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BICHROMATICITY AND DOMATIC NUMBER OF A BIPARTITE GRAPH

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In this paper we shall relate the bichromaticity of a connected finite bipartite graph (shortly bigraph) to its domatic number.

The bichromaticity of a connected bigraph was introduced by F. Harary, D. Hsu and Z. Miller [2]. Let B be a bigraph on the vertex sets C, D. A bicomplete homomorphism of B is a homomorphic mapping φ of B onto the complete bigraph $K_{r,s}$ (where r, s are positive integers) with the property that for any two vertices x, y of B, the identity $\varphi(x) = \varphi(y)$ holds only if either $x \in C$, $y \in C$, or $x \in D$, $y \in D$. The maximal value of $r + s$ for all graphs $K_{r,s}$ with the property that there exists a bicomplete homomorphism of B onto $K_{r,s}$ is called the bichromaticity of B and denoted by $\beta(B)$.

If B is a finite bigraph on sets C, D, then the majority of B is the number $\mu = \max(|C|, |D|)$.

The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi [1]. A dominating set in an undirected graph G is a subset D of the vertex set $V(G)$ of G with the property that for each vertex $x \in V(G) - D$ there exists at least one vertex $y \in D$ adjacent to x. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximal number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$.

First we state a lemma.

Lemma. *Let B be a connected bigraph on sets C, D, let $\mathscr{P}$ be a domatic partition of B. Then either $\mathscr{P} = \{C, D\}$, or $C \cap X \neq \emptyset$, $D \cap X \neq \emptyset$ for each $X \in \mathscr{P}$.*

Proof. Let $X \in \mathscr{P}$. If $\mathscr{P} = \{X\}$, then the assertion is evidently true. If X is a proper subset of C, then $C - X \neq \emptyset$; let $u \in C - X$. As C is an independent set in B, then there is no vertex of X adjacent to u and X is not a dominating set in B, which is a contradiction. Therefore X cannot be a proper subset of C and analogously, it cannot be a proper subset of D. If $X = C$ and $Y \in \mathscr{P}$, $Y \neq X$, then Y is a subset of D. As it cannot be a proper subset of D, we have $Y = D$ and $\mathscr{P} = \{C, D\}$; analogously if $X = D$. Thus the assertion is proved.

Now we prove a theorem.

Theorem 1. *For every connected finite bigraph B we have*

$$\beta(B) \geqq \mu + [\tfrac{1}{2}\, d(B)]\,.$$

This inequality cannot be improved.

Proof. Let the colour sets of B be C, D, let $|C| \geqq |D|$, i.e. $\mu = |C|$. As B is a connected bigraph, it contains no isolated vertices and therefore $d(B) \geqq 2$. If $d(B) \leqq 3$, then $\mu + [\frac{1}{2}\, d(B)] = \mu + 1$ and $\beta(B) \geqq \mu + 1$ according to [2]; therefore the assertion holds. Suppose that $d(B) \geqq 4$. Then there exists a domatic partition $P = \{P_1, \ldots, P_{d(B)}\}$ of B and, by Lemma, $P_i \cap C \neq \emptyset$ and $P_i \cap D \neq \emptyset$ for $i = 1, \ldots, d(B)$. Denote $C_i = P_i \cap C$, $D_i = P_i \cap D$ for $i = 1, \ldots, d(B)$. Further, denote $a = [\frac{1}{2}\, d(B)]$. Now we shall define sets $Q_1, \ldots, Q_a$. For $i = 1, \ldots, a - 1$ let $Q_i = D_{2i-1} \cup D_{2i}$. If $d(B)$ is even, then $Q_a = D_{d(B)-1} \cup D_{d(B)}$; if $d(B)$ is odd, then $Q_a = D_{d(B)-2} \cup D_{d(B)-1} \cup D_{d(B)}$. Let x be an arbitrary vertex of C. If $x \in C - C_1$, then there exists $y \in D_1$ adjacent to x; if $x \in C_1$, then there exists $y \in D_2$ adjacent to x. In both these cases $y \in Q_1$. Quite analogously we can prove that for each $x \in C$ and for each $i \in \{1, \ldots, a\}$ there exists $y \in Q_i$ adjacent to x. Take the complete bigraph $K_{\mu,2}$ on the sets C, $\{Q_1, \ldots, Q_a\}$ and define the mapping φ so that $\varphi(x) = x$ for $x \in C$, $\varphi(x) = Q_i$ for $x \in Q_i$ and $i = 1, \ldots, a$. The mapping φ evidently is a bicomplete homomorphism of B onto $K_{\mu,a}$ and hence $\beta(B) \geqq \mu + a = \mu + [\frac{1}{2}\, d(B)]$. If B is a circuit of the length 6, then $\mu = 3$, $d(B) = 3$ and $\beta(B) = \mu + [\frac{1}{2}\, d(B)] = 4$. Hence the inequality cannot be improved.

Corollary. *For every connected finite bigraph B with $d(B) \geqq 3$ we have*

$$\beta(B) \geqq [\tfrac{3}{2}\, d(B)]\,.$$

Note that Lemma implies that each class of a domatic partition of B with $d(B)$ classes has a non-empty intersection with each colour class and thus $d(B)$ cannot exceed μ. Hence $[\frac{3}{2}\, d(B)] \leqq \mu + [\frac{1}{2}\, d(B)]$. If $d(B) = 2$ the inequality need not hold; we have $\beta(K_{1,1}) = 2$, $d(K_{1,1}) = 2$.

At the end we shall disprove a conjecture from [2]. The authors have conjectured that $\beta(B) = \mu + \delta(B) - x$, where $\delta(B)$ is the minimum degree of a vertex of B and x is a non-negative integer "small" compared with $\delta(B)$.

Theorem 2. Let q be an arbitrary positive integer. Then there exists a connected finite bigraph B for which

$$\beta(B) = \mu + \delta(B) + q\,.$$

Proof. Let B be a bigraph on sets C, D, let $|C| = q + 3$. $|D| = q + 2$. Let $c_1 \in C$, $c_2 \in C$, $d \in D$ be vertices of B such that c_1 is adjacent only to d, c_2 is adjacent to all vertices of D except d and each vertex of $C - \{c_1, c_2\}$ is adjacent to all vertices of D. Obviously $\mu = q + 3$, $\delta(B) = 1$ (this is the degree of c_1). By identifying the vertices c_1, c_2 the complete bigraph $K_{q+2,q+2}$ is obtained and hence $\beta(B) = 2q + 4 = \mu + \delta(B) + q$.

References

[1] *E. J. Cockayne, S. T. Hedetniemi:* Towards a theory of domination in graphs. Networks 7 (1977), 247—261.

[2] *F. Harary, D. Hsu, Z. Miller:* The bichromaticity of a tree. In: Theory and Applications of Graphs, Proc. Michigan 1976, ed. by Y. Alavi and D. R. Lick. Springer Verlag Berlin—Heidelberg—New York 1978.

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