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Renata Burgetová; Dalibor Klucký

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THE SPECTRUM OF A CARTESIAN PRODUCT OF PLURAL ALGEBRAS

Renata Burgetová, Dalibor Klucký, Olomouc

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The ideals of cartesian products of s.c. plural algebras are studied. First, it is shown how the study of these ideals can be reduced to the investigation of ideals of cartesian products of fields. On the basis of this fact, it is proved that every prime ideal of a cartesian product of plural algebras is maximal. Finally, it is shown that the spectrum of such a ring depends only on the index set.

By a plural algebra of order n (n a positive integer) over a field F we mean an F – algebra having an F – basis

$$(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1})$$

where the element ε satisfies the condition $\varepsilon^n = 0$. We denote such an algebra by $F^{[n]}\langle\varepsilon\rangle$.

The algebra $F^{[n]}\langle\varepsilon\rangle$ evidently is a homomorphic map of the polynomial ring $F[X]$ by the F – homomorphism sending X into ε. The kernel of this homomorphism is the ideal (X^n). This remark immediately implies

(a) For any $\xi \in F^{[n]}\langle\varepsilon\rangle$ there exists a uniquely determined n – tuple $(x_0, x_1, \ldots \ldots, x_{n-1}) \in F^n$ such that

$$\xi = x_0 + x_1\varepsilon + \ldots + x_{n-1}\varepsilon^{n-1} . \tag{1}$$

(b) $F^{[n]}\langle\varepsilon\rangle$ has only the ideals $(1), (\varepsilon), \ldots, (\varepsilon^{n-1}), (0)$ for which obviously $(1) \supset \supset (\varepsilon) \supset \ldots \supset (\varepsilon^{n-1}) \supset (0)$ is true.

(c) (ε) is the unique maximal ideal in $F^{[n]}\langle\varepsilon\rangle$, therefore $F^{[n]}\langle\varepsilon\rangle$ is a local ring.

(d) The element $\xi \in F^{[n]}\langle\varepsilon\rangle$ expressed by (1) is a unit of $F^{[n]}\langle\varepsilon\rangle$ if and only if $x_0 \neq 0$.

1. Ideals in a cartesian product of plural algebras. Let a system $\mathsf{S} = \{F_i\}_{i\in I}$ of fields F_i as well as a system $\mathsf{M} = \{F_i^{[n]}\langle\varepsilon\rangle\}$ of plural algebras of the same order n be given. Put

$$\mathbf{S} = \mathop{\mathbf{X}}_{i\in I} F_i\,, \quad \mathbf{M} = \mathop{\mathbf{X}}_{i\in I} F_i^{[n]}\langle\varepsilon\rangle\,. \tag{2}$$

We will consider **S** and **M** as rings in the usual sense, **M** will be considered also as an **S** – algebra. Denote by $\bar{\varepsilon}$ the element of **M** whose all projections $\mathrm{pr}_i\,\bar{\varepsilon}$ are equal to ε. Therefore, for any $\alpha \in \mathbf{M}$ there exists just one n – tuple $(a_0, a_1, \ldots, a_{n-1}) \in \mathbf{S}^n$ such that

$$\alpha = a_0 + a_1\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-1} . \tag{3}$$

This means that **M** is a free module over **S** generated by the set $\{1, \bar{\varepsilon}, \ldots, \bar{\varepsilon}^{n-1}\}$. If (3) is true, we will write

$$a_j = \pi_j(\alpha) .$$

Let $\mathfrak{A}$ be an ideal in **M**. For any $j = 0, 1, \ldots, n-1$, let $\mathfrak{A}_j$ denote the set

$$\{a \in \mathbf{S} \mid \exists\alpha \in \mathfrak{A} : a = \pi_j(\alpha)\} .$$

Clearly, $\mathfrak{A}_j$ is an ideal in **S**.

Lemma. *Let $\mathfrak{A}$ be an ideal in **M**. Then for any $j = 0, 1, \ldots, n-1$, $a_j \in \mathfrak{A}_j$ implies $a_j\bar{\varepsilon}^j \in \mathfrak{A}$.*

We prove our lemma by induction for j. First, suppose $j = 0$. Let $a_0 \in \mathfrak{A}_0$. Then there exist $a_1, \ldots, a_{n-1} \in \mathbf{S}$ for which the element (3) belongs to $\mathfrak{A}$. Considering the projections of α on the components of **M**, we see that

$$\mathrm{pr}_i\,\alpha = \mathrm{pr}_i\,a_0 + \mathrm{pr}_i\,a_1\varepsilon + \ldots + \mathrm{pr}_i\,a_{n-1}\varepsilon^{n-1}$$

(for arbitrary $i \in I$). We shall prove that $\mathrm{pr}_i\,a_0$ is divisible by $\mathrm{pr}_i\,\alpha$ in $F_i^{[n]}\langle\varepsilon\rangle$. This is clear if $\mathrm{pr}_i\,a_0 = 0$. But it the opposite case, $\mathrm{pr}_i\,\alpha$ is a unit of $F_i^{[n]}\langle\varepsilon\rangle$ and therefore also a divisor of $\mathrm{pr}_i\,a_0$. This implies that a_0 is a multiple of α in **M**, hence $a_0 \in \mathfrak{A}$.

Further, suppose that an integer j, $1 \leqq j \leqq n-1$, is given and that the lemma holds for any $k = 0, 1, \ldots, j-1$. Let $a_j \in \mathfrak{A}_j$. Similarly as in the first step, there exist $a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n-1} \in \mathbf{S}$ such that the element

$$\alpha = a_0 + a_1\bar{\varepsilon} + \ldots + a_{j-1}\bar{\varepsilon}^{j-1} + a_j\bar{\varepsilon}^j + a_{j+1}\bar{\varepsilon}^{j+1} + \ldots + a_{n-1}\bar{\varepsilon}^{n-1}$$

belongs to $\mathfrak{A}$. Now, by induction we conclude

$$a_0 + a_1\bar{\varepsilon} + \ldots + a_{j-1}\bar{\varepsilon}^{j-1} \in \mathfrak{A}$$

and hence

$$\beta = (a_j + a_{j+1}\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-j-1})\,\bar{\varepsilon}^j$$

belongs also to $\mathfrak{A}$. In the same way as in the first step we obtain that $\mathrm{pr}_i\,a_j$ for any $i \in I$ is divisible by $\mathrm{pr}_i\,a_j + \mathrm{pr}_i\,a_{j+1}\varepsilon + \ldots + \mathrm{pr}_i\,a_{n-1}\varepsilon^{n-j-1}$ and therefore a_j is a multiple of $a_j + a_{j+1}\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-j-1}$ in **M**. Thus $a_j\bar{\varepsilon}^j$ is a multiple of β, hence $a_j\bar{\varepsilon}^j \in \mathfrak{A}$.

The just proved lemma yields the following

Proposition 1. *Let $\mathfrak{A}$ be an ideal in $\mathbf{M}$,*

$$\alpha = a_0 + a_1\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-1}$$

an arbitrary element of $\mathbf{M}$. Then $\alpha \in \mathfrak{A}$ if and only if $(a_0, a_1, \ldots, a_{n-1}) \in \mathfrak{A}_0 \times \times \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_{n-1}$.

Corollary. *Every ideal $\mathfrak{A}$ in $\mathbf{M}$ can be expressed uniquely in the form*

$$\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1\bar{\varepsilon} \oplus \ldots \oplus \mathfrak{A}_{n-1}\bar{\varepsilon}^{n-1} \tag{4}$$

where $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}$ are ideals in $\mathbf{S}$.

Theorem 1. *Let $\mathfrak{A}$ be an ideal in the ring $\mathbf{M} = \underset{i\in I}{\mathbf{X}} F^{[n]}\langle\varepsilon\rangle$. Let $\mathfrak{A}$ be expressed in the form (4). Then $\mathfrak{A}$ is a prime ideal if and only if $\mathfrak{A}_0$ is a prime ideal in $\mathbf{S}$ and $\mathfrak{A}_1 = \ldots = \mathfrak{A}_{n-1} = \mathbf{S}$ $(\mathbf{S} = \underset{i\in I}{\mathbf{X}} F_i)$.*

Proof. Suppose that $\mathfrak{A}$ is a prime ideal (in $\mathbf{M}$). Let there exist some $j = 1, \ldots, n - 1$ such that $\mathfrak{A}_j \neq \mathbf{S}$. Choose an element a_j of $\mathbf{S} \setminus \mathfrak{A}_j$. Therefore $a_j\bar{\varepsilon}^j \notin \mathfrak{A}$. However, for a sufficiently large positive integer ϱ we get $(a_j\bar{\varepsilon}^j)^\varrho = 0 \in \mathfrak{A}$ which is is a contradiction. Let $a_0, b_0 \in \mathbf{S}$ and let $a_0b_0 \in \mathfrak{A}_0$. By Proposition 1, $a_0b_0 \in \mathfrak{A} \Rightarrow \Rightarrow a_0 \in \mathfrak{A}$ or $b_0 \in \mathfrak{A} \Rightarrow a_0 \in \mathfrak{A}_0$ or $b_0 \in \mathfrak{A}_0$.

Conversely, let $\mathfrak{A}_0$ be a prime ideal in $\mathbf{S}$ and let $\mathfrak{A}_1 = \ldots = \mathfrak{A}_{n-1} = \mathbf{S}$. Finally, let

$$\alpha = a_0 + a_1\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-1},$$

$$\beta = b_0 + b_1\bar{\varepsilon} + \ldots + b_{n-1}\bar{\varepsilon}^{n-1}.$$

For any $j = 1, \ldots, n - 1$ the element a_j belongs to $\mathfrak{A}_j$. Thus $\alpha \in \mathfrak{A}$ if and only if $a_0 \in \mathfrak{A}$ and similarly for any other element of $\mathbf{M}$. Since

$$\alpha\beta = a_0b_0 + (a_0b_1 + a_1b_0)\,\bar{\varepsilon} + \ldots + (a_0b_{n-1} + \ldots + a_{n-1}b_0)\,\bar{\varepsilon}^{n-1},$$

we get

$$\alpha\beta \in \mathfrak{A} \Leftrightarrow a_0b_0 \in \mathfrak{A}_0 \Leftrightarrow a_0 \in \mathfrak{A}_0 \quad \text{or} \quad b_0 \in \mathfrak{A}_0 \Leftrightarrow \alpha \in \mathfrak{A} \quad \text{or} \quad \beta \in \mathfrak{A}.$$

2. The spectra of $\underset{i\in I}{\mathbf{X}} F_i^{[n]}\langle\varepsilon\rangle$ and of $\underset{i\in I}{\mathbf{X}} F_i$. Theorem 1 makes it possibile to reduce the study of Spec $(\underset{i\in I}{\mathbf{X}} F_i^{[n]}\langle\varepsilon\rangle)$ to the study of Spec $(\underset{i\in I}{\mathbf{X}} F_i)$. We leave without change the notation $\mathbf{M}$ and $\mathbf{S}$ of the rings $\underset{i\in I}{\mathbf{X}} F_i^{[n]}\langle\varepsilon\rangle$ and $\underset{i\in I}{\mathbf{X}} F_i$, respectively.

Theorem 2. *The spectrum of $\underset{i\in I}{\mathbf{X}} F_i$ is the maximal spectrum.*

Proof. Let $\mathfrak{p}$ be an arbitrary prime ideal in $\mathbf{S}$. Let a be an element of $\mathbf{S} \setminus \mathfrak{p}$. Define the elements b and x of $\mathbf{S}$ by the conditions

$$\mathrm{pr}_i\, b = 1 \Leftrightarrow \mathrm{pr}_i\, a = 0\,; \qquad \mathrm{pr}_i\, b = 0 \Leftrightarrow \mathrm{pr}_i\, 0 \neq 0\,,$$

$$\mathrm{pr}_i\, x = (\mathrm{pr}_i\, a)^{-1} \Leftrightarrow \mathrm{pr}_i\, a \neq 0\,; \quad \mathrm{pr}_i\, x = 0 \Leftrightarrow \mathrm{pr}_i\, a = 0\,.$$

It is easy to verify that $ax + b = 1$ (1 denotes here the unit element of $\mathbf{S}$), hence $\mathfrak{p}$ is maximal.

Corollary. *The spectrum of* $\mathop{\mathbf{X}}\limits_{i\in I} F_i^{[n]}\langle\varepsilon\rangle$ *is the maximal spectrum.*

Remark. The connection between $\mathrm{Spec}\,(\mathbf{M})$ and $\mathrm{Spec}\,(\mathbf{S})$ can be characterized also by following way: Let φ be the $\mathbf{S}$ – epimorphism $\mathbf{M} \to \mathbf{S}$ carrying $\bar{\varepsilon}$ into 0. The kernel of φ is just the ideal $(\bar{\varepsilon})$ – nilradical of $\mathbf{M}$. The φ induces a mapping ${}^a\varphi : \mathrm{Spec}\,(\mathbf{S}) \to \mathrm{Spec}\,(\mathbf{M})$ given by ${}^a\varphi(\mathfrak{A}_0) = \mathfrak{A} \Leftrightarrow \varphi^{-1}(\mathfrak{A}_0) = \mathfrak{A}$. (see f.e. I. G. Macdonald: Algebraic Geometry (Introduction to Schemes), W. A. Benjamin, inc., New York, Amsterdam, 1968, pg. 24–25). Moreover, ${}^a\varphi$ is a continuous mapping with respect to spectral topology. As $\mathbf{M}/(\bar{\varepsilon})$ is $\mathbf{S}$ – isomorphic to $\mathbf{S}$ and the $\mathrm{Spec}\,(\mathbf{M}/(\bar{\varepsilon}))$ and $\mathrm{Spec}\,(\mathbf{M})$ are the same, we conclude, that $\mathrm{Spec}\,(\mathbf{M})$ is homeomorphic to $\mathrm{Spec}\,(\mathbf{S})$. Since the ideal $\mathfrak{A}$ in $\mathbf{M}$ is a prime ideal if and only if there exists a (uniquely determined) prime ideal $\mathfrak{A}_0$ in $\mathbf{S}$ such that $\mathfrak{A} = {}^a\varphi(\mathfrak{A}_0)$ ($\Leftrightarrow \mathfrak{A} = \varphi^{-1}(\mathfrak{A}_0)$), then

$$\mathfrak{A} = \{a_0 + a_1\bar{\varepsilon} + \ldots + a_{n-1}\bar{\varepsilon}^{n-1} \in \mathbf{M} \mid a_0 \in \mathfrak{A}_0\}\,.$$

3. The connection between ideals in $\mathop{\mathbf{X}}\limits_{i\in I} F_i$ and filters on I. In this part, we will show that $\mathrm{Spec}\,(\mathop{\mathbf{X}}\limits_{i\in I} F_i)$ and consequently also $\mathrm{Spec}\,(\mathop{\mathbf{X}}\limits_{i\in I} F_i^{[n]}\langle\varepsilon\rangle)$ depends only on the index set I.

For any $x \in \mathbf{S} = \mathop{\mathbf{X}}\limits_{i\in I} F_i$ we define

$$x(0) = \{i \in I \mid \mathrm{pr}_i\, x = 0\}\,,$$

$$x(1) = \{i \in I \mid \mathrm{pr}_i\, x \neq 0\}\,.$$

Evidently, $x(1)$ is the complement of $x(0)$ in the set I.

Proposition 2. (i) *For any* $a, b \in \mathbf{S}$, $b(0) \subset a(0)$ *if and only if there exists* $c \in \mathbf{S}$ *such that* $b = c\,.\,a$.

(ii) *For any* $a, b \in \mathbf{S}$, $(ab)(0) - a(0) \cup b(0)$.

(iii) *For any* $a, b \in \mathbf{S}$, $(a + b)(0) \supset a(0) \cap b(0)$.

(iv) *For any* $a, b \in \mathbf{S}$ *there exists an* $x \in \mathbf{S}$, *for which*

$$a(0) \cap b(0) = (a + xb)(0)\,.$$

Proof. Only (iv) is not trivial. We define $x \in \mathbf{S}$ as follows:

$$\mathrm{pr}_i\, x = 1 \quad \text{if} \quad i \in a(0)\,; \quad \mathrm{pr}_i\, x = 0 \quad \text{if} \quad i \in a(1)\,.$$

The verification of the equality $a(0) \cap b(0) = (a + xb)(0)$ is now quite routine.

Now, to every ideal $\mathfrak{A}$ in $\mathbf{S}$ we assign the subset $\Phi(\mathfrak{A})$ of $\exp I$, given by

$$\Phi(\mathfrak{A}) = \{X \in \exp I \mid \exists x \in \mathfrak{A} : X = x(0)\} . \tag{5}$$

It follows without trouble from Proposition 2 that Φ is a $1-1$ correspondence between the lattice of ideals in and the lattice of filters on the index set I and that this correspondence preserves the inclusion relation, i.e. it is an isomorphism of both lattices. According to Theorem 2 we get the following result:

Theorem 3. *The ideal $\mathfrak{A}$ in the ring $\mathbf{X}_{i \in I} F_i$ is a prime ideal if and only if the set (5) is an ultrafilter on the index set I.*

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Author's address: 771 46 Olomouc, Leninova 26 (Přírodovědecká fakulta UP).