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ON THE MONOTONICITY OF SOLUTIONS OF CERTAIN TYPES OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Let $\mathbb{R}$ denote the set of all real numbers and let $I \subset \mathbb{R}$ be an interval. By $C^1(I, \mathbb{R})$ we denote the set of all continuously differentiable real functions defined on I.

We shall consider the functional-differential equation of the first order

$$x'(t) = f(t, x), \tag{1}$$

where

$$f : D \to \mathbb{R}$$

is a given function and D is a subset of the set $I \times C^1(I, \mathbb{R})$.

We will assume that

$$\underset{\substack{(t_1, x) \in D \\ (t_2, x) \in D}}{\forall} \left[x(t_1) = x(t_2) \Rightarrow f(t_1, x) \,.\, f(t_2, x) \geqq 0\right]. \tag{2}$$

(For special cases cf. Corollary 2.)

A function $x \in C^1(I, \mathbb{R})$ is a solution of (1) iff (1) holds for every $t \in I$.

We shall prove the following lemma:

Lemma 1. *Suppose that* $-\infty < a < b < \infty$. *If* $y \in C^1([a, b], \mathbb{R})$ *and*

$$y(a) < y(b)$$

then for every $\bar{y} \in (y(a), y(b))$ *there exists* $c \in (a, b)$ *such that*

$$y(c) = \bar{y}, \quad y'(c) \geqq 0. \tag{3}$$

Proof. Consider the set

$$T := \{t \in [a, b] : y(t) = \bar{y}\}.$$

It follows from the continuity of the function y that T is a non-empty and compact set. Denote

$$t_m = \inf T. \tag{4}$$

The inequality $t_m > a$ holds because $y(t_m) = \bar{y} \neq y(a)$. If for a certain $c \in T$ we have $y'(c) \geqq 0$, then condition (3) is fulfilled. Suppose that the inequality $y'(t) \geqq 0$ does not hold for any $t \in T$. Hence, in particular, $y'(t_m) < 0$ and this yields the existence of a positive number ε such that

$$t_m - \varepsilon \in (a, b)$$

and

$$y(t_m - \varepsilon) > y(t_m).$$

Thus

$$y(a) < \bar{y} = y(t_m) < y(t_m - \varepsilon).$$

Hence and from the continuity of y we infer that there exists an $s \in (a, t_m - \varepsilon)$ such that

$$y(s) = \bar{y}.$$

In particular, $s \in T$ and $s < t_m$. So we have obtained a contradiction with (4).

Lemma 2. *If $x \in C^1(I, \mathbb{R})$ is not a monotonic function then there exist points $t_1, t_2 \in I$ such that*

$$x(t_1) = x(t_2) \tag{5}$$

and

$$x'(t_1) \,.\, x'(t_2) < 0. \tag{6}$$

Proof. Since $x \in C^1(I, \mathbb{R})$ is not a monotonic function it follows that for some $u, v \in I$ such that $u < v$, the inequalities

$$x'(u) > 0, \quad x'(v) < 0 \tag{7a}$$

or

$$x'(u) < 0, \quad x'(v) > 0 \tag{7b}$$

are satisfied. We shall confine ourselves to the case (7a). (In the case (7b) it is enough to consider the function $-x$.)

The function $x|_{[u,v]}$ assumes its maximum at a point w. It follows from the condition (7a) that $w \in (u, v)$ and

$$x(u) < x(w), \quad x(v) < x(w).$$

Consider the case

$$x(u) < x(v). \tag{8}$$

(In the case of equality,

$$x(u) = x(v),$$

putting $t_1 = u$, $t_2 = v$ in (7a) we obtain (6).

If

$$x(u) > x(v)$$

we argue similarly as in the case (8).

Putting

$$\bar{y} = y(v), \quad a = u, \quad b = v$$

and applying Lemma 1 we obtain the existence of a point $c \in (u, v)$ such that

$$x(c) = x(v) \quad \text{and} \quad x'(c) \geqq 0. \tag{9}$$

Now we shall prove that there are points t_1, t_2 in the interval $[u, v]$ such that (5) and (6) hold.

It follows from the inequality

$$x'(v) < 0$$

that there exists a $z \in (w, v)$ such that

$$x'(t) < 0 \quad \text{for} \quad t \in [z, v]. \tag{10}$$

In particular, $x|_{[z,v]}$ is a strictly decreasing function and so

$$x(v) < x(z). \tag{11}$$

Moreover, it follows from (10) and from the choice of the points z and w that

$$x(z) < x(w). \tag{12}$$

Consider the set

$$U := \{t \in [c, w] : x(t) \in [x(v), x(z)]\}.$$

We shall prove that there exists a point $t_1 \in U$ satisfying

$$x'(t_1) > 0. \tag{13}$$

To this end we consider the set

$$V := \{t \in [c, w] : x(t) = x(z)\}.$$

Recalling (9), (11) and (12) we get

$$V \neq \emptyset.$$

Put

$$v_m = \inf V.$$

In view of (9) and (11) as well as from the continuity of the function x we infer that there exists an $\varepsilon > 0$ such that $v_m - \varepsilon \geqq c$ and for every $t \in (v_m - \varepsilon, v_m)$ we have

$$x(v) < x(t) < x(z). \tag{14}$$

Hence

$$(v_m - \varepsilon, v_m) \subset U.$$

Now, if we had $x'(t) \leqq 0$ for every $t \in U$ then $x|_{(v_m - \varepsilon, v_m)}$ would be a decreasing function, which, however, contradicts (14). Hence, there exists a $t_1 \in U$ such that (13) holds.

Finally, since

$$x(v) < x(t_1) \leqq x(z),$$

we can find a point $t_2 \in [z, v]$ such that (5) is true. Recalling (10) and (13) we get (6).

This completes the proof of Lemma 2.

Theorem. *Under the hypothesis* (2), *every solution* $x \in C^1(I, \mathbb{R})$ *of equation* (1) *is a monotonic function.*

Proof. Suppose that a solution x of (1) is not a monotonic function. Then by Lemma 2 there exist points $t_1, t_2 \in I$ such that (5) and (6) holds.

From (5) and assumption (2) we derive that

$$f(t_1, x) \,.\, f(t_2, x) \geqq 0,$$

i.e., since x is a solution of equation (1), we have

$$x'(t_1) \,.\, x'(t_2) \geqq 0,$$

which contradicts (6).

This completes the proof of the theorem.

Remark. A particular case of the theorem was proved in [1], where the equations

$$x'(t) = \frac{1}{x(x(t))}\,.$$

and

$$x'(t) = x(x(t))$$

were considered.

An immediate consequence of the theorem is the following corollary.

Corollary 1. *Under the hypothesis* (2), *equation* (1) *has neither periodic nor oscillatory solutions, except that which is a constant function.*

The theorem is no longer true if we drop assumption (2), which is shown by the following examples.

Example 1. The function

$$x(t) = t^2, \quad t \in (-1, 1) \quad (\text{or } t \in \mathbb{R})$$

is a solution of the equation

$$x'(t) = 2t\,\frac{x(x(t)) + 1}{[x(t)]^2 + 1}$$

in $(-1, 1)$ (or in $\mathbb{R}$, respectively).

A type of equations of the form (1) for which condition (2) need not be fulfilled are ordinary differential equations with a constant but different from zero deviation argument.

Example 2. Consider the equation

$$x'(t) = x(t + \tfrac{1}{2}\pi), \quad t \in \mathbb{R}.$$

The function $x(t) = \sin t$, $t \in \mathbb{R}$, is a solution of this equation.

It is impossible to obtain analogous theorems for systems of differential equations of type (1). Under similar assumptions about each equation of the system the components of the solution need not be monotonic.

Example 3. Consider the system of ordinary differential equations

$$\begin{cases} x'(t) = [x(t)]^2 - y(t) + 1, \\ y'(t) = 2\,x(t), \end{cases}$$

on the real line. It is easily verified that the function

$$(x(t), y(t)) = (t, t^2), \quad t \in \mathbb{R},$$

is a solution of the above system.

Corollary 1 and Example 3 complete a result of J. A. Yorke (see [2]).

For ordinary differential equations without deviation the following is true:

Corollary 2. *Suppose that $f_1 : I \to \mathbb{R}$ is a function which does not change the sign and let $f_2 : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then every solution $x \in C^1(I, \mathbb{R})$ of the equation*

$$x'(t) = f_1(t) \cdot f_2(x(t))$$

is a motonic function.

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[2] *J. A. Yorke:* The period of periodic solutions and charged particles in magnetic fields. Lecture Notes in Mathematics *144* (1970), 267—268.

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