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PHASE MATRIX OF LINEAR DIFFERENTIAL SYSTEMS

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1. INTRODUCTION

In the present paper we deal with a linear differential system of the second order

$$y'' + P(x)\, y = 0\,, \tag{1.1}$$

where $y(x)$ is an n-dimensional vector function and $P(x)$ is a continuous symmetric $n \times n$ matrix. Simultaneously with the system (1.1) we consider the associated matrix system

$$Y'' + P(x)\, Y = 0\,. \tag{1.2}$$

It is known that for the scalar differential equation

$$y'' + p(x)\, y = 0 \tag{1.3}$$

there exists a function $\alpha(x)$ such that two linearly independent solutions $u(x)$, $v(x)$ of (1.3) can be expressed in the form

$$\begin{aligned} u(x) &= k(|\alpha'(x)|)^{-1/2} \sin \alpha(x)\,, \\ v(x) &= k(|\alpha'(x)|)^{-1/2} \cos \alpha(x)\,, \end{aligned} \tag{1.4}$$

where k is a real constant. The aim of the present paper is to show that linearly independent solutions of (1.2) can be expressed in a similar way. We also derive some results concerning the oscillation properties of the solutions of (1.1).

Throughout the paper the system (1.1) is investigated on an interval $I = [a, \infty)$, and it is assumed that $P(x)$ is continuous on I. All solutions are considered in the classical sense, i.e. they have continuous second derivatives.

Notation. $C^m(I)$ denotes the space of functions which are m-times continuously differentiable on I, $C^0(I)$ means continuity. If $F(x)$ is an arbitrary matrix, we write $F(x) \in C^m(I)$ if all elements of $F(x)$ belong to $C^m(I)$, F^T denotes the transpose of a matrix F, F^* means the conjugate transpose of F. If F, G are symmetric matrices of the same dimension, $F < G$ means that the matrix $F - G$ is negative definite.

Let $Y(x)$, $Z(x)$ be arbitrary solutions of (1.2). Then $Y^{\mathrm{T}}(x)\,Z'(x) - Y^{\mathrm{T}\prime}(x)\,Z(x) = K$ on I, where K is a constant $n \times n$ matrix (see e.g. [5]). A solution $Y(x)$ of (1.2) is said to be *isotropic* if $Y^{\mathrm{T}}(x)\,Y'(x) - Y^{\mathrm{T}\prime}(x)\,Y(x) = 0$. Terminology concerning the solution with the above property is not unified. For example, Sternberg [9] calls this solution conjugate, Hartman [5] prepared.

Let $Y_1(x)$, $Y_2(x)$ be two solutions of (1.2). Then

$$\det\begin{pmatrix} Y_1(x) & Y_2(x) \\ Y_1'(x) & Y_2'(x) \end{pmatrix} = \text{const} = c\,.$$

Indeed, (1.2) can be rewritten as the system of the first order

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}' = \begin{pmatrix} 0 & E \\ -P(x) & 0 \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}, \tag{2.1}$$

where E is the unit $n \times n$ matrix. If we denote

$$X = \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}, \quad \mathfrak{P} = \begin{pmatrix} 0 & E \\ -P & 0 \end{pmatrix}$$

then the Jacobi formula yields $\det X(x) = \det X(a) \exp\{\int_a^x \operatorname{Tr} \mathfrak{P}(s)\,\mathrm{d}s\} = \det X(a) = \text{const}$, where $\operatorname{Tr} \mathfrak{P}$ denotes the trace of the matrix $\mathfrak{P}$. If const $\neq 0$, solutions $Y_1(x)$, $Y_2(x)$ are called linearly independent and every solution of (1.1) can be expressed in the form

$$y(x) = Y_1(x)\,c_1 + Y_2(x)\,c_2\,, \tag{2.2}$$

where c_1, c_2 are n-dimensional constant vectors. It can be proved (see [7]) that $Y_1(x)$, $Y_2(x)$ are linearly independent if and only if the constant matrix $Y_1^{\mathrm{T}}(x)\,Y_2'(x) - Y_1^{\mathrm{T}\prime}(x)\,Y_2(x)$ is nonsingular. Let $Y(x)$ be isotropic solution of (1.2) which is nonsingular on I. Then $Z(x) = Y(x) \int_a^x (Y^{\mathrm{T}}(s)\,Y(s))^{-1}\,\mathrm{d}s$ is also a solution of (1.2) and $Y^{\mathrm{T}}(x)\,Z'(x) - Y^{\mathrm{T}\prime}(x)\,Z(x) = E$.

Now let us consider the linear Hamiltonian system

$$\begin{aligned} S' &= Q(x)\,C \\ C' &= -Q(x)\,S\,, \end{aligned} \tag{2.3}$$

where $Q(x) \in C^0(I)$ is a symmetric $n \times n$ matrix. The matrices $S(x)$, $C(x)$ which satisfy the identities

$$\begin{aligned} S^{\mathrm{T}}(x)\,S(x) + C^{\mathrm{T}}(x)\,C(x) &= E \\ S^{\mathrm{T}}(x)\,C(x) - C(x^{\mathrm{T}})\,S(x) &= 0 \end{aligned} \tag{2.4}$$

are called the matrix sine and cosine and were for the first time used by Barrett [2] in the extension of the Prüfer transformation to matrix systems. It can be proved that

(2.4) holds if and only if

$$S(x)\,S^{\mathsf T}(x) + C(x)\,C^{\mathsf T}(x) = E \qquad (2.5)$$
$$S(x)\,C^{\mathsf T}(x) - C(x)\,S^{\mathsf T}(x) = 0$$

(see e.g. Barrett [2]).

3. MAIN RESULTS

In this section it is shown that with the aid of matrix sine and cosine the expressions (1.4) can be extended to the system (1.2). First, let us recall that the relations (1.4) imply that $(|\alpha'(x)|)^{-1} = k^{-2}(u^2(x) + v^2(x))$. If we denote

$$r(x) = (u^2(x) + v^2(x))^{1/2} \qquad (3.1)$$

then $r(x) = k(|\alpha'(x)|)^{-1/2}$ and (1.4) can be rewritten

$$u(x) = r(x)\sin\alpha(x)\,, \quad v(x) = r(x)\cos\alpha(x)\,. \qquad (3.2)$$

Theorem 1. *Let $U(x)$, $V(x)$ be two isotropic solutions of (1.2) satisfying*

$$U^{\mathsf T\prime}(x)\,V(x) - U^{\mathsf T}(x)\,V'(x) = E\,. \qquad (3.3)$$

Then there exist a nonsingular $n \times n$ matrix $R(x)$ and a symmetric positive definite $n \times n$ matrix $Q(x) \in C^2(I)$ satisfying

$$(R^{\mathsf T}(x)\,R(x))^{-1} = Q(x) \qquad (3.4)$$
$$R(x)\,R^{\mathsf T}(x) = U(x)\,U^{\mathsf T}(x) + V(x)\,V^{\mathsf T}(x)$$
$$R^{\mathsf T\prime}(x)\,R(x) - R^{\mathsf T}(x)\,R'(x) = 0$$

and such that

$$U(x) = R(x)\,S(x)\,, \quad V(x) = R(x)\,C(x)\,, \qquad (3.5)$$

where $(S(x), C(x))$ is a solution of (2.3) for which (2.4) holds.

Proof. Since $U(x)$, $V(x)$ are isotropic and (3.3) holds, we have

$$\begin{pmatrix} U^{\mathsf T}(x) & U^{\mathsf T\prime}(x) \\ V^{\mathsf T}(x) & V^{\mathsf T\prime}(x) \end{pmatrix} \begin{pmatrix} V'(x) & -U'(x) \\ -V(x) & U(x) \end{pmatrix} = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix},$$

hence

$$\begin{pmatrix} V'(x) & -U'(x) \\ -V(x) & U(x) \end{pmatrix} \begin{pmatrix} U^{\mathsf T}(x) & U^{\mathsf T\prime}(x) \\ V^{\mathsf T}(x) & V^{\mathsf T\prime}(x) \end{pmatrix} = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}$$

and thus

$$V'(x)\,U^{\mathsf T}(x) - U'(x)\,V^{\mathsf T}(x) = -E \qquad (3.6)$$
$$V'(x)\,U^{\mathsf T\prime}(x) - U'(x)\,V^{\mathsf T\prime}(x) = 0$$
$$U(x)\,V^{\mathsf T}(x) - V(x)\,U^{\mathsf T}(x) = 0\,.$$

Let $H(x) = U(x)\,U^{\mathrm{T}}(x) + V(x)\,V^{\mathrm{T}}(x)$. The matrix $H(x)$ is nonsingular on I. Indeed, $H(x) = (U(x) + i\,V(x))\,(U^{\mathrm{T}}(x) - i\,V^{\mathrm{T}}(x)) = (U(x) + i\,V(x))\,(U(x) + i\,V(x))^*$. Therefore $H(x)$ is nonsingular if and only if the complex matrix $U(x) + i\,V(x)$ is nonsingular. Let $c = c_1 + ic_2$ be a constant complex n-dimensional vector for which $(U + iV)(c_1 + ic_2) = 0$. Then $Uc_1 - Vc_2 = 0$ and $Uc_2 + Vc_1 = 0$. As $U(x)$, $V(x)$ are linearly independent solutions, the last identity gives $c_1 = 0 = c_2$, hence $U(x) + i\,V(x)$ is nonsingular. Furthermore $H(x)$ obviously is symmetric and positive definite.

Let $D(x)$ denote the (unique) symmetric positive definite matrix for which $D^2(x) = H(x)$ and let us set $K(x) = D'(x)\,D(x) - D(x)\,D'(x)$, $L(x) = K(x)\,D^{-2}(x) - D^{-2}(x)\,K(x)$. It is obvious that $K(x)$ is antisymmetric (i.e. $K^{\mathrm{T}}(x) = -K(x)$) and $L(x)$ is symmetric. Let $M(x)$ be a solution of the matrix equation

$$D^{-2}(x)\,M(x) + M(x)\,D^{-2}(x) = L(x)\,. \tag{3.7}$$

Since $D^{-2}(x)$ is symmetric and positive definite, it is known (see [3]) that the matrix $M(x)$ is uniquely determined by (3.7) and $M^{\mathrm{T}}(x) = M(x)$.

Let $T(x)$ be the solution of

$$T' = \tfrac{1}{2}\,D^{-2}(x)\,(K(x) + M(x))\,T\,, \quad T(a) = E\,. \tag{3.8}$$

As $D^{-2}(K + M) + (D^{-2}(K + M))^{\mathrm{T}} = D^{-2}K + D^{-2}M + (-K + M)\,D^{-2} = D^{-2}K + D^{-2}M - KD^{-2} + MD^{-2} = D^{-2}M + MD^{-2} - L = 0$, the matrix $T(x)$ is orthonormal on I (i.e. $T^{-1}(x) = T^{\mathrm{T}}(x)$). If we set

$$R(x) = D(x)\,T(x) \tag{3.9}$$

then $R^{\mathrm{T}'}R - R^{\mathrm{T}}R' = (T^{\mathrm{T}}D' + T^{\mathrm{T}'}D)\,DT - T^{\mathrm{T}}D(D'T + DT') = T^{\mathrm{T}}D'DT + \tfrac{1}{2}T^{\mathrm{T}}(M - K)\,D^{-2}D^2T - T^{\mathrm{T}}DD'T - \tfrac{1}{2}T^{\mathrm{T}}D^2D^{-2}(K + M)\,T = T^{\mathrm{T}}D'DT - \tfrac{1}{2}T^{\mathrm{T}}KT - T^{\mathrm{T}}DD'T - \tfrac{1}{2}T^{\mathrm{T}}KT = T^{\mathrm{T}}D'DT - T^{\mathrm{T}}DD'T - T^{\mathrm{T}}(D'D - DD')\,T = 0$ and $RR^{\mathrm{T}} = DTT^{\mathrm{T}}D = D^2 = UU^{\mathrm{T}} + VV^{\mathrm{T}}$.

Now let $Q(x) = (R^{\mathrm{T}}(x)\,R(x))^{-1}$. Then $Q(x)$ is symmetric, positive definite and belongs to $C^2(I)$ since $U(x), V(x) \in C^2(I)$. We shall show that the matrices $S(x) = R^{-1}(x)\,U(x)$, $C(x) = R^{-1}(x)\,V(x)$ are a solution of (2.3) and (2.4) holds.

$SS^{\mathrm{T}} + CC^{\mathrm{T}} = R^{-1}UU^{\mathrm{T}}R^{\mathrm{T}-1} + R^{-1}VV^{\mathrm{T}}R^{\mathrm{T}-1} = R^{-1}(UU^{\mathrm{T}} + VV^{\mathrm{T}})\,R^{\mathrm{T}-1} = E$, $SC^{\mathrm{T}} - CS^{\mathrm{T}} = R^{-1}UV^{\mathrm{T}}R^{\mathrm{T}-1} - R^{-1}VU^{\mathrm{T}}R^{\mathrm{T}-1} = R^{-1}(UV^{\mathrm{T}} - VU^{\mathrm{T}})\,R^{\mathrm{T}-1} = 0$, we have (2.5) and hence (2.4) holds. Further $S'S^{\mathrm{T}} + C'C^{\mathrm{T}} = (R^{-1}U)'\,U^{\mathrm{T}}R^{\mathrm{T}-1} + (R^{-1}V)'\,V^{\mathrm{T}}R^{\mathrm{T}-1} = (-R^{-1}R'R^{-1}U + R^{-1}U')\,U^{\mathrm{T}}R^{\mathrm{T}-1} + (-R^{-1}R'R^{-1}V + R^{-1}V')\,V^{\mathrm{T}}R^{\mathrm{T}-1} = -R^{-1}R'R^{-1}(UU^{\mathrm{T}} + VV^{\mathrm{T}})\,R^{\mathrm{T}-1} + R^{-1}U'U^{\mathrm{T}}R^{\mathrm{T}-1} + R^{-1}V'V^{\mathrm{T}}R^{\mathrm{T}-1} = R^{-1}(-R'R^{\mathrm{T}} + U'U^{\mathrm{T}} + V'V^{\mathrm{T}})\,R^{\mathrm{T}-1}$. Now using (3.3) and (3.6) we obtain $RR^{\mathrm{T}}(U'U^{\mathrm{T}} + V'V^{\mathrm{T}}) = (UU^{\mathrm{T}} + VV^{\mathrm{T}})(U'U^{\mathrm{T}} + V'V^{\mathrm{T}}) = UU^{\mathrm{T}}U'U^{\mathrm{T}} + UU^{\mathrm{T}}V'V^{\mathrm{T}} + VV^{\mathrm{T}}U'U^{\mathrm{T}} + VV^{\mathrm{T}}V'V^{\mathrm{T}} = UU^{\mathrm{T}'}UU^{\mathrm{T}} + U(-E + U^{\mathrm{T}'}V)\,V^{\mathrm{T}} + V(E + V^{\mathrm{T}'}U)\,U^{\mathrm{T}} + VV^{\mathrm{T}'}VV^{\mathrm{T}} = (UU^{\mathrm{T}'} + VV^{\mathrm{T}'})(UU^{\mathrm{T}} + VV^{\mathrm{T}}) = (UU^{\mathrm{T}'} + VV^{\mathrm{T}'})\,RR^{\mathrm{T}}$. Simultaneously, the third relation of (3.4) yields $RR^{\mathrm{T}}R'R^{\mathrm{T}} = RR^{\mathrm{T}'}RR^{\mathrm{T}}$. Let us denote $X_1 = R'R^{\mathrm{T}}$, $Y_1 = RR^{\mathrm{T}'}$, $X_2 = U'U^{\mathrm{T}} + V'V^{\mathrm{T}}$, $Y_2 =$

$= UU^{\mathsf{T}\prime} + VV^{\mathsf{T}\prime}$. Then we have

$$(3.10)\qquad \begin{aligned} RR^{\mathsf{T}}X_i - Y_iR^{\mathsf{T}}R &= 0 \\ X_i + Y_i &= (RR^{\mathsf{T}})', \quad i = 1, 2 . \end{aligned}$$

and hence

$$(3.11)\qquad \begin{aligned} RR^{\mathsf{T}}X_i + X_iRR^{\mathsf{T}} &= (RR^{\mathsf{T}})'\, RR^{\mathsf{T}} \\ RR^{\mathsf{T}}Y_i + Y_iRR^{\mathsf{T}} &= RR^{\mathsf{T}}(RR^{\mathsf{T}})' . \end{aligned}$$

Since the matrix RR^{T} is positive definite, it is known that both matrix equations (3.11) have unique solution, and hence $X_1 = X_2$ and $Y_1 = Y_2$, i.e. $R'R^{\mathsf{T}} = U'U^{\mathsf{T}} + V'V^{\mathsf{T}}$ and $RR^{\mathsf{T}} = UU^{\mathsf{T}\prime} + VV^{\mathsf{T}\prime}$. Thus

$$(3.12)\qquad S'(x)\, S^{\mathsf{T}}(x) + C'(x)\, C^{\mathsf{T}}(x) = 0 .$$

Further using (3.6) we have $S'C^{\mathsf{T}} - C'S^{\mathsf{T}} = (-R^{-1}R'R^{-1}U + R^{-1}U')\, V^{\mathsf{T}}R^{\mathsf{T}-1} - (-R^{-1}R'R^{-1}V + R^{-1}V')\, U^{\mathsf{T}}R^{\mathsf{T}-1} = -R^{-1}R'R^{-1}(UV^{\mathsf{T}} - VU^{\mathsf{T}})\, R^{-1} + R^{-1}(U'V^{\mathsf{T}} - V'U^{\mathsf{T}})\, R^{\mathsf{T}-1} = (R^{\mathsf{T}}R)^{-1} = Q$. Thus

$$(3.13)\qquad S'(x)\, C^{\mathsf{T}}(x) - C'(x)\, S^{\mathsf{T}}(x) = Q(x) .$$

Now multiplying (3.12) and (3.13) on the right by $S(x)$ and $C(x)$ respectively, and adding these equations we get $S'(x) = Q(x)\, C(x)$. Similarly we obtain $C'(x) = -Q(x)\, S(x)$. Thus $(S(x), C(x))$ is a solution of (2.3) for which (2.4) holds. It remains to verify that if $U(x)$, $V(x)$ are expressed by (3.5) then (3.3) holds. Indeed, $U^{\mathsf{T}\prime}V - U^{\mathsf{T}}V' = (S^{\mathsf{T}}R^{\mathsf{T}\prime} + C^{\mathsf{T}}QR^{\mathsf{T}})\, RC - S^{\mathsf{T}}R^{\mathsf{T}}(R'C - RQS) = S^{\mathsf{T}}(R^{\mathsf{T}\prime}R - R^{\mathsf{T}}R')\, C + C^{\mathsf{T}}QR^{\mathsf{T}}RC + S^{\mathsf{T}}R^{\mathsf{T}}RQS = C^{\mathsf{T}}C + S^{\mathsf{T}}S = E$. The last computation shows that the third identity of (3.4) is essential and cannot be removed. The proof is complete.

Remark 1. Let $U(x)$, $V(x)$ be two isotropic solutions of (1.2) for which $U^{\mathsf{T}}(x) . V'(x) - U^{\mathsf{T}\prime}(x)\, V(x) = K$, where K is a constant symmetric, either positive definite or negative definite $n \times n$ matrix. In this case $U(x)$ and $V(x)$ can also be expressed in the form (3.5) but (2.4) and (2.5) must be replaced by $S(x)\, K^{-1}\, S^{\mathsf{T}}(x) + C(x)\, K^{-1}\, C^{\mathsf{T}}(x) = E$, $S(x)\, K^{-1}\, C^{\mathsf{T}}(x) - C(x)\, K^{-1}\, S^{\mathsf{T}}(x) = 0$, $S^{\mathsf{T}}(x)\, S(x) + C^{\mathsf{T}}(x) . C(x) = K$, $S^{\mathsf{T}}(x)\, C(x) - C^{\mathsf{T}}(x)\, S(x) = 0$. If $U(x)$, $V(x)$ are arbitrary linearly independent isotropic solutions of (1.2) then both the symmetry and the definiteness of the constant matrix $U^{\mathsf{T}}(x)\, V'(x) - U^{\mathsf{T}\prime}(x)\, V(x)$ are also nesessary conditions for $U(x)$, $V(x)$ to be expressed in the form (3.5).

Remark 2. If we consider a more general system

$$(3.14)\qquad (F(x)\, Y')' + P(x)\, Y = 0 ,$$

where $F(x)$ is a symmetric positive definite $n \times n$ matrix, then e.g. in [1] it is proved that the transformation $Y(x) = H(x)\, U(x)$ where $H(x)$ is a nonsingular $n \times n$ matrix for which

$$(3.15) \qquad H^{\mathrm{T}}(x)\,F(x)\,H'(x) - H^{\mathrm{T}\prime}(x)\,F(x)\,H(x) = 0\,,$$

transforms the system

$$(F_1(x)\,U')' + P_1(x)\,U = 0\,,$$

where

$$(3.16) \qquad \begin{aligned} F_1(x) &= H^{\mathrm{T}}(x)\,F(x)\,H(x) \\ P_1(x) &= H^{\mathrm{T}}(x)\,(F(x)\,H'(x))' + H^{\mathrm{T}}(x)\,P(x)\,H(x)\,, \end{aligned}$$

to the system (3.14). Theorem 1 shows that the transformation $U(x) = R(x)\,S(x)$ transforms the system $(Q^{-1}(x)\,S')' + Q(x)\,S = 0$ to (1.2) and the third relation of (3.4) corresponds to (3.15).

Now we wish to find the connection between the matrices $R(x)$ and $Q(x)$ from Theorem 1 and the matrix $P(x)$ from (1.1). Recall that for the function $\alpha(x)$ from (1.4) we have $\{\alpha, x\} + \alpha'^2(x) = p(x)$, where $\{a, x\} = -((|\alpha'(x)|)^{-1/2})''\,(|\alpha'(x)|)^{1/2}$ is the so called Schwartzian derivative of $\alpha(x)$ and thus $p(x) = -r''(x)\,r^{-1}(x) + r^{-4}(x)$, where $r(x)$ is determined by (3.1) and $r(x) = (|\alpha'(x)|)^{-1/2}$. If we denote $s(x) = \sin \alpha(x)$, $c(x) = \cos \alpha(x)$ then

$$(3.17) \qquad \begin{aligned} s' &= \alpha'(x)\,c \\ c' &= -\alpha'(x)\,s\,. \end{aligned}$$

Hence we see that the matrix $Q(x)$ in (2.3) plays the same role as the function $\alpha'(x)$ in (3.17). The function $\alpha(x)$ is called the phase function of (1.3) (see [4]) and therefore the matrix $A(x) = \int_a^x Q(s)\,\mathrm{d}s$ will be called *the phase matrix* of (1.1).

It is known (see [4]) that for every function $\alpha(x) \in C^3(I)$, $\alpha'(x) \neq 0$ on I, there exists a unique function $p(x)$ such that $\alpha(x)$ is the phase function of $y'' + p(x)\,y = 0$. For the phase matrices we have the following statement.

Theorem 2. *Let $A(x) \in C^3(I)$ be a symmetric $n \times n$ matrix for which $A'(x)$ is positive on I. Then there exists a symmetric $n \times n$ matrix $P(x) \in C^0(I)$ such that $A(x)$ is the phase matrix of $Y' + P(x)\,Y = 0$, i.e. there exist two isotropic solutions $U(x)$, $V(x)$ of this system which can be expressed in the form (3.5), where $(S(x), C(x))$ is a solution of (2.3) with $Q(x) = A'(x)$ for which (2.4) holds, and $R(x)$ is a nonsingular $n \times n$ matrix satisfying*

$$(3.18) \qquad \begin{aligned} (R^{\mathrm{T}}(x)\,R(x))^{-1} &= A'(x) \\ R^{\mathrm{T}\prime}(x)\,R(x) - R^{\mathrm{T}}(x)\,R'(x) &= 0\,. \end{aligned}$$

The matrix $P(x)$ is determined by the relation

$$(3.19) \qquad P(x) = -R''(x)\,R^{-1}(x) + (R(x)\,R^{\mathrm{T}}(x))^{-2}\,.$$

Proof. Let $B(x)$ be the symmetric positive definite $n \times n$ matrix for which $B^2(x) = (A'(x))^{-1}$ and let $N(x) = B'(x)B^{-1}(x) - B^{-1}(x)B'(x)$. Since $N(x)$ is antisymmetric, the matrix $G(x)$ which is the solution of

$$G' = \tfrac{1}{2}N(x)\,G\,, \quad G(a) = E\,, \tag{3.20}$$

is orthonormal. If we set

$$R(x) = G^{\mathsf T}(x)\,B(x) \tag{3.21}$$

then we obtain $R^{\mathsf T\prime}R - R^{\mathsf T}R' = (BG' + B'G)\,G^{\mathsf T}B - BG(G^{\mathsf T\prime}B + G^{\mathsf T}B') = \tfrac{1}{2}BNGG^{\mathsf T}B + B'B - \tfrac{1}{2}BGG^{\mathsf T}N^{\mathsf T}B - BB' = B'B - BB' + \tfrac{1}{2}B(B'B^{-1} - B^{-1}B')\,B - \tfrac{1}{2}B(B^{-1}B' - B'B^{-1})\,B = 0$ and $R^{\mathsf T}R = BGG^{\mathsf T}B = B^2 = A'^{-1}$.

Now let $(S(x), C(x))$ be a solution of (2.3) with $Q(x) = A'(x)$ for which (2.4) holds. Let us put $U(x) = R(x)\,S(x)$, $V(x) = R(x)\,C(x)$ and

$$P(x) = U''(x)\,V^{\mathsf T\prime}(x) - V''(x)\,U^{\mathsf T\prime}(x)\,. \tag{3.22}$$

Then $U^{\mathsf T\prime}V - U^{\mathsf T}V' = (S^{\mathsf T}R^{\mathsf T})'\,RC - S^{\mathsf T}R^{\mathsf T}(RC)' = C^{\mathsf T}QR^{\mathsf T}RC + S^{\mathsf T}R^{\mathsf T\prime}RC + S^{\mathsf T}R^{\mathsf T}RQS - S^{\mathsf T}R^{\mathsf T}R'C = S^{\mathsf T}S + C^{\mathsf T}C = E$,
$U^{\mathsf T\prime}U - U^{\mathsf T}U' = (S^{\mathsf T}R^{\mathsf T})'\,RS - S^{\mathsf T}R^{\mathsf T}(RS)' = S^{\mathsf T}R^{\mathsf T\prime}RS + C^{\mathsf T}QR^{\mathsf T}RS - S^{\mathsf T}R^{\mathsf T}R'S - S^{\mathsf T}R^{\mathsf T}RQC = S^{\mathsf T}(R^{\mathsf T\prime}R - R^{\mathsf T}R')\,S + C^{\mathsf T}S - S^{\mathsf T}C = 0$ and
$V^{\mathsf T\prime}V - V^{\mathsf T}V' = (C^{\mathsf T}R^{\mathsf T})'\,RC - C^{\mathsf T}R^{\mathsf T}(RC)' = C^{\mathsf T}R^{\mathsf T\prime}RC - C^{\mathsf T}QR^{\mathsf T}RS - C^{\mathsf T}R^{\mathsf T}R'C + C^{\mathsf T}R^{\mathsf T}RQS = 0$, thus (3.6) holds. The matrix $P(x)$ is symmetic. In fact, $P = (U'V^{\mathsf T\prime} - V'U^{\mathsf T\prime})' - (U'V^{\mathsf T\prime\prime} - V'U^{\mathsf T\prime\prime}) = -(U'V^{\mathsf T\prime\prime} - V'U^{\mathsf T\prime\prime}) = P^{\mathsf T}$. According to (3.6) we have

$$\begin{pmatrix} U & V \\ U' & V' \end{pmatrix}' \begin{pmatrix} V' & -V^{\mathsf T} \\ -U & U^{\mathsf T} \end{pmatrix} = -\begin{pmatrix} 0 & E \\ -P & 0 \end{pmatrix}$$

hence

$$\begin{pmatrix} U & V \\ U' & V' \end{pmatrix}' = \begin{pmatrix} 0 & E \\ -P & 0 \end{pmatrix} \begin{pmatrix} U & V \\ U' & V' \end{pmatrix}$$

and we see that $U(x)$ and $V(x)$ are solutions of $Y'' + P(x)\,Y = 0$. If we substitute (3.5) into (3.22), we obtain
$P = U''V^{\mathsf T\prime} - V''U^{\mathsf T\prime} = (R''S + 2R'QC + RQ'C - RQ^2S)(C^{\mathsf T}R^{\mathsf T\prime} - S^{\mathsf T}QR^{\mathsf T}) - (R''C - 2R'QS - RQ'S - RQ^2C)(S^{\mathsf T}R^{\mathsf T\prime} + C^{\mathsf T}QR^{\mathsf T}) = R''SC^{\mathsf T}R^{\mathsf T\prime} + 2R'QCC^{\mathsf T}R^{\mathsf T\prime} + RQ'CC^{\mathsf T}R^{\mathsf T\prime} - RQ^2SC^{\mathsf T}R^{\mathsf T\prime} - R''SS^{\mathsf T}QR^{\mathsf T} - 2R'QCS^{\mathsf T}QR^{\mathsf T} - RQ'CS^{\mathsf T}QR^{\mathsf T} + RQ^2SS^{\mathsf T}QR^{\mathsf T} - R''CS^{\mathsf T}R^{\mathsf T\prime} + 2R'QSS^{\mathsf T}R^{\mathsf T\prime} + RQ'SS^{\mathsf T}R^{\mathsf T\prime} + RQ^2CS^{\mathsf T}R^{\mathsf T\prime} - R''CC^{\mathsf T}QR^{\mathsf T} + 2R'QSC^{\mathsf T}QR^{\mathsf T} + RQ'SC^{\mathsf T}QR^{\mathsf T} + RQ^2CC^{\mathsf T}QR^{\mathsf T} = 2R'QR^{\mathsf T\prime} + RQ'R^{\mathsf T\prime} - R''QR^{\mathsf T} + RQ^3R^{\mathsf T} = 2R'(R^{\mathsf T}R)^{-1}R^{\mathsf T\prime} + R((R^{\mathsf T}R)^{-1})'\,R^{\mathsf T\prime} - R''R^{-1}R^{\mathsf T-1}R^{\mathsf T} + R(R^{\mathsf T}R)^{-3}\,R^{\mathsf T} = 2R'R^{-1}R^{\mathsf T-1}R^{\mathsf T\prime} - R''R^{-1} + (RR^{\mathsf T})^{-2} - R(R^{\mathsf T}R)^{-1}(R^{\mathsf T\prime}R + R^{\mathsf T}R')(R^{\mathsf T}R)^{-1}R^{\mathsf T\prime} = 2R'R^{-1}R^{\mathsf T-1}R^{\mathsf T\prime} - R''R^{-1} + (RR^{\mathsf T})^{-2} - 2RR^{-1}R^{\mathsf T-1}R^{\mathsf T}R'R^{-1}R^{\mathsf T-1}R^{\mathsf T\prime} = -R''R^{-1} + (RR^{\mathsf T})^{-2}$. The proof is complete.

Remark 3. If $A(x)$ is the phase matrix of $y'' + P(x)\,y = 0$, then $A(x)$ is also the phase matrix of $y'' + T^{\mathrm{T}} P(x)\,Ty = 0$, where T is a constant orthonormal matrix. In fact, if the matrix $R(x)$ satisfies (3.18), then the matrix $R_1(x) = T^{\mathrm{T}} R(x)$ also satisfies these relations and $-R_1''R_1^{-1} + (R_1R_1^{\mathrm{T}})^{-2} = T^{\mathrm{T}} P(x)\,T$.

Remark 4. The relation (3.19) can be also derived from the statement of Remark 2. By (3.16) we have $Q^{-1} = R^{\mathrm{T}}R$ and $Q = R^{\mathrm{T}}R'' + R^{\mathrm{T}}PR$. Multiplying the last identity by $R^{\mathrm{T}-1}$ on the left and by R^{-1} on the right we obtain (3.19).

4. APPLICATIONS

Using the results of the preceding section we shall derive several statements concerning the oscillation properties of (1.1) and (1.2). First we recall some definitions and auxiliary results.

The linear Hamiltonian system

(4.1)
$$\begin{aligned} y' &= B(x)\,z \\ z' &= C(x)\,y\,, \end{aligned}$$

where $B(x)$, $C(x)$ are symmetric $n \times n$ matrices, $B(x)$ is positive definite, is said to be *oscillatory for large* x if there exists a solution $(y(x), z(x))$ of (4.1) and an infinite sequence $x_n \to \infty$ such that $y(x_n) = 0$. In the opposite case the system (4.1) is said to be *nonoscillatory for large* x (see e.g. [9]).

Lemma 1. *The system* (4.1) *is oscillatory for large* x *if and only if there exists a solution* $(Y(x), Z(x))$ *of the associated matrix system*

(4.2)
$$\begin{aligned} Y' &= B(x)\,Z \\ Z' &= C(x)\,Y \end{aligned}$$

for which $Y^{\mathrm{T}}(x)\,Z^{\mathrm{T}}(x) - Z(x)\,Y(x) = 0$ *on* I *and* $Y(x)$ *is nonsingular for large* x.

Proof. See [8].

Lemma 2. *The linear Hamiltonian system*

(4.3)
$$\begin{aligned} u' &= Q(x)\,v \\ v' &= -Q(x)\,u\,, \end{aligned}$$

where $Q(x) \in C^0(I)$ *is a symmetric positive definite* $n \times n$ *matrix, is nonoscillatory for large* x *if and only if at least one of the following conditions holds.*

(4.4)
$$\text{i)} \int^{\infty} \|Q(x)\|_e \,\mathrm{d}x < \infty$$

$$\text{ii)} \int^{\infty} \operatorname{Tr} Q(x)\,\mathrm{d}x < \infty$$

$$\text{iii)} \int^{\infty} q_{ij}(x)\,\mathrm{d}x \quad < \infty\,, \quad 1 < i, j < n\,,$$

where $\| \ \|_e$ *denotes the Euclidean matrix norm and* $q_{ij}(x)$ *are the elements of* $Q(x)$.

Proof. See [8].

Theorem 3. *The system* (1.1) *is nonoscillatory for large* x *if and only if there exist two isotropic solutions* $U(x)$, $V(x)$ *of* (1.2) *for which* (3.3) *holds and*

$$\int^{\infty} (U(x)\,U^{\mathrm{T}}(x) + V(x)\,V^{\mathrm{T}}(x))^{-1}\,\mathrm{d}x < \infty\,. \tag{4.5}$$

Proof. Let (1.1) be nonoscillatory for large x. Then by Lemma 1 there exists an isotropic solution $U(x)$ of (1.2) which is nonsingular on the interval $[b, \infty)$ for some real b. For $V(x) = U(x) \int_b^x (U^{\mathrm{T}}(s)\,U(s))^{-1}\,\mathrm{d}s$ we have $U^{\mathrm{T}}(x)\,V'(x) - U^{\mathrm{T}\prime}(x)\,V(x) = E$. By Theorem 1 there exists a nonsingular $n \times n$ matrix $R(x)$ such that $U(x) = R(x)\,S(x)$, $V(x) = R(x)\,C(x)$, where $(S(x), C(x))$ is a solution of (2.3) with $Q(x) = (R^{\mathrm{T}}(x)\,R(x))^{-1}$. Let us denote $H(x) = R(x)\,R^{\mathrm{T}}(x) = U(x)\,U^{\mathrm{T}}(x) + V(x)\,V^{\mathrm{T}}(x)$. From (3.9) and (3.21) we see that $H^{-1}(x) = T(x)\,Q(x)\,T^{\mathrm{T}}(x)$, where $T(x)$ is the solution of (3.8). The matrix $T(x)$ is orthonormal, hence $\|Q(x)\|_e = \|H^{-1}(x)\|_e$. Since $U(x)$ is nonsingular, $S(x) = R^{-1}(x)\,U(x)$ is also nonsingular and $S^{\mathrm{T}}(x)\,C(x) - C^{\mathrm{T}}(x)\,S(x) = 0$. Therefore by Lemma 1 (4.3) is nonoscillatory for large x and by Lemma 2 $\int^{\infty} \|Q(x)\|_e\,\mathrm{d}x = \int^{\infty} \|H^{-1}(x)\|_e\,\mathrm{d}x < \infty$. This implies $\int^{\infty} H^{-1}(x)\,\mathrm{d}x = \int^{\infty} (U(x)\,U^{\mathrm{T}}(x) + V(x)\,V^{\mathrm{T}}(x))^{-1}\,\mathrm{d}x < \infty$.

Conversely, let $U(x)$, $V(x)$ be two isotropic solutions of (1.2) for which (3.3) and (4.5) hold. These solutions can be expressed in the form (3.5) and by the above argument $\int^{\infty} \|Q(x)\|_e\,\mathrm{d}x < \infty$, where $Q(x) = (R^{\mathrm{T}}(x)\,R(x))^{-1}\,\mathrm{d}x$. It follows that (4.3) is nonoscillatory for large x and hence (1.1) is also nonoscillatory for large x. The proof is complete.

For the scalar equation (1.3) we have the Sturm comparison theorem. A similar statement is valid for systems (1.1) (see [6]). Due to this result many comparison criteria have been derived for the oscillation and nonoscillation properties of (1.1). It is known that for the scalar equations there exists no "ideal" comparison criterion in the sense that for every oscillatory equation $y'' + p(x)\,y = 0$ there exists a function $p_1(x) < p(x)$ such that the equation $y'' + p_1(x)\,y = 0$ is also oscillatory. The following theorem shows that the same is true for systems (1.1).

Theorem 4. *Let* (1.1) *be oscillatory for large* x. *Then there exists a symmetric* $n \times n$ *matrix* $P_1(x)$ *such that* $P_1(x) < P(x)$ *on* I *and the system* $y'' + P_1(x)\,y = 0$ *is oscillatory for large* x.

Proof. Let $A(x)$ be the phase matrix of (1.1) and let us set $A_1(x) = k\,A(x)$, where $k \in (0, 1)$. We denote by $B(x)$ and $B_1(x)$ the symmetric positive definite matrices for which $B^2(x) = (A'(x))^{-1}$ and $B_1^2(x) = (A_1'(x))^{-1}$. For $R_1(x) = G_1(x)\,B_1(x)$, where

$G_1(x)$ is the solution of $G_1' = \frac{1}{2}(B_1'(x) B_1^{-1}(x) - B_1^{-1}(x) B'(x)) G_1$, $G(a) = E$, we have $R_1^T(x) R_1'(x) - R_1^{T'}(x) R_1(x) = 0$ and $(R_1^T(x) R_1(x))^{-1} = A_1'(x) = Q_1(x)$, as in the proof of Theorem 2. If $R(x) = G(x) B(x)$, where $G(x)$ is determined by (3.20), we see that $R_1(x) = k^{-1/2} R(x)$. Let us put $P_1(x) = -R_1''(x) R_1^{-1}(x) + (R_1(x) . R_1^T(x))^{-2} = -R''(x) R^{-1}(x) + k^2(R(x) R^T(x))^{-2} < P(x)$. Since $\int^\infty Q_1(x)\,dx = \infty$ the system $y'' + P_1(x) y = 0$ is oscillatory for large x.

The following statement generalizes the known result for the scalar equations.

Theorem 5. *If every solution of* (1.1) *is bounded for large x, then* (1.1) *is oscillatory for large x.*

Proof. Let $U(x)$, $V(x)$ be linearly independent isotropic solutions of (1.2) for which (3.3) holds. Then $U(x)$, $V(x)$ can be expressed in the form (3.5) and every solution of (1.1) is bounded if there exists $k > 0$ such that $\|U(x)\|_e < k$, $\|V(x)\|_e < k$. From (2.4) we see that $\|S(x)\|_e \leq 1$, $\|C(x)\|_e \leq 1$, hence $U(x)$, $V(x)$ are bounded if and only if $\|R(x)\|_e < k_1$ for some positive real k_1. If $R(x)$ is bounded then $\liminf_{x\to\infty} (R(x) . R^T(x))^{-1} > 0$, where 0 denotes the zero $n \times n$ matrix. On the other hand, if (1.2) is nonoscillatory for large x then by Theorem 3 we have $\int^\infty (U(x) U^T(x) + V(x) . V^T(x))^{-1}\,dx = \int^\infty (R(x) R^T(x))^{-1}\,dx < \infty$. Since $R(x) R^T(x)$ is positive definite, the necessary condition for the convergence of the last integral is $\liminf_{x\to\infty} (R(x) . R^T(x))^{-1} = 0$, which is a contradiction and thus (1.1) is oscillatory for large x.

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