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ON α-CONTINUOUS FUNCTIONS

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1. INTRODUCTION

In 1965, O. Njåstad [12] introduced a weak form of open sets called α-sets. The present author [16] defined a function $f: X \to Y$ to be strongly semi-continuous if $f^{-1}(V)$ is an α-set of X for each open set V of Y and showed that the images of open connected sets are connected under strongly semi-continuous functions. Recently, A. S. Mashhour et. al. [10] have called strongly semi-continuous functions α-continuous and obtained several properties of such functions. In [10], they stated without proofs that α-continuity implies θ-continuity and is independent of almost-continuity in the sense of Singal [19]. On the other hand, in 1980 S. N. Maheshwari and S. S. Thakur [8] defined a function $f: X \to Y$ to be α-irresolute if $f^{-1}(V)$ is an α-set of X for each α-set V of Y and obtained several properties of α-irresolute functions.

The purpose of the present paper is to continue the investigation of α-continuous functions. In Section 3, we shall investigate the relationships between α-continuous functions and several known functions, for example, almost-continuous, η-continuous, δ-continuous or irresolute functions. In the last section, we shall obtain some improvements of the results established in [8] and show that every α-continuous function is α-irresolute if it is either semi-open due to N. Biswas [1] or almost-open due to M. K. Singal and A. R. Singal [19].

2. PRELIMINARIES

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of (X, τ). The closure of S and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. The subset S is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S)) = S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S)) = S$). The subset S is said to be *α-open* [12] (resp. *semi-open* [7], *pre-open* [9]) if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$ (resp. $S \subset \mathrm{Cl}(\mathrm{Int}(S))$, $S \subset \mathrm{Int}(\mathrm{Cl}(S))$). The complement of an α-open (resp. semi-open) set is called *α-closed* (resp. *semi-closed*). The family of all α-open (resp. semi-open, pre-open) sets of (X, τ)

is denoted by τ^α (resp. $SO(X, \tau)$, $PO(X, \tau)$). It is known in [12] that τ^α is a topology for X and $\tau^\alpha \subset SO(X, \tau)$.

Definition 2.1. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be *α-continuous* [10] (resp. *semi-continuous* [7]) if $f^{-1}(V) \in \tau^\alpha$ (resp. $f^{-1}(V) \in SO(X, \tau)$) for every $V \in \sigma$.

In [16], the present author called α-continuous functions strongly semi-continuous. However, in this paper we use the term "α-continuous" following A. S. Mashhour et. al. [10].

Definition 2.2. A function $f: X \to Y$ is said to be *almost-continuous* (briefly, *a.c.H.*) [5] if for each $x \in X$ and each neighborhood V of $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x.

It is obvious that a function $f: (X, \tau) \to (Y, \sigma)$ is a.c.H. if and only if $f^{-1}(V) \in PO(X, \tau)$ for each $V \in \sigma$. It is reasonable that A. S. Mashhour et. al. [9] called a.c.H. functions pre-continuous. Example 3.1 and 3.2 of [11] show that the concepts of "a.c.H." and "semi-continuous" are independent of each other.

Definition 2.3. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be *almost-continuous* (briefly, *a.c.S.*) [19] (resp. *θ-continuous* [4], *weakly-continuous* [6]) if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$ (resp. $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$, $f(U) \subset \mathrm{Cl}(V)$).

Definition 2.4. A function $f: X \to Y$ is said to be *η-continuous* [3] if for every regular open sets U, V of Y,

(1) $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ and

(2) $\mathrm{Int}(\mathrm{Cl}(f^{-1}(U \cap V))) = \mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$.

Remark 2.5. For a function $f: X \to Y$, the following implications are known ([3], [19]):

continuous $\Rightarrow$ *a.c.S.* $\Rightarrow$ *η-continuous* $\Rightarrow$ *θ-continuous* $\Rightarrow$ *weakly-continuous.*

3. α-CONTINUOUS FUNCTIONS

Lemma 3.1. *Let A be a subset of a space (X, τ). Then A is α-open in (X, τ) if and only if A is semi-open and pre-open in (X, τ).*

Proof. *Necessity.* Let $A \in \tau^\alpha$. By the definition of α-open sets, we have $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ and $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. Therefore, we obtain $A \in SO(X, \tau) \cap PO(X, \tau)$.

Sufficiency. Let $A \in SO(X, \tau) \cap PO(X, \tau)$. Since $A \in SO(X, \tau)$, $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ and hence it follows from $A \in PO(X, \tau)$ that

$$A \subset \mathrm{Int}(\mathrm{Cl}(A)) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(A)))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

Therefore, we have $A \in \tau^\alpha$.

In [17, Theorem 1], V. Popa showed that every a.c.H. and semi-continuous function is weakly-continuous. Furthermore, in [10, Theorem 3.2] A. S. Mashhour et. al. obtained the result that every a.c.H. and semi-continuous function is α-continuous. As an improvement of these results, we have

Theorem 3.2. *A function $f : X \to Y$ is α-continuous if and only if f is a.c.H. and semi-continuous.*

Proof. This is an immedaite consequence of Lemma 3.1.

Definition 3.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *strongly η-continuous* if f is a.c.H. and for every $U, V \in \sigma$,

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(U \cap V))).$$

Lemma 3.4. *A function $f : (X, \tau) \to (Y, \sigma)$ is strongly η-continuous if and only if for every $U, V \in \sigma$,*

(1) $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ *and*
(2) $\mathrm{Int}(\mathrm{Cl}(f^{-1}(U \cap V))) = \mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$.

Proof. It is obvious that f is a.c.H. if and only if f satisfies (1). We assume that f is strongly η-continuous, and show equality (2). For any $U, V \in \sigma$, it follows from (1) that

$$f^{-1}(U \cap V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V))).$$

Since the intersection of two regular open sets is regular open, we obtain

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(U \cap V))) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V))).$$

Hence, equality (2) holds.

Lemma 3.5. *Let A and B be subsets of (X, τ). If either $A \in \mathrm{SO}(X, \tau)$ or $B \in \mathrm{SO}(X, \tau)$, then*

$$\mathrm{Int}(\mathrm{Cl}(A \cap B)) = \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B)).$$

Proof. For any subsets $A, B \subset X$, we generally have

$$\mathrm{Int}(\mathrm{Cl}(A \cap B)) \subset \mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B)).$$

Assume that $A \in \mathrm{SO}(X, \tau)$. Then we have $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(A))$. Therefore,

$$\begin{aligned}\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B)) &= \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B)))) \subset \\ &\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(A) \cap \mathrm{Int}(\mathrm{Cl}(B)))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}(A)) \cap \mathrm{Int}(\mathrm{Cl}(B)))) \subset \\ &\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A) \cap \mathrm{Cl}(B))) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A) \cap B)) \subset \mathrm{Int}(\mathrm{Cl}(A \cap B)).\end{aligned}$$

This completes the proof.

Theorem 3.6. *If a function $f : (X, \tau) \to (Y, \sigma)$ is α-continuous, then f is strongly η-continuous.*

Proof. Since f is α-continuous, by Lemma 3.1 $f^{-1}(V) \subset \tau^\alpha \in \mathrm{PO}(X, \tau)$ for any $V \in \sigma$ and hence $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$. Furthermore, $f^{-1}(U)$, $f^{-1}(V) \in \tau^\alpha \subset \subset \mathrm{SO}(X, \tau)$ for any U, $V \in \sigma$, and hence by Lemma 3.5 we have

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(U \cap V))) = \mathrm{Int}(\mathrm{Cl}(f^{-1}(U))) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) .$$

It follows from Lemma 3.4 that f is strongly η-continuous.

A strongly η-continuous function need not be α-continuous as the following example shows.

Example 3.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$. Let $Y = \{x, y, z\}$ and $\sigma = \{\emptyset, \{x\}, \{z\}, \{x, z\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = x$, $f(b) = f(c) = y$ and $f(d) = z$. Then f is strongly η-continuous but is neither α-continuous nor a.c.S.

Theorem 3.8. *Every strongly η-continuous function is η-continuous.*

Proof. Since every regular open set is open, this follows immediately from Lemma 3.4.

Since every a.c.S. function is η-continuous [3, Proposition 3.3], the following example shows that the converse to Theorem 3.8 is not true in general.

Example 3.9 (Singal and Singal [19]). Let X be the set of real numbers and τ the co-countable topology for X. Let $Y = \{a, b\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(x) = a$ if x is rational and $f(x) = b$ if x is irrational. Then, f is a.c.S. [19, Example 2.1]. However, since f is not a.c.H., it is neither strongly η-continuous nor α-continuous.

Examples 3.7 and 3.9 show that "strongly η-continuous" and "a.c.S." are independent of each other. Futhermore, the following example and Example 3.9 show that "α-continuous" and "a.c.S." are independent of each other.

Example 3.10. Let $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\} .$$

Let $Y = \{x, y, z\}$ and $\sigma = \{\emptyset, \{x\}, \{y\}, \{x, y\}, Y\}$. Define a function $f : (X, \tau) \to \to (Y, \sigma)$ as follows: $f(a) = z$ and $f(b) = f(c) = f(d) = y$. Then f is α-continuous but it is not a.c.S.

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *irresolute* [2] if $f^{-1}(V) \in \mathrm{SO}(X, \tau)$ for every $V \in \mathrm{SO}(Y, \sigma)$. We shall show that "$\alpha$-continuous" and "irresolute" are independent of each other.

Example 3.11. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is irresolute but it is not α-continuous.

Theorem 3.12. *Not every α-continuous function is irresolute.*

Proof. Assume that every α-continuous function is necessarily irresolute. Let $f : X \to Y$ be α-continuous. Let $x \in X$ and let V be any open set of Y containing $f(x)$. Since f is irresolute and $\mathrm{Int}(\mathrm{Cl}(V))$ is semi-closed in Y, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is semi-closed and hence

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))).$$

By Theorem 3.2, f is a.c.H. and hence

$$x \in f^{-1}(V) \subset f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))).$$

Put $U = \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))))$, then U is an open set of X containing x and $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that every α-continuous function is a.c.S. This contradicts Example 3.10.

A function $f : X \to Y$ is said to be *δ-continuous* [14] if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}(\mathrm{Cl}(U))) \subset \mathrm{Int}(\mathrm{Cl}(V))$. In [14], it is shown that every δ-continuous function is a.c.S. and δ-continuity and continuity are independent of each other. Example 4.4 of [14] shows that there exists a δ-continuous function without being α-continuous. Furthermore, Example 4.5 of [14] shows that a continuous (hence α-continuous) function is not necessarily δ-continuous. Therefore, we see that the concepts of α-continuity and δ-continuity are independent of each other.

4. α-IRRESOLUTE FUNCTIONS

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *α-irresolute* [8] if $f^{-1}(V) \in \tau^\alpha$ for every $V \in \sigma^\alpha$.

Every α-irresolute function is α-continuous but a continuous function is not necessarily α-irresolute [8, Example 1]. Therefore, the concept of α-continuous functions is strictly weaker than that of α-irresolute functions.

In [16, Theorem 3.6], the present author showed that the images of open connected sets are connected under α-continuous (strongly semi-continuous) functions. In [8, Theorem 2], it is shown that if a function $f : X \to Y$ is α-irresolute and A is α-open and closed in X then the restriction $f \mid A : A \to Y$ is α-irresolute. We shall obtain the improvements of these results. For this purpose, the following lemma is very useful.

Lemma 4.2. (Mashhour et. al. [10]). *Let A and V be subsets of (X, τ). If $A \in \mathrm{PO}(X, \tau)$ and $V \in \tau^\alpha$, then $A \cap V \in (\tau/A)^\alpha$, where $(\tau/A)^\alpha$ denotes the family of all α-open sets in the subspace $(A, \tau/A)$.*

Theorem 4.3. *If $f:(X,\tau)\to(Y,\sigma)$ is α-continuous and A is a pre-open and connected set of (X,τ), then $f(A)$ is connected.*

Proof. Let $f_A:(A,\tau/A)\to(f(A),\sigma/f(A))$ be a function defined by $f_A(x)=f(x)$ for every $x\in A$. We show that f_A is α-continuous. For any $V_A\in\sigma/f(A)$, there exists $V\in\sigma$ such that $V_A=V\cap f(A)$. Since f is α-continuous, $f^{-1}(V)\in\tau^\alpha$ and hence by Lemma 4.2, $(f_A)^{-1}(V_A)=f^{-1}(V)\cap A\in(\tau/A)^\alpha$. Therefore, f_A is α-continuous and hence $f_A(A)=f(A)$ is connected [16, Theorem 3.1].

Theorem 4.4. *If $f:(X,\tau)\to(Y,\sigma)$ is α-irresolute and $A\in\mathrm{PO}(X,\tau)$, then the restriction $f\mid A:(A,\tau/A)\to(Y,\sigma)$ is α-irresolute.*

Proof. Let $V\in\sigma^\alpha$. Since f is α-irresolute, $f^{-1}(V)\in\tau^\alpha$. By Lemma 4.2, $(f\mid A)^{-1}(V)=f^{-1}(V)\cap A\in(\tau/A)^\alpha$ because $A\in\mathrm{PO}(X,\tau)$. This shows that $f\mid A$ is α-irresolute.

A point $x\in X$ is said to be a *δ-cluster* point of a subset $S\subset X$ [20] if $S\cap V\neq\emptyset$ for every regular open set V containing x. A subset S is called *δ-closed* if all δ-cluster points of S are contained in S. The graph $G(f)$ of a function $f:X\to Y$ is said to be *δ-closed* if $G(f)$ is δ-closed in the product space $X\times Y$. It is known that if $f:X\to Y$ is δ-continuous and Y is Hausdorff then $G(f)$ is δ-closed [14, Theorem 5.2]. As an improvement of this result, we have

Theorem 4.5. *If a function $f:X\to Y$ is θ-continuous and Y is Hausdorff, then $G(f)$ is δ-closed.*

Proof. Let $(x,y)\notin G(f)$. Then $y\neq f(x)$ and there exist disjoint open sets V, W of Y such that $f(x)\in V$ and $y\in W$. Since V and W are disjoint open, we have $\mathrm{Cl}(V)\cap\cap\,\mathrm{Int}(\mathrm{Cl}(W))=\emptyset$. Since f is θ-continuous, there exists an open set U containing x such that $f(\mathrm{Cl}(U))\subset\mathrm{Cl}(V)$. Therefore, we obtain $f(\mathrm{Int}(\mathrm{Cl}(U)))\cap\mathrm{Int}(\mathrm{Cl}(W))=\emptyset$. It follows from [14, Theorem 5.2] that $G(f)$ is δ-closed.

Corollary 4.6. *If $f:X\to Y$ is α-continuous and Y is Hausdorff, then $G(f)$ is δ-closed.*

Proof. Every α-continuous function is η-continuous by Theorems 3.6 and 3.8 and every η-continuous function is θ-continuous [3, Proposition 3.3]. Thus, this immediately follows from Theorem 4.5.

Corollary 4.7. (Maheshwari and Thakur [8]). *If $f:(X,\tau)\to(Y,\sigma)$ is α-irresolute and (Y,σ^α) is Hausdorff, then $G(f)$ is α-closed.*

Proof. We show that if (Y,σ^α) is Hausdorff then so is (Y,σ). Since (Y,σ^α) is Hausdorff, for distinct points $x,y\in Y$ there exist $U,V\in\sigma^\alpha$ such that $x\in U$, $y\in V$ and $U\cap V=\emptyset$. Then we have $\mathrm{Cl}(\mathrm{Int}(U))\cap\mathrm{Int}(V)=\emptyset$ and hence

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U)))\cap\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(V)))=\emptyset\,.$$

Moreover, $x \in U \subset \text{Int}(\text{Cl}(\text{Int}(U)))$ and $y \in V \subset \text{Int}(\text{Cl}(\text{Int}(V)))$. This shows that (Y, σ) is Hausdorff. It is obvious that δ-closedness implies closedness and closedness implies α-closedness.

Remark 4.8. In [18], I. L. Reilly and M. K. Vamanamurthy showed that if (Y, σ^α) is Hausdorff then so is (Y, σ). However, as their proof is complicated, we gave a simple one.

Theorem 4.9. *If $f, g : (X, \tau) \to (Y, \sigma)$ are α-continuous and (Y, σ) is Hausdorff, then the set $\{x \in X \mid f(x) = g(x)\}$ is α-closed.*

Proof. A function $f : (X, \tau) \to (Y, \sigma)$ is α-continuous if and only if $f_\alpha : (X, \tau^\alpha) \to \to (Y, \sigma)$ is continuous, where f_α is the function defined by $f_\alpha(x) = f(x)$ for every $x \in X$. Since (Y, σ) is Hasudorff, the set $\{x \in X \mid f_\alpha(x) = g_\alpha(x)\}$ is closed in (X, τ^α). Therefore, $\{x \in X \mid f(x) = g(x)\}$ is α-closed in (X, τ).

Corollary 4.10. (Maheshwari and Thakur [8]). *If $f, g : (X, \tau) \to (Y, \sigma)$ are α-irresolute and (Y, σ^α) is Hausdorff, then the set $\{x \in X \mid f(x) = g(x)\}$ is* α-closed in (X, τ).

Proof. Since (Y, σ^α) is Hausdorff, (Y, σ) is Hausdorff. Thus, this is an immediate consequence of Theorem 4.9.

We shall conclude the section by giving two sufficient conditions for an α-continuous function to be α-irresolute. A function $f : X \to Y$ is said to be *almost-open* [19] if $f(U)$ is open in Y for every regular open set U of X. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *semi-open* [1] (resp. *pre-open* [9]) if $f(U) \in \text{SO}(Y, \sigma)$ (resp. $f(U) \in \text{PO}(Y, \sigma)$) for every $U \in \tau$. In [9], it is noted that pre-openness is equivalent to almost-openness in the sense of Wilansky [21]. It is known that every α-continuous pre-open function is α-irresolute [10, Theorem 3.3]. We shall show that an α-continuous function is α-irresolute if it is either almost-open or semi-open. For the relationship between "almost-open", "semi-open" and "pre-open" we have

Remark 4.11. In [15], it is shown that for a function $f : X \to Y$ the concepts of almost-openness, semi-openness and pre-openness are independent of each other.

Lemma 4.12. *Let A and B be subsets of (X, τ). Then*

(1) *$A \in \tau^\alpha$ if and only if there exists $V \in \tau$ such that* $V \subset A \subset \text{Int}(\text{Cl}(V))$.

(2) *If $A \in \tau^\alpha$ and $A \subset B \subset \text{Int}(\text{Cl}(A))$, then $B \in \tau^\alpha$.*

Proof. Since (1) is obvious, we prove (2). Since $A \in \tau^\alpha$,

$$B \subset \text{Int}(\text{Cl}(A)) \subset \text{Int}(\text{Cl}(\text{Int}(\text{Cl}(\text{Int}(A))))) = \\ = \text{Int}(\text{Cl}(\text{Int}(A))) \subset \text{Int}(\text{Cl}(\text{Int}(B))) .$$

This shows that $B \in \tau^\alpha$.

Theorem 4.13. *If $f:(X,\tau)\to(Y,\sigma)$ is almost-open and α-continuous, then f is α-irresolute.*

Proof. Let B be any α-open set of (Y,σ). By Lemma 4.12, there exists $V\in\sigma$ such that $V\subset B\subset \mathrm{Int}(\mathrm{Cl}(V))$. Since f is α-continuous, $f^{-1}(V)\in\tau^\alpha\subset \mathrm{SO}(X,\tau)$ and hence $f^{-1}(V)\subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$. Put

$$F = Y - f(X - \mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))\,.$$

Then F is closed in Y because f is almost-open and $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$ is regular closed. Furthermore, we obtain $V\subset F$ and $f^{-1}(F)\subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$. Thus, $f^{-1}(\mathrm{Cl}(V))\subset$ $\subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))$ which implies

$$f^{-1}(V)\subset f^{-1}(B)\subset f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))\subset$$
$$\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))))\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))\,.$$

It follows from Lemma 4.12 that $f^{-1}(B)\in\tau^\alpha$. This shows that f is α-irresolute.

Let S be a subset of X. The intersection of all semi-closed sets containing S is called the *semi-closure* of S and denoted by $\mathrm{sCl}(S)$.

Lemma 4.14. *If S is a subset of X, then* $\mathrm{Int}(\mathrm{Cl}(S))\subset \mathrm{sCl}(S)$.

Proof. Let $x\in \mathrm{Int}(\mathrm{Cl}(S))$ and let G be any semi-open set of X containing x. There exists an open set U of X such that $U\subset G\subset \mathrm{Cl}(U)$. Since $x\in G\subset \mathrm{Cl}(U)$ and $x\in \mathrm{Int}(\mathrm{Cl}(S))$,

$$\emptyset \neq \mathrm{Int}(\mathrm{Cl}(S))\cap U\subset \mathrm{Cl}(S)\cap U\subset \mathrm{Cl}(S\cap U)\,.$$

Therefore, we have $S\cap U\neq\emptyset$ and hence $S\cap G\neq\emptyset$. This shows that $x\in \mathrm{sCl}(S)$

Lemma 4.15. (*Noiri* [13]). *A function $f: X\to Y$ is semi-open if and only if $f^{-1}(\mathrm{sCl}(B))\subset \mathrm{Cl}(f^{-1}(B))$ for every subset B of Y.*

Theorem 4.16. *If $f:(X,\tau)\to(Y,\sigma)$ is semi-open and α-continuous, then f is α-irresolute.*

Proof. Let B be any α-open set of (Y,σ). By Lemma 4.12, there exists $V\in\sigma$ such that $V\subset B\subset \mathrm{Int}(\mathrm{Cl}(V))$. Since f is α-continuous, $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))\in\tau^\alpha$. It follows from Lemmas 4.14 and 4.15 that

$$f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))))\subset$$
$$\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{sCl}(V)))))\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))\,.$$

Therefore, we obtain $f^{-1}(V)\subset f^{-1}(B)\subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$ and $f^{-1}(V)\in\tau^\alpha$. By Lemma 4.12, $f^{-1}(B)\in\tau^\alpha$. This shows that f is α-irresolute.

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