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REMARK ON THE THEOREM OF EGOROFF

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I. VRKOČ in [1] has proved the following theorem: *there exists a real function f defined and measurable on $[0, 1]$ such that there does not exist a countable family $\{A_n\}$ of sets fulfilling $\bigcup_n A_n = [0, 1]$ such that the restricted function $f \mid A_n$ is continuous for every n.* The theorem of Vrkoč is a refinement of the well known theorem of Lusin. In this short note we shall prove the theorem which can be considered as a similar refinement of the theorem of Egoroff.

Before stating the theorem we shall prove the following lemma:

Lemma. *Let $\{n(k, i)\}$ be a double sequence of natural numbers, which for every k is increasing with respect to the variable i. There exists an increasing sequence $\{n(i)\}$ of natural numbers such that for every k and for every $i \geqq k$*

$$n(i) > n(k, i).$$

Proof. Put $n(i) = 1 + \max(n(1, i), n(2, i), \ldots, n(i, i))$ for every natural i. It is easy to see that the sequence $\{n(i)\}$ fulfills all required conditions.

Theorem. *For every set A of the power of continuum there exists a sequence of real functions $\{f_n\}$ defined on A such that $f_n(x)$ tends to zero for every $x \in A$ and there does not exist a countable family $\{A_k\}$ of sets fulfilling $\bigcup_k A_k = A$ such that the restricted sequence $\{f_n \mid A_k\}$ is uniformly convergent for every k.*

Proof. Let N be a set of all increasing sequences of natural numbers. Of course, N is a set of the power of continuum. Let $\Phi : A \rightarrow_{\text{onto}} N$ be a one-to-one correspondence.

For $x \in A$ let us put $f_{n(1)}(x) = 1^{-1}$, $f_{n(2)}(x) = 2^{-1}, \ldots, f_{n(i)}(x) = i^{-1}, \ldots$ and $f_j(x) = 0$ for remaining natural j, where $\{n(1), n(2), \ldots, n(i), \ldots\} = \Phi(x)$.

So we have defined a sequence of real functions $\{f_n\}$ and it is easy to verify that $f_n(x) \rightarrow 0$ for every $x \in A$.

Suppose that there exists a sequence $\{A_k\}$ of sets such that $\bigcup_k A_k = A$ and $\{f_n \mid A_k\}$ tends uniformly to 0 for every k.

Let for fixed k the sequence $\{n(k, i)\}$ of variable i be a sequence of natural numbers corresponding to $\varepsilon = 1^{-1}, \varepsilon = 2^{-1}, \ldots, \varepsilon = i^{-1}, \ldots$ and to uniform convergence of $\{f_n\}$ on A_k, i.e. for every i, for every $j > n(k, i)$ and for every $x \in A_k$ we have $|f_j(x)| < i^{-1}$. Obviously we can choose $\{n(k, i)\}$ to be increasing with respect to i. If k changes in the set of natural numbers, we obtain a double sequence $\{n(k, i)\}$. In virtue of the lemma there exists an increasing sequence $\{n(i)\}$ such that for every k and for every $i \geqq k$ $n(i) > n(k, i)$. Let $x = \Phi^{-1}(\{n(i)\})$. There exists a natural number k_0 such that $x \in A_{k_0}$. So for $i \geqq k_0$ we have $n(i) > n(k_0, i)$ and $|f_{n(i)}(x)| < i^{-1}$ and simultaneously from the definition we have $f_{n(i)}(x) = i^{-1}$, a contradiction. The theorem is proved.

Corollary. *There exists a sequence of measurable real functions $\{f_n\}$ defined on $[0, 1]$, which tends to zero at every point and such that there does not exist a sequence $\{A_k\}$ of sets fulfilling $\bigcup_k A_k = [0, 1]$ such that the restricted sequence $\{f_n \mid A_k\}$ is uniformly convergent for every k.*

Proof. It suffices to take in the theorem the set $A \subset [0, 1]$ of the power of continuum and of measure zero and to define additionally $f_n(x) = 0$ for every n and for every $x \notin A$. Then we obtain a sequence of functions which are equal almost everywhere to zero and hence measurable.

References

[1] *Vrkoč, Ivo:* Remark about the relation between measurable and continuous functions, Čas. pro pěst. mat. *96* (1971) p. 225—228.

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