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REMARK ON UGAHERI'S MAXIMUM PRINCIPLE

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By a kernel on R^m we mean a non-negative function G on $R^m \times R^m$ such that $G(x, \cdot)$ is Borel-measurable for each $x \in R^m$. For every Radon measure μ on R^m the corresponding potential is defined by

$$G\,\mu(x) = \int_{R^m} G(x, y)\, \mathrm{d}\mu(y)\,.$$

G is said to satisfy Ugaheri's maximum principle if there is a positive constant M such that

$$\sup_{x \in R^m} G\,\mu(x) \leqq M \sup_{x \in \operatorname{spt}\mu} G\,\mu(x)$$

for every finite nonzero Borel measure μ (whose support is denoted by $spt\ \mu$).

An important example of a kernel satisfying this principle is given by the classical Riesz kernel

$$\begin{aligned} G(x, y) = |x - y|^{-s} \quad &\text{for} \quad x \neq y\,, \\ \infty \qquad\quad &\text{for} \quad x = y\,, \end{aligned}$$

where $|\ |$ denotes the Euclidean norm and $s > 0$. This follows from Ugaheri's theorem guaranteeing the validity of Ugaheri's principles for kernels of the form

$$\begin{aligned} G(x, y) = L(|x - y|) \quad &\text{for} \quad x \neq y\,, \\ L(0) \qquad\quad &\text{for} \quad x = y \end{aligned}$$

for every non-increasing function $L: \langle 0, \infty) \to \langle 0, \infty \rangle$.

An important generalization of Riesz kernels is given by

$$\begin{aligned} G(x, y) = \left(\sum_{i=1}^{m} |x_i - y_i|^{r_i}\right)^{-s} \quad &\text{for} \quad x \neq y\,, \\ \infty \qquad\qquad &\text{for} \quad x = y\,, \end{aligned}$$

where $s > 0$, $r_i > 0$ $(i = 1, \ldots, m)$. It appears in many investigations, in particular, those connected with estimates of fundamental solutions of semielliptic partial differential equations. One may naturally ask whether this kernel also satisfies the Ugaheri's principle. We shall show that this is indeed the case. For the proof of this fact we employ Theorem 2 to be proved below and the following generalization of Ugaheri's result established in [1].

Theorem 1. *Let G be a kernel on R^m and suppose there are a positive constant c and a seminorm q on R^m such that*

$$q(x - y) \geqq q(z - y) \Rightarrow G(x, y) \leqq c\, G(z, y)$$

whenever $x, y, z \in R^m$. Then G satisfies Ugaheri's principle.

Of course, Theorem 1 is far from characterizing all kernels satisfying Ugaheri's principle. In order to illustrate this we show in Example 1 below that the characteristic function of the "chess-board", considered as a kernel on R^1, satisfies Ugaheri's principle although it does not fulfil the assumption of Theorem 1.

Theorem 2. *Let K be a kernel on R^n. Further, let f be a Borel-measurable mapping of R^m into R^n such that $f(B)$ is a Borel set for every closed set $B \subset R^m$. Let $G(x, y) = K(f(x), f(y))$ for each $x, y \in R^m$. If K satisfies Ugaheri's maximum principle then the nonnegative kernel G satisfies Ugaheri's maximum principle as well.*

Proof. G is a kernel, because $G(x, \cdot) = K(f(x), f(\cdot))$ for each $x \in R^m$ and $K(f(x), \cdot), f$ are Borel-measurable mappings.

Since K satisfies Ugaheri's principle, there is a positive constant M such that

$$\sup_{x \in R^m} K\, \nu(x) \leqq M \sup_{x \in \operatorname{spt} \nu} K\, \nu(x)$$

for every nonzero finite Borel measure ν on R^n. If μ is a nonzero finite Borel measure on R^m, we define a measure ν on R^n by $\nu(A) = \mu(f^{-1}(A))$, where $f^{-1}(A)$ is the inverse image of a set A. The measure ν is concentrated on the set $f(\operatorname{spt} \mu)$, because $\nu(R^n - f(\operatorname{spt} \mu)) \leqq \mu(R^m - \operatorname{spt} \mu) = 0$. Since the set $f(\operatorname{spt} \mu)$ is Borel-measurable, there is a non-decreasing sequence F_k of closed sets such that $F_k \subset f(\operatorname{spt} \mu)$ and $\nu(f(\operatorname{spt} \mu) - \bigcup F_k) = 0$. Denote $\nu_k(A) = \nu(A \cap F_k)$ for every Borel set A. Then

$$G\, \mu(x) = K\, \nu(f(x)) = \lim_{k \to \infty} K\, \nu_k(f(x)) \leqq \limsup_{k \to \infty} M \sup_{y \in F_k} K\, \nu_k(y) \leqq$$

$$\leqq M \sup_{y \in f(\operatorname{spt} \mu)} K\, \nu(y) = M \sup_{y \in \operatorname{spt} \mu} G\, \mu(y)$$

for each $x \in R^m$. Thus G satisfies Ugaheri's maximum principle.

Remark 1. Let G, K be kernels on R^m. Let $0 < c_1 < c_2 < \infty$ be such that $c_1 K \leqq G \leqq c_2 K$. If K satisfies Ugaheri's maximum principle then G satisfies Ugaheri's maximum principle as well.

Proof. Let K satisfy Ugaheri's maximum principle with a constant M. Let μ be a nonzero finite Borel measure on R^m. Then

$$G\,\mu(x) \leqq c_2 K\,\mu(x) \leqq c_2 M \sup_{y \in \operatorname{spt}\mu} K\,\mu(y) \leqq c_2 M c_1^{-1} \sup_{y \in \operatorname{spt}\mu} G\,\mu(y)$$

is valid for each $x \in R^m$.

Example 1. There is a kernel which satisfies Ugaheri's maximum principle but does not fulfil conditions of Theorem 1. Such kernel is the characteristic function of the "chess board"

$$G(x, y) = \begin{cases} 1 & \text{if there are integers } k, n \text{ such that } x \in \langle 2k, 2k+1\rangle,\ y \in \langle 2n, 2n+1\rangle, \\ 1 & \text{if there are integers } k, n \text{ such that } x \in \langle 2k-1, 2k\rangle, \\ & y \in \langle 2n-1, 2n\rangle, \\ 0 & \text{in all remaining cases.} \end{cases}$$

Theorem 1 yields that the kernel

$$K(x, y) = \begin{cases} 1 & \text{for} \quad x = y\,, \\ 0 & \text{for} \quad x \neq y \end{cases}$$

satisfies Ugaheri's maximum principle in R^1. If we define

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in \bigcup_{k=-\infty}^{\infty} \langle 2k, 2k+1\rangle\,, \\ 0 & \text{in all remaining cases}\,, \end{cases}$$

then $G(x, y) = K(f(x), f(y))$. Theorem 2 implies that the kernel G satisfies Ugaheri's maximum principle. At the same time we have $q(1, 5) \leqq q(2)$ for every seminorm q on R^1, but $G(1, 5; 0) = 0$, $G(2; 0) = 1$.

Example 2. Let $s > 0$, $r_i > 0$ for $i = 1, \ldots, m$. We define a kernel G on R^m by

$$G(x, y) = \begin{cases} \left(\sum_{i=1}^{m} |x_i - y_i|^{r_i}\right)^{-s} & \text{for} \quad x \neq y\,, \\ \infty & \text{for} \quad x = y\,. \end{cases}$$

The kernel G satisfies Ugaheri's maximum principle. Let

$$r = \min_{i=1,\ldots,m} r_i\,, \qquad R = \max_{i=1,\ldots,m} r_i\,, \qquad a_i = r_i/r\,.$$

We define a kernel G_1 on R^m by

$$G_1(x, y) = \begin{cases} \left(\sum_{i=1}^{m} |x_i - y_i|^{r}\right)^{-s} & \text{for} \quad x \neq y\,, \\ \infty & \text{for} \quad x = y\,. \end{cases}$$

Let us define by $\|x\| = \max |x_i|$ a norm in R^m. If $x = y$ then $G_1(x, y) \geqq G_1(z, y)$ for every z. If $x \neq y$ and $\|x - y\| \leqq \|z - y\|$ then $z \neq y$ and

$$G_1(x, y) = (\sum_{i=1}^{m} |x_i - y_i|^r)^{-s} \geqq (m\|x - y\|^r)^{-s} \geqq m^{-s}(\|z - y\|^r)^{-s} \geqq$$

$$\geqq m^{-s}(\sum_{i=1}^{m} |z_i - y_i|^r)^{-s} = m^{-s} G_1(z, y).$$

Thus $\|x - y\| \leqq \|z - y\| \Rightarrow G_1(z, y) \leqq m^s G_1(x, y)$. Theorem 1 yields that G_1 satisfies Ugaheri's maximum principle. Further, we define a kernel G_2 on R^m by

$$G_2(x, y) = (\sum_{i=1}^{m} \big| |x_i|^{a_i} \operatorname{sgn} x_i - |y_i|^{a_i} \operatorname{sgn} y_i \big|^r)^{-s}.$$

Since the mapping

$$f(x_1, \ldots, x_m) = (|x_1|^{a_1} \operatorname{sgn} x_1, \ldots, |x_m|^{a_m} \operatorname{sgn} x_m)$$

from R^m into R^m is continuous, Theorem 2 implies that the kernel G_2 satisfies Ugaheri's maximum principle. Therefore there is a positive constant M such that

$$\sup_{x \in R^m} G_2\, \mu(x) \leqq M \sup_{x \in \operatorname{spt} \mu} G_2\, \mu(x)$$

for every nonzero finite Borel measure μ on R^m. Let us choose a fixed i. Since $a_i \geqq 1$, the function

$$(y + t)^{a_i} - t^{a_i}$$

is non-decreasing on the interval $\langle 0, \infty)$ for an arbitrary nonnegative y. Hence

$$(y + t)^{a_i} - t^{a_i} \geqq y^{a_i}$$

for every nonnegative t, y. If $(\operatorname{sgn} x_i)(\operatorname{sgn} y_i) \geqq 0$, $|x_i| \geqq |y_i|$ then

$$\big| |x_i|^{a_i} \operatorname{sgn} x_i - |y_i|^{a_i} \operatorname{sgn} y_i \big| = (|x_i| - |y_i| + |y_i|)^{a_i} - |y_i|^{a_i} \geqq |x_i - y_i|^{a_i}.$$

If $(\operatorname{sgn} x_i)(\operatorname{sgn} y_i) \leqq 0$, $|x_i| \geqq |y_i|$ then

$$\big| |x_i|^{a_i} \operatorname{sgn} x_i - |y_i|^{a_i} \operatorname{sgn} y_i \big| \geqq |x_i|^{a_i} \geqq 2^{-a_i}(|x_i| + |y_i|)^{a_i} \geqq 2^{-R/r}|x_i - y_i|^{a_i}.$$

Evidently

$$\sum_{i=1}^{m} \big| |x_i|^{a_i} \operatorname{sgn} x_i - |y_i|^{a_i} \operatorname{sgn} y_i \big|^r \geqq \sum_{i=1}^{m} (2^{-R/r}|x_i - y_i|^{a_i})^r = 2^{-R} \sum_{i=1}^{m} |x_i - y_i|^{r_i}$$

for each $x, y \in R^m$. Thus

$$G_2 \leqq 2^{Rs} G.$$

Let μ be a nonzero finite Borel measure on R^m, $x \in R^m$. Let us define a measure ν in R^m by $\nu(A) = \mu(A + x)$ for every Borel set A. Since $G(0, y) = G_2(0, y)$ for each

$y \in R^m$, we obtain

$$G\,\mu(x) = G\,\nu(0) = G_2\,\nu(0) \leqq M \sup_{y \in \operatorname{spt} \nu} G_2\,\nu(y) \leqq M\,2^{Rs} \sup_{y \in \operatorname{spt} \nu} G\,\nu(y) =$$
$$= M\,2^{Rs} \sup_{y \in \operatorname{spt} \mu} G\mu(y)\,.$$

Thus G satisfies Ugaheri's maximum principle.

References

[1] *D. Křivánková:* On a maximum principle in potential theory. Čas. pěst. mat. *107* (1982), 346—359.

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