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ON THE RANGE OF VALUES OF THE SUM OF A CONTINUOUS AND A DARBOUX FUNCTIONS

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The following assertion will be used in the proof of Theorem 1.

Assertion. *There exist real functions g, h defined on E_1 with the property: For any interval $J \subset E_1$ and $y \in E_1$,*

$$h(g^{-1}(y) \cap J) = E_1 . \tag{1}$$

The proof of this assertion can be found in [1].

Theorem 1. *Let the set $M \subset E_1$ be at most countable. Let f be a continuous function defined in an interval $I \subset E_1$ and such that*

$$f^{-1}(y) \quad \text{is at most countable for any} \quad y \in E_1 . \tag{2}$$

Then there exists a function d defined in I with the properties:

(i) *$d(J) = E_1$ for any interval $J \subset I$;*

(ii) *$(f + d)(I) \subset E_1 - M$;*

(iii) *$f + d$ is unbounded both from above and from below on any interval $J \subset I$.*

Proof. Let $k \in E_1 - M$. Let us define:

$$d(x) = \begin{cases} -f(x) + k & \text{in case } x \in I \text{ and there exists } m \in M \text{ such that } g(x) = -f(x) + m; \\ g(x) & \text{for the other } x \in I. \end{cases} \tag{3}$$

g is the function from (1).

Let $J \subset I$, $y \in E_1$. According to (1), the set $g^{-1}(y) \cap J$ is uncountable. Let us define:

$$A_m = \{z; z \in g^{-1}(y) \cap J \,\&\, f(z) = m - y\} \tag{4}$$

for any $m \in M$. A_m is at most countable according to (2), therefore $A = \bigcup_{m \in M} A_m$ is also at most countable. That is why $g^{-1}(y) \cap J - A \neq \emptyset$ and (according to (3), (4)) if $t \in g^{-1}(y) \cap J - A$, then $d(t) = y$. This completes the proof of (i).

The following equality holds according to (3):

$$f(x) + d(x) = \begin{cases} k & \text{in case } x \in I \text{ and there exists } m \in M \text{ such that } g(x) = -f(x) + m; \\ f(x) + g(x) & \text{in case } x \in I \text{ and } g(x) \neq -f(x) + m \text{ for any } m \in M. \end{cases}$$

Hence the validity of the condition (ii) can be easily seen.

The validity of the condition (iii) is evident.

Theorem 2. *Let the set $M \subset E_1$ be nowhere dense in E_1. Let f be a continuous function in an interval $I \subset E_1$ such that*

(5) $$f \text{ is not constant in any interval } J \subset I\,.$$

Then there exists a function d defined in I having the following properties:

(i) *$d(J) = E_1$ for any interval $J \subset I$;*

(ii) *$(f + d)(I) \subset E_1 - M$;*

(iii) *$f + d$ is unbounded both from above and from below on any interval $J \subset I$.*

Proof. Let us define the function d by the formula (3). Let $J \subset I$, $y \in E_1$. It follows from (1) that

(6) $$g^{-1}(y) \cap J \quad \text{is dense in } J.$$

The function F defined by the formula

(7) $$F(x) = f(x) + y \quad \text{for any} \quad x \in I$$

is continuous in J. According to (5) there exist two points $x_1 < x_2$, $x_1, x_2 \in J$ such that $F(x_1) \neq F(x_2)$. Let (y', y'') be the open interval with the end points $F(x_1)$, $F(x_2)$. Since M is nowhere dense in E_1, there exists an interval $(w', w'') \subset (y', y'')$ such that $(w', w'') \cap M = \emptyset$. Since (6) holds and the function F is continuous, the set $g^{-1}(y) \cap \cap J \cap F^{-1}(w', w'')$ is non-empty. Let t be an element of this set. Then $d(t) = y$ according to (3) and (7). This completes the proof of (i).

The proof of the conditions (ii) and (iii) is similar to that of the conditions (ii) and (iii) of Theorem 1.

Remark 1. The set of all rationals fulfils the assumptions of Theorem 1 but does not fulfil the assumptions of Theorem 2. Cantor set fulfils the assumptions of Theorem

2 but does not fulfil the assumptions of Theorem 1 with respect to the set M. Moreover

$$(8) \qquad (2) \Rightarrow (5)$$

but there exists a continuous function fulfilling (5) and not fulfilling (2).

Remark 2. If in the assumptions of Theorem 1 the set M is not countable but $M = M_1 \cup M_2$ holds, $M_1 \subset E_1$ being at most countable and $M_2 \subset E_1$ nowhere dense in E_1, Theorem 1 holds again.

To prove this assertion, we define the function d again by (3) and a suitable combination of the corresponding parts of the proofs of Theorem 1 and 2. The proof of the conditions (ii) and (iii) is identical with the proof of the conditions (ii) and (iii) of Theorem 1.

Remark 3. The function g in the proof of Theorem 2 can be replaced by any function d' with the property: $d'(J) = E_1$ for any $J \subset I$.

References

[1] *H. Fast:* Une remarque sur la propriété de Weierstrass, Colloquium Mathematicum 7 (1959), p. 75—77.

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