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MINIMUM FUNCTORS ON CATEGORIES OF NEUMAN TREES

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1. In this paper graphs are supposed to be undirected and without loops and multiple edges. An infinite graph is a graph with a denumerable infinite set of vertices. We denote a graph G by an ordered pair $G = (V(G), E(G))$ where $V(G)$ means the set of vertices of G and $E(G)$ the set of edges of G. For graphs G, G', a triple (f, G, G') is called a homomorphic mapping on G into G' iff f is a mapping on $V(G)$ into $V(G')$ with the property that $(a, b) \in E(G)$ always implies $(f(a), f(b)) \in E(G')$; if the converse of this implication is valid as well and f is one-to-one, (f, G, G') is an isomorphic mapping (isomorphism) on G into G'.

For a category $\boldsymbol{G}$ we denote by $\text{Ob}\, \boldsymbol{G}$ the class of objects and by $\text{Mor}\, \boldsymbol{G}$ the class of morphisms of $\boldsymbol{G}$. A category $\boldsymbol{G}$ is called an *I-category* iff $\boldsymbol{G}$ satisfies the following conditions:

(α) Each $G \in \text{Ob}\, \boldsymbol{G}$ is a connected graph;

(β) if $G \in \text{Ob}\, \boldsymbol{G}$ and G' is isomorphic to G, then $G' \in \text{Ob}\, \boldsymbol{G}$;

(γ) $\gamma \in \text{Mor}\, \boldsymbol{G}$ iff $\gamma = (f, G, G')$ is a one-to-one homomorphic mapping on G into G' with $G, G' \in \text{Ob}\, \boldsymbol{G}$.

Let $\boldsymbol{G}, \boldsymbol{H}$ be I-categories and $F : \boldsymbol{G} \to \boldsymbol{H}$ a covariant functor on $\boldsymbol{G}$ into $\boldsymbol{H}$. F is called an *I-functor* on $\boldsymbol{G}$ into $\boldsymbol{H}$ iff:

(δ) $F(G) = (V(G), E(F(G)))$ with $E(G) \subseteqq E(F(G))$ for every $G \in \text{Ob}\, \boldsymbol{G}$;

(ε) $F(f, G, G') = (f, F(G), F(G'))$ for each $(f, G, G') \in \text{Mor}\, \boldsymbol{G}$.

In the class $I(\boldsymbol{G}, \boldsymbol{H})$ of all I-functors on $\boldsymbol{G}$ into $\boldsymbol{H}$ we define a partial ordering $\leqq$: : $F \leqq F'$ iff $E(F(G)) \subseteqq E(F'(G))$ for every $G \in \text{Ob}\, \boldsymbol{G}$.

In [1] we describe minimum elements (with respect to this partial ordering) in the class of I-functors between I-categories of finite graphs with certain Hamiltonian properties; in [2] such elements are constructed on the I-category of infinite connected graphs with sufficient binding into I-categories of Hamiltonian graphs; in this paper we shall prove the existence of minimum functors on I-categories of Neuman trees into I-categories of Hamiltonian graphs.

2. Let G be a graph, x_i $(i = 1, \ldots, n)$ vertices of G. Then we denote by $v_G(x_i)$ the degree of x_i in G and by $d_G(x_i, x_j)$ the distance of x_i, x_j in G. $G(x_1, \ldots, x_n)$ means the subgraph of G obtained from G by deleting the vertices of degree 1 with the exception of $x_1, \ldots, x_n$. In the sequence $w = x_1 \ldots x_n$ we call x_1 and x_n the endvertices of w; each x_i with $1 < i < n$ is said to be between x_1 and x_n or to be an inner vertex of w. If $w' = x'_1 \ldots x'_k$ is another sequence of vertices of G, we understand by ww' the sequence $x_1 \ldots x_n x'_1 \ldots x'_k$. Analogously we consider sequences of the form $w = x_0 x_1 x_2 \ldots$ (one-way infinite sequence starting with x_0) or $w = \ldots x_{-1} x_0 x_1 \ldots$ (two-way infinite sequence). A sequence w of vertices of G is called a path in G iff for consecutive members x, y of w we have $(x, y) \in E(G)$ and each vertex of G is occuring at most once in w; it is said to be a Hamiltonian path of G iff it is a path in G and each vertex of G occurs at least once in it.

For a graph U we write $U \subseteqq G$ iff U is a subgraph of G. Let G be an infinite graph. G is said to be *Hamiltonian* iff there is a one-way infinite sequence of G which is a Hamiltonian path of G. G is called *strong-Hamiltonian* iff for each vertex x of G there is a one-way infinite sequence of G starting with x which is a Hamiltonian path of G. If $U \subseteqq G$ and w is a path of G containing each vertex of U exactly once and no other vertices, we call w a *U-Hamiltonian* path of G.

An infinite (A finite) *Neuman tree* T is an infinite (a finite) tree the square T^2 of which is Hamiltonian (has a Hamiltonian path). Such trees have been characterized in [4] and [5]; it has been proved:

Theorem 1. *For a finite tree T there is a Hamiltonian path in T^2 with endvertices a, b iff the tree $T(a, b)$ satisfies*

(i) $v_{T(a,b)}(x) \leqq 4$ *for each* $x \in V(T(a, b))$;

(ii) *each* $x \in V(T(a, b))$ *with* $v_{T(a,b)}(x) \geqq 3$ *is an inner vertex of the path connecting* a, b;

(iii) *between each two vertices of degree* 4 *(in $T(a, b)$), there is at least one vertex of degree* 2 *(in T); if $v_T(a) > 1$, then for every vertex x with $v_{T(a,b)}(x) = 4$ there is at least one vertex of degree* 2 *(in T) between a and x, and similarly for the vertex b; if both $v_T(a) > 1$ and $v_T(b) > 1$, then there is at least one vertex of degree* 2 *(in T) between a and b.*

Theorem 2. *Let T be an infinite tree, $a \in V(T)$.*

(iv) *There is no Hamiltonian path of T^2 starting with a if T contains more than one vertex of infinite degree;*

(v) *let be a vertex of infinite degree in T. Denote by Z the set of all those vertices of T of degree* 1 *(with the exception of a) which are adjacent to b. Then there is a Hamiltonian path of T^2 starting with a iff Z is not the empty set and for every $z \in Z$ the subtree generated by $V(T) - (Z - \{z\})$ is a finite tree the square of which has a Hamiltonian path with endvertices a, z;*

(vi) *for every* $x \in V(T)$, *let* $v_T(x)$ *be finite. Then there is a Hamiltonian path of* T^2 *starting with a iff* $T(a)$ *satisfies*

(vi.i) *there is exactly one one-way infinite path* w *of* T *starting with* a;

(vi.ii) $v_{T(a)}(x) \leqq 4$ *for every* $x \in V(T(a))$;

(vi.iii) *each* $x \in V(T(a))$ *with* $v_{T(a)}(x) \geqq 3$ *is an inner vertex of* w;

(vi.iv) *between each two vertices of degree* 4 *in* $T(a)$ *there is at least one vertex of degree* 2 *in* T; *if* $v_T(a) > 1$, *then, for every vertex* x *with* $v_{T(a)}(x) = 4$, *there is at least one vertex of degree* 2 *in* T *between* a *and* x.

Now we consider the following three cases:

(a) T is a finite tree and $w_T = y \dots$ a Hamiltonian path in T^2 with $v_T(y) = 1$;

(b) T is a finite tree and $w_T = y' \dots y''$ a Hamiltonian path in T^2;

(c) T is an infinite tree without vertices of infinite degree, not a one-way infinite path, and with Hamiltonian T^2.

In order to avoid repetitions we will define a *tree* T' *associated with* T. To this end we choose

in the case (a): a vertex x in T such that $d_T(x, y)$ becomes maximum with respect to the condition that there is a Hamiltonian path in T^2 with the endvertices x, y;

in the case (b): vertices x, y in T such that $d_T(x, y)$ becomes maximum under the condition that there is a Hamiltonian path in T^2 with the endvertices x, y;

in the case (c): such a vertex x in T that $d_T(x, x')$ becomes maximum with respect to the condition that there is a Hamiltonian path in T^2 starting with x where x' is the first vertex of the one-way infinite path of T starting with x and satisfying $v_{T(x)}(x') \geqq 3$.

For $u \in V(T)$, we denote by M_u the set of all those vertices of degree 1 in T which are adjacent to u. We choose a set M_T of vertices of T so that M_T contains no other vertices than exactly one element of M_u for each $u \in V(T)$ with the following properties: M_u is not the empty set, and in the cases (a), (b) it is $v_{T(x,y)}(u) = 1$ or u is between x and y with $v_{T(x,y)}(u) = 2$, but in the case (c) it is $v_{T(x)}(u) = 1$ or u is a vertex of the one-way infinite path in T starting with x so that $v_{T(x)}(u) = 2$. The subgraph of T generated by $V(T(x, y)) \cup M_T$ in the cases (a), (b) and by $V(T(x)) \cup M_T$ in the case (c) is called a tree associated with T and denoted by T'. Because of Theorems 1 and 2, it is obvious that there is also a Hamiltonian path in T'^2 with the endvertices x, y in the cases (a), (b) and starting with x in the case (c). In connection with such a tree T' (associated with T) we make the following agreements: Let U be the set of all vertices u of T' with $v_{T(x,y)}(u) \geqq 3$ in the cases (a), (b) and with $v_{T(x)}(u) \geqq 3$ in the case (c). Then because of Theorems 1 and 2 there is a path of T with the endvertices x, y in the cases (a), (b) or starting with x in the case (c) such that the vertices of U are inner vertices of this path. Let $w = x_0 \dots x_n$ with $x_0 = x$, $x_n = y$ in the cases (a), (b) and $w = x_0 x_1 x_2 \dots$ with $x_0 = x$ in the case (c) be such paths. For each x_i, $i \geqq 0$, there are at most two non-trivial paths in T' starting with x_i and having no other common vertex with w. If there are exactly two paths of this kind, we denote them by $\bar{w}(x_i)$ and $\bar{\bar{w}}(x_i)$ supposing that the length of $\bar{\bar{w}}(x_i)$ does not exceed the length of $\bar{w}(x_i)$. If there is only one such a path, it is denoted by $\bar{w}(x_i)$, and $\bar{\bar{w}}(x_i)$ means the trivial path x_i; if there is no such path, we define $\bar{w}(x_i) = \bar{\bar{w}}(x_i) = x_i$. Let $\bar{x}_i^j$, $\bar{\bar{x}}_i^j$ be the vertices of $\bar{w}(x_i)$, $\bar{\bar{w}}(x_i)$, respectively, with $d_T(x_i, \bar{x}_i^j) = j$, $d_T(x_i, \bar{\bar{x}}_i^j) = j$. $\bar{w}^+(x_i)$ means the sequence $\bar{x}_i^1 \bar{x}_i^3 \dots \bar{x}_i^4 \bar{x}_i^2$ in which the upper indices first increase through odd numbers for as long as possible and then decrease through even numbers so that each vertex of $\bar{w}(x_i)$ except x_i is contained exactly once in this sequence; analogously $\bar{w}_+(x_i)$ is the sequence $\bar{x}_i^2 \bar{x}_i^4 \dots \bar{x}_i^3 \bar{x}_i^1$ and $\bar{\bar{w}}_+(x_i)$ the sequence $\bar{\bar{x}}_i^2 \bar{\bar{x}}_i^4 \dots \bar{\bar{x}}_i^3 \bar{\bar{x}}_i^1$. For $i \geqq 0$ we define further

$$w_i = \begin{cases} x_i, \text{ if } v_{T'}(x_i) \leqq 2, \\ \bar{w}^+(x_i)\, x_i\, \bar{\bar{w}}_+(x_i), \text{ if } v_{T'}(x_i) = 4, \\ \bar{w}^+(x_i)\, x_i, \text{ if } v_{T'}(x_i) = 3 \text{ and there is a } j < i \text{ with } v_{T'}(x_j) \leqq 2 \text{ and} \\ \text{between } x_i \text{ and } x_j \text{ there is no vertex with degree 4 in } T', \\ x_i\, \bar{w}_+(x_i) \text{ in the other cases.} \end{cases}$$

3. We recall the results of [1] needed in what follows.

Theorem 3. *Let $\boldsymbol{G}$, $\boldsymbol{H}$ be I-categories, M a subclass of* Ob $\boldsymbol{G}$ *and* $F : \text{Ob}\,\boldsymbol{G} \to \text{Ob}\,\boldsymbol{H}$ *a function on* Ob $\boldsymbol{G}$ *into* Ob $\boldsymbol{H}$ *which satisfies the following condition:*
For every $G \in \text{Ob}\,\boldsymbol{G}$,

(I) $V(G) = V(F(G))$,

(II) $(a, b) \in E(F(G))$ *iff* $(a, b) \in E(G)$ *or there are a* $G' \in M$, *a subgraph* U *of* G, *and an isomorphic mapping* $(g, G', U) \in \text{Mor}\,\boldsymbol{G}$ *on* G' *onto* U *with* $(a, b) \in E(U)$ *and* $(g^{-1}(a), g^{-1}(b)) \in E(F(G'))$.

Then F_0 *with* $F_0(G) = F(G)$ *for* $G \in \text{Ob}\,\boldsymbol{G}$, $F_0(f, G, \bar{G}) = (f, F(G), F(\bar{G}))$ *for* $(f, G, \bar{G}) \in \text{Mor}\,\boldsymbol{G}$ *is an I-functor.*

The following lemma is a slight generalization of the corresponding result in [1].

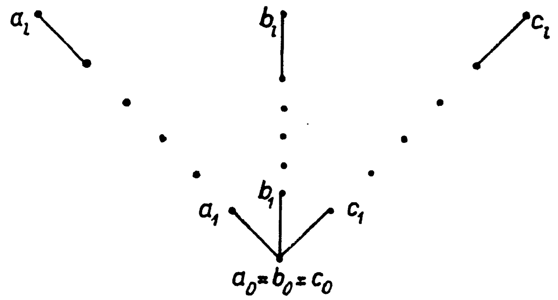
Lemma 1. *Let $\boldsymbol{G}$, $\boldsymbol{H}$ be I-categories,* $F : \boldsymbol{G} \to \boldsymbol{H}$ *an I-functor on $\boldsymbol{G}$ into $\boldsymbol{H}$. Furthermore, let* $\bar{G} \in \text{Ob}\,\boldsymbol{G}$ *and* $G \in \text{Ob}\,\boldsymbol{G}$ *such that there is a morphism* $(f, \bar{G}, G)$ *in $\boldsymbol{G}$. If* (x, y) *is an edge of* $F(\bar{G})$, *then* $(f(x), f(y))$ *is an edge of* $F(G)$.

Theorem 4. *Let $\boldsymbol{G}$, $\boldsymbol{H}$ be I-categories,* $F : \boldsymbol{G} \to \boldsymbol{H}$, $F' : \boldsymbol{G} \to \boldsymbol{H}$ *I-functors. Moreover, let* $M \subseteqq \text{Ob}\,\boldsymbol{G}$ *be chosen so that condition* (II) *of Thoerem* 3 *is satisfied for* F *and every* $G \in \text{Ob}\,\boldsymbol{G}$. *If always* $E(F(G')) \subseteqq E(F'(G'))$ *for* $G' \in M$, *then* $F \leqq F'$.

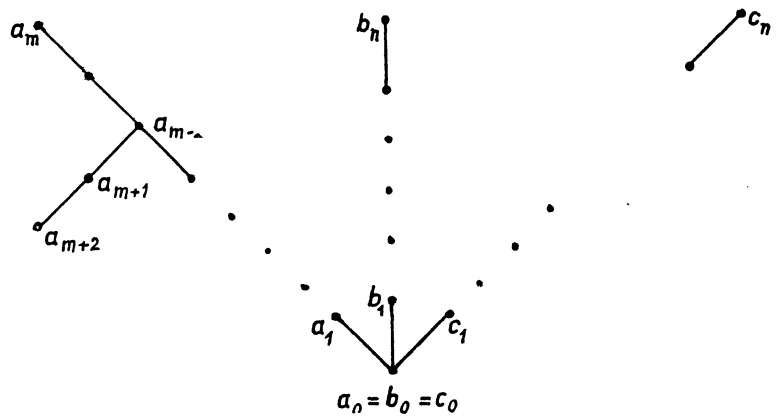
4. In this section we shall construct a minimum I-functor on the I-category $\boldsymbol{N}$ of finite Neuman trees into the I-category $\boldsymbol{H}$ of all finite graphs in which a Hamiltonian path exists.

Let l, m, n be natural numbers, G a finite graph. We define:

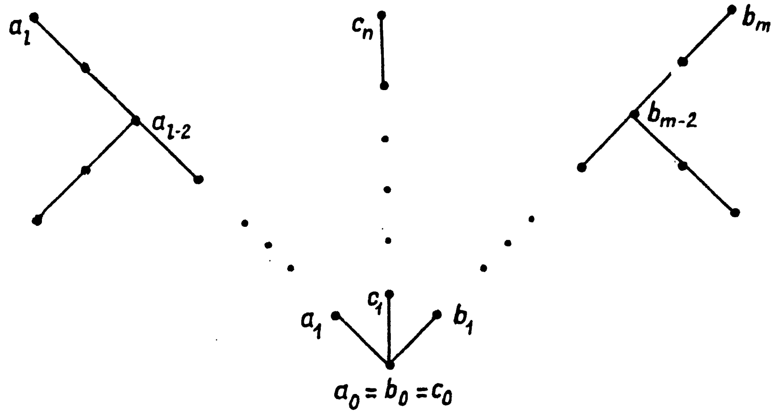
(1) G is an l-graph iff G is isomorphic to the graph



(2) G is an (mT, n)-graph iff G is isomorphic to the graph



(3) G is an (lT, mT, n)-graph iff G is isomorphic to the graph



Obviously, in each of these three cases G is an element of Ob N.

Now, for every finite tree $T = (V(T), E(T))$, we define a finite graph $W(T)$ by

$$V(W(T)) = V(T);$$

$(x, y) \in E(W(T))$ if either $g_T(x, y) = 2$ and one of the following conditions (W1), (W2), (W3) is satisfied or $(x, y) \in E(T)$.

(W1) There is an $l \geqq 1$ and a subgraph T^* of T such that T^* is an l-graph and $x, y \in V(T^*)$.

(W2) There is an $m \geqq 3$, and $n \geqq m + 1$ and a subgraph T^* of T such that T^* is an (mT, n)-graph and $x, y \in V(T^*)$.

(W3) There is an $l \geqq 3$, an $m \geqq 1$, an $n \geqq m + 1$ and a subgraph T^* of T such that T^* is an (lT, mT, n)-graph and $x, y \in V(T^*)$.

Lemma 2. *Let T be a finite tree, $x \neq y$ vertices of T, and w_T a Hamiltonian path with the endvertices x and y in $W(T)$. The tree B may arise from T by adding some endedges to given vertices $u_1, \ldots, u_q$ $(q \geqq 0)$ under the condition that in w_T there are consecutive vertices s_i, t_i $(i = 1, \ldots, q)$ with $(s_i, u_i) \in E(T)$ and $(t_i, u_i) \in E(T)$. Then there is also a Hamiltonian path w_B with the endevertices x, y in $W(B)$ with the following property: If x' and y' are consecutive in w_T, and if there is no u_i $(i = 1, \ldots, q)$ with $(x', u_i) \in E(T)$ and $(y', u_i) \in E(T)$, then x' and y' are consecutive in w_B, too.*

This lemma can be proved easily by induction on q.

Theorem 5. *$T \in \mathrm{Ob}\, N$ implies $W(T) \in \mathrm{Ob}\, H$.*

Proof. Let $T \in \mathrm{Ob}\, N$. Then there is a Hamiltonian path in T^2. In a fixed tree T' associated with T (introduced in Section 2, case (b)) the sequence $w_T = w_0 w_1 \ldots w_n$ can be shown to be a Hamiltonian path in $W(T')$. By the preceding lemma, there is a Hamiltonian path in $W(T)$, too. Q.e.d.

We denote by M the subclass of $\mathrm{Ob}\, N$ consisting of all l-graphs with $l \geqq 1$, all (mT, n)-graphs with $n \geqq m + 1 \geqq 4$, and all (lT, mT, n)-graphs with $n \geqq m + 1 > l \geqq 3$. Hence by Theorem 3, there is a unique I-functor on N into H with the object function $\mathrm{Ob}\, N \ni T \to W(T) \in \mathrm{Ob}\, H$. We shall denote this I-functor by W.

Theorem 6. *W is a minimum element in the class $I(N, H)$ of I-functors on N into H.*

Proof. Let $F \in I(N, H)$ with $F \leqq W$. We consider the elements of the set M defined above.

For $l \geqq 1$ let G_l be an l-graph; the vertices may be denoted as in (1). If none of (a_1, b_1), (b_1, c_1), (a_1, c_1) were an edge of $F(G_1)$, $F(G_1)$ could not be an element of $\mathrm{Ob}\, H$ because of $F \leqq W$. Therefore we can assume without loss of generality $(a_1, b_1) \in E(F(G_1))$. By the structure of isomorphisms of G_l onto itself and lemma we have

$E(W(G_l)) \subseteqq E(F(G_l))$. Let $E(W(G_l)) \subseteqq E(F(G_l))$ be proved for $l \leqq k - 1$; $k \geqq 2$. Let $(x, y) \in E(W(G_k))$. If $(x, y) = (a_k, a_{k-2})$ or $(x, y) = (b_k, b_{k-2})$ or $(x, y) = (c_k, c_{k-2})$, then $(x, y) \in E(F(G_k))$ follows by $F(G_k) \in \text{Ob}\, \boldsymbol{H}$, $F \leqq W$, the structure of isomorphisms of G_k onto itself and lemma 1.

If (x, y) is none of these pairs of vertices and $(x, y) \neq (a_k, a_{k-1})$, $(x, y) \neq (b_k, b_{k-1})$, $(x, y) \neq (c_k, c_{k-1})$, there is an $l \leqq k - 1$, an $(x', y') \in E(W(G_l))$, and a morphism (f, G_1, G_k) with $(f(x'), f(y')) = (x, y)$. By lemma 1 and the induction assumption, we get $(x, y) \in E(F(G_k))$. Therefore $E(W(G_k)) \subseteqq E(F(G_k))$ for every $k \geqq 1$. Now for $m \geqq 3$, $n \geqq m + 1$ let G_{mn} be an (mT, n)-graph; the vertices may be denoted as in (2). For all $n \geqq m + 1$ we have: If neither (b_n, b_{n-2}) nor (c_n, c_{n-2}) belonged to $E(F(G_{mn}))$ then we should have a contradiction with $F(G_{mn}) \in \text{Ob}\, \boldsymbol{H}$ because $F \leqq W$ and the square of the subgraph generated by $\{a_i / i = m - 4, \ldots, m + 2\}$ (if $m = 3$, define $a_{-1} = b_1$) is not Hamiltonian ([4]). Therefore at least one of these pairs must be an edge of $F(G_{mn})$, and the structure of isomorphisms of G_{mn} onto itself and lemma 1 imply that (b_n, b_{n-2}) and (c_n, c_{n-2}) must be edges of $F(G_{mn})$. Now it is easy to show by induction on n that $E(W(G_{mn})) \subseteqq E(F(G_{mn}))$.

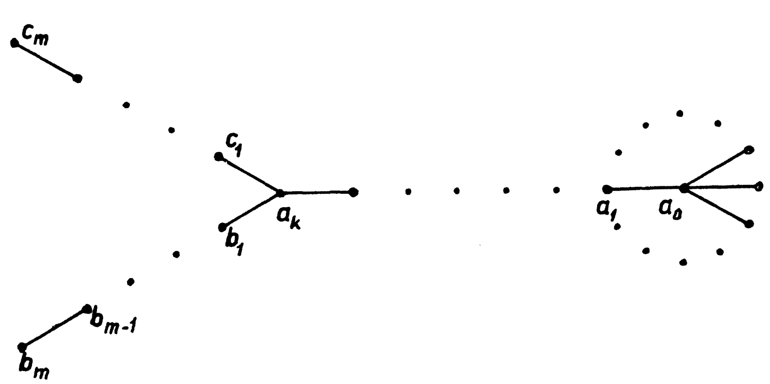
Analogously $E(W(G_{lmn})) \subseteqq E(F(G_{lmn}))$ can be proved for all $l \geqq 3$, $m \geqq 3$, $n \geqq m + 1$ where G_{lmn} is an (lT, mT, n)-graph.

Thus we get from Theorem 4 and the above facts that $W \leqq F$. Q.e.d.

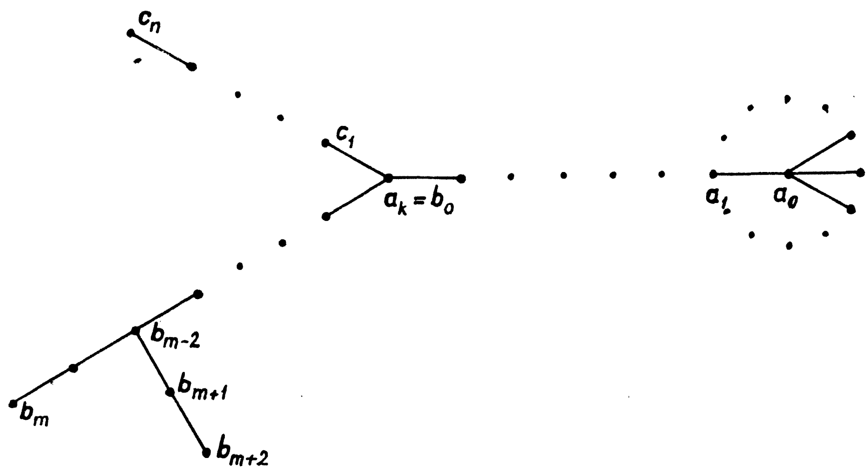
Remark. We denote by $Q : \boldsymbol{N} \to \boldsymbol{H}$ the I-functor with the object function $\text{Ob}\, \boldsymbol{N} \ni G \to G^2 \in \text{Ob}\, \boldsymbol{H}$. Then W is the least element in the subclass of $I(\boldsymbol{N}, \boldsymbol{H})$ consisting of the functors $F \leqq Q$.

5. Let $\boldsymbol{NT}$ be the I-category of all infinite Neuman trees, $\boldsymbol{HT}$ the I-category with all infinite Hamiltonian graphs as objects. In this part we shall describe a minimum element in $I(\boldsymbol{NT}, \boldsymbol{HT})$. Let k, l, m, n be natural numbers.

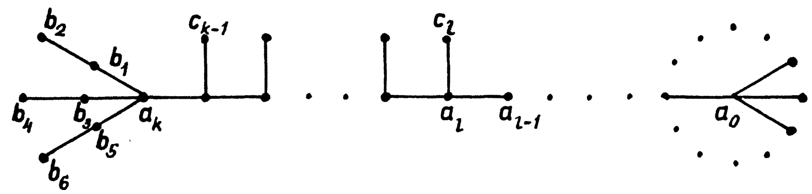
(4) An infinite graph G is called an (m, k)-star, if it is isomorphic to the graph



(5) G is called an (mT, n, k)-star $(m \geqq 2)$, if it is isomorphic to the graph



(6) G is an (lW, k)-star $(l < k)$, if it is isomorphic to the graph



An infinite graph G is called an $m - U$-graph [an $(mT, n) - U$-graph, an $lW - U$-graph, respectively] if G is isomorphic to the graph which we get from (4) [(5, (6)] by replacing the subgraph generated by a_0 and all vertices of degree 1 adjacent to a_0 by a one-way infinite path starting with a_0. All those graphs are objects of $\boldsymbol{NT}$.

Now we define for each $T \in \mathrm{Ob}\, \boldsymbol{NT}$ an infinite graph $N(T)$: $V(N(T)) = V(T)$; $(x, y) \in E(N(T))$ iff either $d_T(x, y) = 2$ and one of the following conditions (N1), ..., (N8) is satisfied or $(x, y) \in E(T)$.

(N1) There is an (m, k)-star $B \subseteqq T$ with $k \geqq 0$, $m \geqq 1$ such that $d_B(x, z) \geqq k$ and $d_B(y, z) \geqq k$ if $v_B(z)$ is infinite.

(N2) There is an (mT, n, k)-star $B \subseteqq T$ with $k \geqq 0$, $n \geqq m + 1 \geqq 4$ such that $d_B(x, z) \geqq k$ and $d_B(y, z) \geqq k$ if $v_B(z)$ is infinite.

(N3) There is a $(2T, 2, k)$-star $B \subseteqq T$ with $k \geqq 1$ such that $(x, z) \in E(B)$ and $(y, z) \in E(B)$ if $v_B(z) = 4$.

(N4) There is an (lW, k)-star $B \subseteqq T$ with $k \geqq 2$, $l \geqq 1$ such that $v_B(x) = 1$ or $v_B(y) = 1$.

(N5) There is an $m - U$-graph $B \subseteqq T$ with $m \geqq 1$ such that neither x nor y is a vertex of the one-way infinite path of B starting with the vertex of degree 3 in B.

(N6) There is an $(mT, n) - U$-graph $B \subseteqq T$ with $n \geqq m + 1 \geqq 4$ such that neither x nor y is a vertex of the one-way infinite path of B starting with the vertex of degree 3 in B.

(N7) There is a $(2T, 2) - U$-graph $B \subseteqq T$ with $(x, z) \in E(B)$ and $(y, z) \in E(B)$ if $v_B(z) = 4$.

(N8) There is an $lW - U$-graph $B \subseteqq T$ with $l \geqq 1$ such that $v_B(x) = 1$ or $v_B(y) = 1$.

We shall prove that $N(T)$ is a Hamiltonian graph.

Lemma 3. *Let $T^* \in \mathrm{Ob}\, \boldsymbol{NT}$, let $(T^*)'$ be a tree associated with T^* (according to 2, (c)), $T \subseteqq T^*$, $x \in V(T)$. Suppose there is a T-Hamiltonian path $w_T(x)$ of $N(T^*)$ starting with x. Moreover, let $B \subseteqq T^*$ arise from T by adding finitely many end-edges to given vertices u_i, $i = 1, \ldots, q$, under the following condition: There is a one-way infinite path $w(i)$ starting with u_i [there is a $(0, k)$-star $S(i)$, $k \geqq 1$, with $v_{S(i)}(u_i)$ infinite or $v_{S(i)}(u_i) = 1$] in T and there are consecutive vertices s_i, t_i in $w_T(x)$ with $d_T(s_i, u_i) = d_T(t_i, u_i) = 1$ such that none of s_i, t_i is a vertex of $w(i)$ [$S(i)$, respectively] if $v_{(T^*)'}(u_i) = 4$ and at least one of these vertices is not a vertex of $w(i)$ [$S(i)$] in the other cases.*

Then there is a B-Hamiltonian path $w_B(x)$ of $N(T^*)$ starting with x such that consecutive members y, z of $w_T(x)$ are also consecutive members of $w_B(x)$ if there is no u_i, $i = 1, \ldots, q$ with $d_T(u_i, y) = d_T(u_i, z) = 1$.

This lemma can be proved by induction on q.

Theorem 7. *If $T^* \in \mathrm{Ob}\, \boldsymbol{NT}$, then $N(T^*) \in \mathrm{Ob}\, \boldsymbol{HT}$.*

Proof. First, T^* is supposed to have a vertex c with infinite degree. Let Z be the set of all vertices of T^* with degree 1 in T^* adjacent to c; choose $z \in Z$. The tree $T^* - (Z - \{z\})$ will be denoted by B. Then Z is an infinite set, say $Z - \{z\} = \{z_1, z_2, \ldots\}$. By Theorem 2 there is a Hamiltonian path in T^{*2} starting with z and also a Hamiltonian path in B^2 with the endvertices z and some $y' \in V(B)$. Let B' be a tree associated with B according to 2(a). The sequence $w_{B'} = w_0 w_1 \ldots w_n$ can be proved to be a B'-Hamiltonian path in $N(T^*)$ starting with z. Applying the preceding lemma we get a B-Hamiltonian path w_B starting with z in $N(T^*)$; therefore $w^* = w_B z_1 z_2 \ldots$ is a Hamiltonian path in $N(T^*)$.

Now we suppose that T^* contains no vertex of infinite degree. Clearly $N(T^*) \in \mathrm{Ob}\, \boldsymbol{HT}$ if T^* is a one-way infinite path. We suppose, therefore, that T^* is not a one-way infinite path. If $(T^*)'$ is again a tree associated with T^* (see 2, (c)), we have a Hamiltonian $(T^*)'$-path of $N(T^*)$, namely $w_{(T^*)'} = w_0 w_1 w_2 \ldots$. We define subtrees $B_i \subseteqq T^*$, $i = 0, 1, 2, \ldots : B_0 = (T^*)'$; if $i \geqq 1$, let B_i be the subgraph of T^* generated by $V(B_{i-1})$ and all those vertices of T^* with degree 1 in T^* adjacent to one of the vertices of $\overline{w}(x_i)$ or $\overline{\overline{w}}(x_i)$. Then $w(0) = w_{(T^*)'}$ is a Hamiltonian B_0-path of $N(T^*)$; if $w(i - 1)$ for $i \geqq 1$ is already defined as a B_{i-1}-Hamiltonian path of $N(T^*)$, let $w(i)$ be a B_i-Hamiltonian path of $N(T^*)$ arising from $w(i - 1)$ by using the preceding lemma. Consequently: If y, z are vertices belonging to $\overline{w}(x_{i-1})$ or to $\overline{\overline{w}}(x_{i-1})$ or adjacent to a vertex of these paths different from x_i and if y, z are subsequent in $w(i)$, then they remain subsequent in all $w(j)$ with $j \geqq i$. Therefore the following sequence w^* is a Hamiltonian path in $N(T^*)$: w^* starts with the first vertex of $w(0)$; y, z are subsequent in w^* iff there is an $l \geqq 0$ such that y, z are subsequent in $w(j)$ for all $j \geqq l$.

Thus in all cases we have demonstrated the existence of a Hamiltonian path in $N(T^*)$. Q.e.d.

Now we define a set M of graphs: $G \in M$ iff G is an (m, k)-star or an $m - U$-graph $(m \geqq 1, k \geqq 0)$ or G is an (mT, n, k)-star or an $(mT, n) - U$-graph $(k \geqq 0, n \geqq m + 1 \geqq 4)$ or G is a $(2T, 2, k)$-star $(k \geqq 1)$ or G is a $(2T, 2) - U$-graph or G is an (lW, k)-star or an $lW - U$-graph $(l \geqq 1, k \geqq 2)$. Then $M \subseteqq$ Ob $\boldsymbol{NT}$ and by Theorem 3 we have a unique I-functor on $\boldsymbol{NT}$ into $\boldsymbol{HT}$ with the object function Ob $\boldsymbol{NT} \ni T \to N(T) \in$ Ob $\boldsymbol{HT}$. We denote this functor by N.

In the following considerations let F be an I-functor on $\boldsymbol{NT}$ into $\boldsymbol{HT}$ with $F \leqq N$.

Lemma 4. *Let $k \geqq 0$, $m \geqq 1$, and let B be an (m, k)-star [or an $m - U$-graph]. Then $E(N(B)) \subseteqq E(F(B))$.*

Proof. We proceed by induction on m. Suppose that the lemma is proved for all (l, k)-stars [$l - U$-graphs, respectively] with $l < m$ and let B be an (m, k)-star [an $m - U$-graph, respectively]. The vertices of B may be denoted as in (4). If $m = 1$ and (b_1, c_1) were not an edge of $F(B)$, we should have $v_{F(B)}(b_1) = v_{F(B)}(c_1) = 1$ because $F \leqq N$. But this is a contradiction to $F(B) \in$ Ob $\boldsymbol{HT}$; therefore we get $(b_1, c_1) \in E(F(B))$ and thus $E(N(B)) \subseteqq E(F(B))$.

Now let $m > 1$ and let (x, y) be an edge of $N(B)$. As $(x, y) \in E(F(B))$ if $(x, y) \in E(B)$ is evident, we assume $d_T(x, y) = 2$. Then there is an l with $1 \leqq l \leqq m$ such that $(x, y) = (b_l, b_{l-2})$ or $(x, y) = (c_l, c_{l-2})$ where we define $b_{-1} = c_1$, $c_{-1} = b_1$ and $b_0 = c_0 = a_k$. If $l < m$ we have an (l, k)-star [an $l - U$-graph, respectively] B' and a morphism $(f, B', B) \in$ Mor $\boldsymbol{NT}$ with $(f(b'_l), f(b'_{l-2})) = (x, y)$. By the induction assumptions and lemma 1 we get $(x, y) \in E(F(B))$. If $l = m$ it follows analogously to the case $m = 1$ that either (b_m, b_{m-2}) or (c_m, c_{m-2}) has to be an edge of $F(B)$. Because of the existence of an isomorphism $(g, B, B) \in$ Mor $\boldsymbol{NT}$ on B onto B with $g(b_m) = c_m$ and $g(b_{m-2}) = c_{m-2}$ we get again by lemma 1 $(b_m, b_{m-2}) \in E(F(B))$ and $(c_m, c_{m-2}) \in E(F(B))$. Thus in all cases $(x, y) \in E(F(B))$ is proved; hence $E(N(B)) \subseteqq E(F(B))$.

Lemma 5. *Let $k \geqq 0$, $n \geqq m + 1 \geqq 4$ and let B be an (mT, n, k)-star [or an $(mT, n) - U$-graph]. Then $E(N(B)) \subseteqq E(F(B))$.*

Proof. We denote the vertices of B as in (5). If $(c_n, c_{n-2}) \notin E(F(B))$, then $F \leqq N$ implies $v_{F(B)}(c_n) = 1$. But this is impossible because the square of the subgraph of B generated by $\{b_{m-4}, \ldots, b_{m+2}\}$ (if $m = 3$, define $b_{-1} = c_1$) is not Hamiltonian. Therefore (c_n, c_{n-2}) has to be an edge of $F(B)$. By lemma 4, lemma 1 and by induction on n it follows easily that $E(N(B)) \subseteqq E(F(B))$. Q.e.d.

Lemma 6. *Let $k \geqq 1$ and B be a $(2T, 2, k)$-star $[$or a $(2T, 2) - U$-graph$]$. Then $E(N(B)) \subseteqq E(F(B))$.*

Proof. We label the vertices of B as in (5). One of the pairs (c_1, a_{k-1}), (b_1, a_{k-1}), (b_3, a_{k-1}) has to be an edge of $F(B)$, otherwise $F(B) \notin \mathrm{Ob}\, \boldsymbol{HT}$. By the structure of isomorphisms on B onto B and lemma 1 it follows that all these pairs are edges of $F(B)$. Q.e.d.

Lemma 7. *Let $k \geqq 2$, $l \geqq 1$ and let B be an (lW, k)-star $[$or an $lW - U$-graph$]$. Then $E(N(B)) \subseteqq E(F(B))$.*

Proof. Suppose the vertices of B are denoted as in (6). The Hamiltonian path in $F(B)$ has to start with one of the vertices b_2, b_4, b_6. Therefore each of the pairs (a_j, c_{j+1}), $j = l - 1, \ldots, k - 2$, has to be an edge of $F(B)$. Using lemma 4 and lemma 1 we get $E(N(B)) \subseteqq E(F(B))$. Q.e.d.

Theorem 8. *N is a minimum element in $I(\boldsymbol{NT}, \boldsymbol{HT})$.*

Proof. Let $F \in I(\boldsymbol{NT}, \boldsymbol{HT})$ with $F \leqq N$. Choosing $M \subseteqq \mathrm{Ob}\, \boldsymbol{NT}$ as before lemma 4 and using the preceding lemmas and Theorem 4 we get immediately $N \leqq F$. Q.e.d.

Remarks 1. Let $B \in \mathrm{Ob}\, \boldsymbol{NT}$. Then for each edge (x, y) of $N(B)$ there is a finite subgraph B' of B such that (x, y) is an edge of $W(B')$.

2. An analogous statement as after Theorem 6 is valid.

6. By the same methods one can prove: The I-functor with the object function $T \to T^2$ is a minimum element in the class of I-functors on the I-category $\boldsymbol{NST}$ of the infinite trees with strong-Hamiltonian squares into the I-category of strong-Hamiltonian infinite graphs. $T \in \mathrm{Ob}\, \boldsymbol{NST}$ iff the tree obtained from T by deleting all vertices of degree 1 is a finite path or a one-way infinite path.

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