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(24) $$A_i \int_0^t (t-\tau)^{i-1} \mathscr{V}(\tau-\sigma, h(\sigma))\, d\sigma\, d\tau =$$
$$= \int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{V}(\sigma, h(\tau))\, d\sigma\, d\tau\,.$$

for every $h \in L_{\text{loc}}(R^+, E)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

By (20) and (24)

(25) for every $h \in L_{\text{loc}}(R^+, E)$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t-\tau)^{i-1} \cdot$

$\cdot \int_0^\tau \mathscr{V}(\tau-\sigma, h(\sigma))\, d\sigma\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$.

Finally, by (8) and (24)

$$(26)\qquad \int_0^t \mathscr{V}(t-\tau, h(\tau))\,\mathrm{d}\tau + A_1 \int_0^t \int_0^\tau \mathscr{V}(\tau-\sigma, h(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau +$$

$$+ A_2 \int_0^t (t-\tau) \int_0^\tau \mathscr{V}(\tau-\sigma, h(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau + \ldots$$

$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \int_0^\tau \mathscr{V}(\tau-\sigma, h(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}\tau =$$

$$= \frac{1}{(m+1)!} \int_0^t (t-\tau)^{m+1}\, h(\tau)\,\mathrm{d}\tau$$

for every $h \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$ and $t \in R^+$.

By 6 and 7

$$(27)\qquad \frac{1}{(m+1)!} \int_0^t (t-\tau)^{m+1}\, h'(\tau)\,\mathrm{d}\tau = \frac{1}{m!} \int_0^t (t-\tau)^m\, h(\tau)\,\mathrm{d}\tau$$

for every $h \in R^+ \to E$ such that h is disintegrable in R^+,

$h(0_+) = 0$ and $h' \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$.

Now we see easily from (1), (16), (17), (23), (25) and (26) that

(28) for every $h \in R^+ \to E$ such that h is disintegrable in R^+, $h(0_+) = 0$ and $h' \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$, the function $w(t) = \mathscr{F}(t, h) - \int_0^t \mathscr{V}(t-\tau, h'(\tau))\,\mathrm{d}\tau$ has the properties [1] 7.10 (1)–(4).

Since by our assumptions and by 22, Lemma [1] 7.10 is applicable, we obtain from (28) that

$$(29)\qquad \mathscr{F}(t, h) = \int_0^t \mathscr{V}(t-\tau, h'(\tau))\,\mathrm{d}\tau$$

for every $h \in R^+ \to E$ such that h is disintegrable in R^+,

$h(0_+) = 0$ and $h' \in \boldsymbol{L}_{\mathrm{loc}}(R^+, E)$.

Let us denote

$$(30)\qquad \Phi = \{\varphi : \varphi \in R^+ \to R,\ \varphi \text{ is integrable in } R^+,\ \varphi(0_+) = 0$$

$$\text{and } \varphi' \in \boldsymbol{L}_{\mathrm{loc}}(R^+, R)\}.$$

As a particular case of (29) we can write with regard to (30)

$$(31)\qquad \mathscr{F}(t, \varphi x) = \int_0^t \mathscr{V}(t-\tau, x)\,\varphi'(\tau)\,\mathrm{d}\tau \quad \text{for every} \quad \varphi \in \Phi,$$

$$x \in E \quad \text{and} \quad t \in R^+.$$

By virtue of [1] 2.8, it follows from (5) and (15) that

(32) $$\mathscr{V}(0_+, x) = 0 \quad \text{for every} \quad x \in D_1(A_1, A_2, \dots, A_n),$$

(33) for every $x \in D_1(A_1, A_2, \dots, A_n)$, the function $\mathscr{V}(., x)$ is differentiable on R^+,

(34) $$\frac{d}{dt}\mathscr{V}(t, x) = -\mathscr{V}(t, A_1x) - \int_0^t \mathscr{V}(\tau, A_2x)\,d\tau - \dots$$
$$\dots - \frac{1}{(n-2)!}\int_0^t (t-\tau)^{n-2}\,\mathscr{V}(\tau, A_nx) + \frac{t^m}{m!}x$$
$$\text{for every} \quad x \in D_1(A_1, A_2, \dots, A_n) \quad \text{and} \quad t \in R^+.$$

Denote

(35) $$\mathscr{W}_0(t, x) = \frac{d}{dt}\mathscr{V}(t, x) \quad \text{for} \quad x \in D_1(A_1, A_2, \dots, A_n) \quad \text{and} \quad t \in R^+.$$

It is clear from (9) and (35) that

(36) for every $t \in R^+$ the function $\mathscr{W}_0(t, .)$ is a linear mapping.

By [1] 2.7 we obtain from (5), (34) and (35) that

(37) for every $x \in D_1(A_1, A_2, \dots, A_n)$, the function $\mathscr{W}(., x)$ is continuous on R^+ and bounded on $(0, 1)$.

By (32), (33), (35) and (37)

(38) $$\mathscr{V}(t, x) = \int_0^t \mathscr{W}_0(\tau, x)\,d\tau \quad \text{for every} \quad x \in D_1(A_1, A_2, \dots, A_n) \quad \text{and} \quad t \in R^+.$$

By (34) and (35)

(39) $$\mathscr{W}_0(t, x) = -\mathscr{V}(\tau, A_1x)\,d\tau - \int_0^t \mathscr{V}(\tau, A_2x)\,d\tau - \dots$$
$$\dots - \frac{1}{(n-2)!}\int_0^t (t-\tau)^{n-2}\,\mathscr{V}(\tau, A_nx)\,d\tau + \frac{t^m}{m!}x$$
$$\text{for every} \quad x \in D_1(A_1, \dots, A_n) \quad \text{and} \quad t \in R^+.$$

It follows from (6), (12) and (39) by virtue of [1] 2.9 that

(40) $$\int_0^t (t-\tau)^{i-1}\,\mathscr{W}_0(\tau, x)\,d\tau \in D(A_i)$$
$$\text{for every} \quad x \in D_1(A_1, A_2, \dots, A_n), \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \dots, n\},$$

(41) $$A_i\frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\,\mathscr{W}_0(\tau, x) = -A_i\frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1}\,\mathscr{V}(\tau, A_1x) -$$
$$- A_i\frac{1}{i!}\int_0^t (t-\tau)^i\,\mathscr{V}(\tau, A_2x)\,d\tau - \dots$$

$$\ldots - A_i \frac{1}{(n+i-2)!} \int_0^t (t-\tau)^{n+i-2} \mathscr{V}(\tau, A_n x) + \frac{t^{m+i}}{(m+i)!} A_i x$$

for every $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

By (7), (13) and (41)

(42) for every $x \in D_1(A_1, A_2, \ldots, A_n)$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t-\tau)^{i-1} \mathscr{W}_0(\tau, x)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$.

Finally, after some suitable regroupings of sums, we get from (8), (39) and (41) that

(43)
$$\mathscr{W}_0(t, x) + A_1 \int_0^t \mathscr{W}_0(\tau, x)\, d\tau + A_2 \int_0^t (t-\tau) \mathscr{W}_0(\tau, x)\, d\tau + \ldots$$
$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} \mathscr{W}_0(\tau, x)\, d\tau = \frac{t^m}{m!} x$$

for every $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$.

On the other hand, using (9) we obtain from (37) and (42) that

(44)
$$\int_0^t \mathscr{W}_0(t-\tau, x)\, \varphi(\tau)\, d\tau \quad \text{exists for every} \quad \varphi \in L_{\text{loc}}(R^+, R),$$
$$x \in D_1(A_1, A_2, \ldots, A_n) \quad \text{and} \quad t \in R^+,$$

(45) for every $\varphi \in L_{\text{loc}}(R^+, R)$ and $x \in D_1(A_1, A_2, \ldots, A_n)$, the function $\int_0^t \mathscr{W}_0(t-\tau, x)\, \varphi(\tau)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,

(46)
$$\int_0^t (t-\tau)^{i-1} \int_0^\tau \mathscr{W}_0(\tau-\sigma, x)\, \varphi(\sigma)\, d\tau =$$
$$= \int_0^t \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, d\sigma\, d\tau$$

for every $\varphi \in L_{\text{loc}}(R^+, R)$, $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(47)
$$\int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau$$

exists for every $\varphi \in L_{\text{loc}}(R^+, R)$, $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(48) for every $\varphi \in L_{\text{loc}}(R^+, R)$, $x \in E$ and $i \in \{1, 2, \ldots, n\}$, the function $\int_0^t A_i \int_0^{t-\tau} (t-\tau-\sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$.

By means of [1] 2.4, we get from (44)–(48) that

$$\int_0^t \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau \in D(A_i) \tag{49}$$

for every $\varphi \in L_{loc}(R^+, R)$, $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} A_i \int_0^t \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau = \\ = \int_0^t A_i \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau \end{aligned} \tag{50}$$

for every $\varphi \in L_{loc}(R^+, R)$, $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

It follows from (46), (49) and (50) that

$$\int_0^t (t - \tau)^{i-1} \int_0^\tau \mathscr{W}_0(\tau - \sigma, x)\, \varphi(\sigma)\, d\sigma\, d\tau \in D(A_i) \tag{51}$$

for every $\varphi \in L_{loc}(R^+, R)$. $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} A_i \int_0^t (t - \tau)^{i-1} \int_0^\tau \mathscr{W}_0(\tau - \sigma, x)\, \varphi(\sigma)\, d\sigma\, d\tau = \\ = \int_0^t A_i \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, \varphi(\tau)\, d\sigma\, d\tau \end{aligned} \tag{52}$$

for every $\varphi \in L_{loc}(R^+, R)$, $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

On the other hand, integration by parts together with (30), (37) and (38) yields

$$\int_0^t \mathscr{W}_0(t - \tau, x)\, \varphi(\tau)\, d\tau = \int_0^t \mathscr{V}(t - \tau, x)\, \varphi'(\tau)\, d\tau \tag{53}$$

for every $\varphi \in \Phi$, $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$.

Now by (31) and (53)

$$\int_0^t \mathscr{W}_0(t - \tau, x)\, \varphi(\tau)\, d\tau = \mathscr{F}(t, \varphi x) \tag{54}$$

for every $\varphi \in \Phi$, $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$.

It follows immediately from (2), (51), (52) and (54) that

$$\left\| \int_0^t \left[A_i \frac{1}{(i-1)!} \int_0^{t-\tau} (t - \tau - \sigma)^{i-1} \mathscr{W}_0(\sigma, x)\, d\sigma \right] \varphi(\tau)\, d\tau \right\| \leq \tag{55}$$

$$\leqq Me^{\omega t}\|x\| \int_0^t |\varphi(\tau)| \, d\tau$$

for every $\varphi \in \Phi$, $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Using 5, we see from (30) that

(56) the set Φ is dense in $L_{loc}(R^+, R)$.

Taking into account (48) and (56) and applying 8 to (55) we get immediately

$$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} \mathscr{W}_0(\tau, x) \, d\tau \right\| \leqq Me^{\omega t}\|x\| \tag{57}$$

for every $x \in D_1(A_1, A_2, \ldots, A_n)$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Further, it follows from (43) and (57) that

$$\|\mathscr{W}_0(t, x)\| \leqq \left[nMe^{\omega t} + \frac{t^m}{m!} \right] \|x\| \tag{58}$$

for every $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$.

Since by the assumption the operators $A_1, A_2, \ldots, A_n$ are closed and the set $D_1(A_1, A_2, \ldots, A_n)$ is dense in E, it is an easy matter to show by means of (36), (37), (40), (42), (43), (57) and (58) that there exists an extension $\mathscr{W} \in R^+ \times E \to E$ such that

(59) $\mathscr{W}(t, x) = \mathscr{W}_0(t, x)$ for every $x \in D_1(A_1, A_2, \ldots, A_n)$ and $t \in R^+$,

(60) the function $\mathscr{W}$ possesses the properties 2.13 (a)–(f).

We see from our assumptions, from Proposition 22 and from the just proved property (60) that Theorem [2] 2.17 is applicable and according to it, the system of operators $A_1, A_2, \ldots, A_n$ is correct of class m.

The proof is complete.

29. Remark. The only difference in apriori assumptions of Theorems 27 and 28 is in the density of certain domains of the operators $A_1, A_2, \ldots, A_n$. It is clear that under the assumptions 28 (α)–(γ), the system of operators $A_1, A_2, \ldots, A_n$ is converse of class m if and only if it is correct of class m.

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