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INTRODUCTORY REMARKS

Let R^m stand for the m-dimensional Euclidean space $(m \geqq 1)$ and let $D \subset R^m$ be an open set with a compact boundary B. Given $T_1, T_2 \in R^1$, $T_1 < T_2$, let

$$C = B \times \langle T_1, T_2 \rangle, \quad E = D \times (T_1, T_2).$$

In [11] J. Král has defined a generalized heat flow for heat potentials derived from measures concentrated on C. The flow of heat is considered "leaking through C", that is, "through" some part of the boundary of E. That definition makes it possible to solve the second boundary value problem for the heat equation on E with boundary values prescribed on C by means of integral equations under very general conditions. As a byproduct an integral representation of the solution of the first boundary value problem for the adjoint heat equation is obtained – in the sense of integral equations the first and the second boundary value problem for the heat equation are adjoint in this case. In [11] the assumption that E is of the form $D \times (T_1, T_2)$ is essential.

The case of time moving boundary is considered in [3], [4], [5] but with the restriction that $m = 1$ and E is of the form

$$E = \{[x, t] \in R^2;\ t \in (a, b),\ x > \varphi(t)\},$$

where φ is a continuous function of bounded variation on an interval $\langle a, b \rangle$, or of the form

$$E = \{[x, t] \in R^2;\ t \in (a, b),\ \varphi_1(t) < x < \varphi_2(t)\},$$

where φ_1, φ_2 are continuous functions of bounded variation on the interval $\langle a, b \rangle$, $\varphi_1(t) < \varphi_2(t)$ for $t \in \langle a, b \rangle$. Only the first boundary value problem is considered in [3], [4]. In [5] it is shown that in the case of time moving boundary the first and second boundary value problems are not adjoint to each other in the sense of itnegral equations but the first boundary value problem and some special type of the third boundary value problem are. The assumption $m = 1$ is essential in [3], [4], [5].

In the present paper the case of time moving boundary in R^{m+1}, $m \geqq 1$, is considered. This is made possible essentially by the work [21] of J. Veselý, where a generalized heat potential is defined and studied which is just suitable for our purposes and corresponds (in an adjoint form) to the flows of heat defined by J. Král. Using the method applied by I. Netuka in [16], [17] in connection with the third boundary value problem for the heat equation on a set of the form $E = D \times (T_1, T_2)$, we shall investigate the third boundary value problem for the heat equation but on a set in R^{m+1} with time moving boundary (note that the same method is used in [5] in the case $m = 1$ – we shall only show here that this method is applicable also if $m > 1$). The solution is found in the form of a heat potential by means of an integral equation. Considering a special type of the third boundary value problem the solution of the first boundary value problem for the adjoint heat equation can be obtained by the adjoint integral equation.

Let us introduce some notations we shall use in the following.

$m \geqq 1$ will be a given integer, $R^{m+1} = R^m \times R^1$ will be the Euclidean $(m + 1)$-space. Points in R^{m+1} will be usually denoted by $[x, t], [\xi, \tau]$ etc., where $x = [x_1, \ldots, x_m]$, $\xi = [\xi_1, \ldots, \xi_m] \in R^m$, $t, \tau \in R^1$. We shall write, for $[x, t] \in R^{m+1}$, $\delta > 0$,

$$\Omega(x, t; \delta) = \{[\xi, \tau] \in R^{m+1};\ |[x, t] - [\xi, \tau]| < \delta\}\,,$$

$$\Gamma(x, t; \delta) = \partial\Omega(x, t; \delta)\,, \quad \Omega^*(x; \delta) = \{\xi \in R^m;\ |x - \xi| < \delta\}\,,$$

$$\Gamma^*(x; \delta) = \partial\Omega^*(x; \delta)\,, \quad \Gamma^* = \Gamma^*(0; 1)$$

(the Euclidean norms in R^m and in R^{m+1} are denoted by the same symbol $|\ldots|$ – a misunderstanding is out of question).

For $\alpha \in R^1$ we denote

$$R_\alpha = \{[x, t] \in R^{m+1};\ t < \alpha\}\,;$$

for $\alpha, \beta \in R^1$, $\alpha < \beta$ we put

$$R_{\alpha\beta} = R_\beta - \bar{R}_\alpha\,.$$

G will stand for the heat kernel in R^{m+1}, that is, for $[x, t] \in R^{m+1}$ we have $G(x, t) = 0$ if $t \leqq 0$ and

$$G(x, t) = (4\pi t)^{-m/2} \exp\left(-\frac{|x|^2}{4t}\right) \tag{0.1}$$

for $t > 0$. Let us note that G defined in this way differs from the kernel used in [11] or in [21] by a multiplicative constant. G^* stands for the adjoint heat kernel, that is, $G^*(x, t) = G(x, -t)$, $[x, t] \in R^{m+1}$.

By the term measure on R^k we shall always mean a finite signed Borel measure on R^k. If μ is a measure on R^k, then $\mu^+, \mu^-, |\mu| = \mu^+ + \mu^-$ stand for the positive, negative and total variation of μ, respectively, and $\|\mu\| = |\mu|\,(R^k)$ denotes the norm of μ. Support of μ is denoted by spt μ. If $K \subset R^k$ is a compact set then by a measure on K we mean a measure μ on R^k with spt $\mu \subset K$. If $K \subset R^k$ is closed, μ is a measure on R^k, then $\mu|_K$ denotes the restriction of μ to K, that is, the measure defined by

$$\mu|_K(M) = \mu(M \cap K)$$

for any Borel $M \subset R^k$.

For a measure μ on R^{m+1} with a compact support define the heat potential U_μ by

$$U_\mu(x, t) = \int_{R^{m+1}} G(x - \xi, t - \tau)\, d\mu(\xi, \tau) \tag{0.2}$$

for $[x, t] \in R^{m+1}$ for which this integral exists (finite or infinite; if μ is non-negative then U_μ is defined on the whole of R^{m+1}). Similarly U_μ^* stands for the adjoint heat potential of μ, that is,

$$U_\mu^*(x, t) = \int_{R^{m+1}} G^*(x - \xi, t - \tau)\, d\mu(\xi, \tau) \tag{0.3}$$

for $[x, t] \in R^{m+1}$ fot which integral in (0.3) exists. The potentials U_μ, U_μ^* are infinitely differentiable on R^{m+1} − spt μ; U_μ and U_μ^* solve the heat and the adjoint heat equation, respectively, on R^{m+1} − spt μ, that is

$$\sum_{j=1}^{m} \partial_j^2 U_\mu - \partial_{m+1} U_\mu = 0\,, \quad \sum_{j=1}^{m} \partial_j^2 U_\mu^* + \partial_{m+1} U_\mu^* = 0$$

on R^{m+1} − spt μ (∂_j denotes the derivative with respect to the j-th variable).

Let us state now an assertion and a simple consequence concerning the continuity of the heat and adjoint heat potentials which will be useful in the following (see, for instance, [12], [13], [15]).

0.1. Theorem. *Let μ be a non-negative measure on R^{m+1} with compact support. If $\emptyset \neq K \subset R^{m+1}$ is compact then the restriction $U_\mu|_K$ of U_μ to K is finite and continuous on K if and only if*

$$\lim_{d \to +\infty} \left(\sup \left\{ \int_d^\infty \mu(A(x, t; c))\, dc;\ [x, t] \in K \right\} \right) = 0\,, \tag{0.4}$$

where (for $[x, t] \in R^{m+1}$, $c > 0$)

$$A(x, t; c) = \{[\xi, \tau] \in R^{m+1};\ G(x - \xi, t - \tau) > c\}\,.$$

Similarly, the restriction of U_μ^* to K is continuous on K if and only if

$$\lim_{d \to +\infty} \left(\sup \left\{ \int_d^\infty \mu(A^*(x, t; c))\, dc;\ [x, t] \in K \right\} \right) = 0\,, \tag{0.5}$$

where

$$A^*(x, t; c) = \{[\xi, \tau] \in R^{m+1};\ G^*(x - \xi, t - \tau) > c\}\,.$$

0.2. Corollary. *Let μ be a non-negative measure on R^{m+1} with compact support and suppose that $U_\mu|_K$ $(U^*_\mu|_K)$ is continuous on a compact $K \subset R^{m+1}$. If ν is a non-negative measure, $\nu \leqq \mu$ (that is, $\nu(M) \leqq \mu(M)$ for any Borel $M \subset R^{m+1}$) then $U_\nu|_K$ $(U^*_\nu|_K$, respectively) is continuous on K, too. Particularly, if $D \subset R^{m+1}$ is closed then $U_{\mu|D}|_K$ $(U^*_{\mu|D}|_K)$ is continuous on K.*

In what follows let $a, b \in R^1$ be fixed numbers, $a < b$. Further, fix an open set $E \subset R_{ab}$. We shall denote

$$B = \overline{\partial E \cap R_{ab}} \tag{0.6}$$

and we shall always suppose that $B \neq \emptyset$, B is compact.

$\mathscr{C} = \mathscr{C}(B)$ will stand for the set of all continuous functions on B, $\mathscr{B} = \mathscr{B}(B)$ for the set of all bounded Baire functions on B and $\mathscr{B}' = \mathscr{B}'(B)$ for the set of all (finite, signed, Borel) measures on B. Further, denote

$$(0.7)\quad B_0 = B \cap R_b\,, \quad \mathscr{B}_0 = \mathscr{B}_0(B) = \{f \in \mathscr{B};\ f(x,t) = 0\ \forall [x,t] \in B - B_0\}\,,$$
$$\mathscr{C}_0 = \mathscr{C}_0(B) = \mathscr{C}(B) \cap \mathscr{B}_0(B)\,, \quad \mathscr{B}'_0 = \mathscr{B}'_0(B) = \{\mu \in \mathscr{B}';\ |\mu|\,(B - B_0) = 0\}\,.$$

$\mathscr{C}$ and $\mathscr{C}_0$ endowed with the supremum norm (which we shall denote by $\|\ldots\|$) as well as $\mathscr{B}'$ and $\mathscr{B}'_0$ with the norm $\|\ldots\|$ are Banach spaces.

By $\mathscr{D} = \mathscr{D}(R^{m+1})$ we denote the set of all infinitely differentiable functions with compact support in R^{m+1}. Further, for $[x, t] \in R^{m+1}$, $\alpha \in R^1$, we shall denote

$$\mathscr{D}(x,t) = \{\varphi \in \mathscr{D};\ [x,t] \notin \operatorname{spt} \varphi\}\,, \quad \mathscr{D}_\alpha = \{\varphi \in \mathscr{D};\ \operatorname{spt} \varphi \subset R_\alpha\}\,,$$
$$\mathscr{D}_\alpha(x,t) = \mathscr{D}_\alpha \cap \mathscr{D}(x,t)\,.$$

In what follows we shall also use the notation ∇, $\hat{\nabla}$, where

$$\nabla = [\partial_1, \ldots, \partial_m, \partial_{m+1}]\,, \quad \hat{\nabla} = [\partial_1, \ldots, \partial_m]\,.$$

For a given measure $\mu \in \mathscr{B}'(B)$ let us define a functional (a distribution) H_μ on $\mathscr{D}_b$ by

$$\langle \varphi, H_\mu \rangle = - \iint_E \{\hat{\nabla} U_\mu(x,t)\, \hat{\nabla}\varphi(x,t) - U_\mu(x,t)\, \partial_{m+1}\varphi(x,t)\}\, \mathrm{d}x\, \mathrm{d}t\,, \tag{0.8}$$

$\varphi \in \mathscr{D}_b$. In virtue of the estimates ($\alpha, \beta \in R^1$, $\alpha < \beta$)

$$\iint_{R_{\alpha\beta}} |\partial_j G(x,t)|\, \mathrm{d}x\, \mathrm{d}t \leqq \frac{2}{\sqrt{\pi}} \sqrt{(\beta - \alpha)}\,, \quad 1 \leqq j \leqq m\,, \tag{0.9}$$

$$\iint_{R_{\alpha\beta}} G(x,t)\, \mathrm{d}x\, \mathrm{d}t \leqq \beta - \alpha \tag{0.10}$$

the integral in (0.8) always exists and is finite so that we can define a functional H_μ by (0.8) indeed.

For $k \geqq 0$ let $\mathscr{H}_k$ stand for the k-dimensional Hausdorff measure; in the case $k = 0$ we mean by $\mathscr{H}_k$ the counting measure.

For $t \in (a, b)$ let us write, for a while,

$$E(t) = \{x \in R^m;\ [x, t] \in E\}$$

and suppose that E has a smooth boundary (in R^{m+1}) and that for any $t \in (a, b)$ either $E(t) = \emptyset$ or $E(t) = R^m$ or $E(t)$ has a smooth boundary (in R^m). Further, suppose that the potential U_μ and its first derivatives can be extended continuously from E to $\bar{E}$. Let $\varphi \in \mathscr{D}_b$ and let $\mathbf{F}_1$, $\mathbf{F}_2$ be defined on $\bar{E}$ by

$$\mathbf{F}_1 = \mathbf{F}_1(x, t) = [\varphi\, \partial_1 U_\mu, \ldots, \varphi\, \partial_m U_\mu]\,, \quad \mathbf{F}_2 = \mathbf{F}_2(x, t) = [0, \ldots, \varphi U_\mu]\,,$$

where $[\varphi(x, t)\, \partial_1 U_\mu(x, t), \ldots, \varphi(x, t)\, \partial_m U_\mu(x, t)] \in R^m$, while $[0, \ldots, \varphi(x, t)\, U_\mu(x, t)] \in R^{m+1}$. Further, let $N = [N_1, \ldots, N_m, N_t]$ stand for the exterior normal of E and $n = [n_1, \ldots, n_m]$ for the exterior normal of $E(t)$ (for a given $t \in (a, b)$, provided $E(t) \neq \emptyset$, $E(t) \neq R^m$). We see that

$$\hat{\nabla}\mathbf{F}_1 = \sum_{j=1}^{m} \partial_j U_\mu\, \partial_j \varphi - U_\mu\, \partial_{m+1}\varphi + (U_\mu\, \partial_{m+1}\varphi + \varphi\, \partial_{m+1} U_\mu)\,.$$

Since $\varphi \in \mathscr{D}_b$ and $U_\mu(x, t) = 0$ for $t \leqq a$, we obtain

$$\text{(0.11)} \qquad \langle \varphi, H_\mu \rangle = -\iint_E \hat{\nabla}\mathbf{F}_1\, \mathrm{d}x\, \mathrm{d}t + \iint_E \nabla \mathbf{F}_2\, \mathrm{d}x\, \mathrm{d}t =$$

$$= -\int_a^b \left(\int_{E(t)} \hat{\nabla}\mathbf{F}_1\, \mathrm{d}x \right) \mathrm{d}t + \iint_E \nabla \mathbf{F}_2\, \mathrm{d}x\, \mathrm{d}t =$$

$$= -\int_a^b \left(\int_{\partial E(t)} \varphi \sum_{j=1}^{m} \partial_j U_\mu n_j\, \mathrm{d}\mathscr{H}_{m-1} \right) \mathrm{d}t + \int_B \varphi N_t U_\mu\, \mathrm{d}\mathscr{H}_m\,.$$

Consequently, H_μ can be regarded as a weak characterization of a combination of the normal derivative of U_μ "in the x-direction" and a certain multiple of U_μ on B (on B we do not consider directly the values of U_μ but "boundary limits of U_μ from within E"!).

If E is of the form $E = D \times (a, b)$ ($D \subset R^m$ open) the functional H_μ is a weak characterization of the normal derivative of U_μ and H_μ is termed the heat flow then – compare [11]. For the expression of H_μ in the case $m = 1$ see also [5].

For $[\xi, \tau] \in R^{m+1}$ let $\delta_{\xi,\tau}$ stand for the Dirac measure (on R^{m+1}) concentrated at $[\xi, \tau]$. Take notice of the fact, which follows from the Fubini theorem, that

$$\text{(0.12)} \qquad \langle \varphi, H_\mu \rangle =$$

$$= -\int_B \left(\iint_E \{ \sum_{j=1}^{m} \partial_j G(x - \xi, t - \tau)\, \partial_j \varphi(x, t) - G(x - \xi, t - \tau)\, \partial_{m+1}\varphi(x, t) \}\, \mathrm{d}x\, \mathrm{d}t \right) .$$

$$.\ \mathrm{d}\mu(\xi, \tau) = \int_B \langle \varphi, H_{\delta_{\xi,\tau}} \rangle\, \mathrm{d}\mu(\xi, \tau)\,.$$

We see that it will be useful to investigate the behaviour of $H_{\delta_{\xi,\tau}}$ ($[\xi, \tau] \in B$). This is one of the reasons for considering the operator $\tilde{W}$ defined in this way: for $\varphi \in \mathscr{D}_b$, $[\xi, \tau] \in R^{m+1}$, put

(0.13)
$$\tilde{W}\varphi(\xi, \tau) = = -\iint_E \{\sum_{j=1}^m \partial_j G(x - \xi,\ t - \tau)\, \partial_j \varphi(x, t) - G(x - \xi,\ t - \tau)\, \partial_{m+1}\varphi(x, t)\}\, \mathrm{d}x\, \mathrm{d}t\,.$$

Suppose, for a while, that $\varphi \in \mathscr{D}_b(\xi, \tau)$ and employ the change of variables $\tilde{t} = -t$ in the integral in (0.13). At the same time denote $\tilde{\tau} = -\tau$, $\tilde{\varphi}(x, \tilde{t}) = \varphi(x, -\tilde{t})$ and

$$E_- = \{[x, t];\ [x, -t] \in E\}\,.$$

Then we have

(0.14)
$$\tilde{W}\varphi(\xi, \tau) = = -\iint_{E_-} \left\{\sum_{j=1}^m \partial_j G(\xi - x,\ \tilde{\tau} - \tilde{t})\, \partial_j \varphi(x, -\tilde{t}) + G(\xi - x,\ \tilde{\tau} - \tilde{t})\, \frac{\partial \varphi(x, -t)}{\partial \tilde{t}}\right\} \mathrm{d}x\, \mathrm{d}\tilde{t} = = (4\pi)^{-m/2}\, T\tilde{\varphi}(\xi, \tilde{\tau})\,,$$

where T is the operator defined in [21] and considered here for E_-. In [21] $T\tilde{\varphi}(\xi, \tilde{\tau})$ is defined for each $\tilde{\varphi} \in \mathscr{D}(\xi, \tilde{\tau})$. Considering now only $\varphi \in \mathscr{D}_b(\xi, \tau)$ then, in fact, we ignore the part of ∂E where $t = b$. If we consider only $[\xi, \tau] \in R^{m+1}$ for which $\tau \geqq a$ then, by the definition of the kernel G, the part of ∂E, where $t = a$, is ignored. We see that in this case the condition (A) from [21], that is, the compactness of ∂E, can be replaced by the condition that B defined by (0.6) is compact. With respect to the relation (0.14) between $\tilde{W}$ and T we may transfer to our situation many assertions from [21] only after a slight modification without detailed proofs.

1. PARABOLIC VARIATION AND OPERATORS $\tilde{W}$, W, $\overline{W}$, H

As in the introduction, let $E \subset R_{ab}$ be a fixed open set such that B defined by (0.6) is compact and $B \neq \emptyset$. This section is devoted to the investigation of basic properties of operators $\tilde{W}$, W, H and concepts which are necessary in this connection. Assertions stated here are, for greater part, essentialy known, but they were stated either for the case of cylindrical sets in R^{m+1} ([11]) or for the case $m = 1$ ([3], [4], [5]) or in an adjoint form ([21]). For our purposes it will be possibly useful to give here a survey of some assertions concerning $\tilde{W}$, W, H and related concepts (and their definitions).

One of the questions concerning the operator $\tilde{W}$ is the following: Given $[\xi, \tau] \in R^{m+1}$. Under which conditions can $\tilde{W}\varphi(\xi, \tau)$, as a functional on $\mathscr{D}_b$, be represented by a measure? That is, under which conditions is there a measure $\mu_{\xi,\tau}$ on R^{m+1} such that

$$\tilde{W}\varphi(\xi, \tau) = \int \varphi(x, t)\, \mathrm{d}\mu_{\xi,\tau}(x, t)$$

for any $\varphi \in \mathscr{D}_b$?

First let us show another form of expression of $\tilde{W}\varphi(\xi, \tau)$. For this purpose we introduce the following notation. For a given $[\xi, \tau] \in R^{m+1}$ let $\tilde{S}_{\xi,\tau}$ be a mapping from $(0, \infty) \times (0, \infty) \times \Gamma^*$ (Γ^* is the boundary of the unit ball in R^m) into R^{m+1} defined by

$$(1.1) \qquad \tilde{S}_{\xi,\tau}(\varrho, \eta, \theta^*) = \left[\xi + \varrho\theta^*, \tau + \frac{\varrho^2}{4\eta}\right]$$

($\varrho > 0$, $\eta > 0$, $\theta^* \in \Gamma^*$). For a given $[\xi, \tau] \in R^{m+1}$, $\eta > 0$, $\theta^* \in \Gamma^*$, denote

$$\tilde{E}(\eta, \theta^*) = \{\varrho > 0;\ \tilde{S}_{\xi,\tau}(\varrho, \eta, \theta^*) \in E\}\,.$$

1.1. Lemma. *For any* $[\xi, \tau] \in R^{m+1}$, $\varphi \in \mathscr{D}_b$, *we have*

$$(1.2) \quad \tilde{W}\varphi(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1} \int_{\tilde{E}(\eta,\theta^*)} \frac{\partial}{\partial\varrho}\, \varphi(\tilde{S}_{\xi,\tau}(\varrho, \eta, \theta^*))\, \mathrm{d}\varrho\,.$$

Proof of this lemma is quite analogous to the proof of Lemma 1.2 from [21] and we omit it.

Let us introduce the following notations. For $[\xi, \tau] \in R^{m+1}$, $\eta > 0$, $\theta^* \in \Gamma^*$, let

$$(1.3) \qquad \boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta) = \left\{\left[\xi + \varrho\theta^*, \tau + \frac{\varrho^2}{4\eta}\right];\ \varrho > 0\right\}.$$

A point $[x, t] \in \boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ is called a hit of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E if for any $r > 0$

$$(1.4) \qquad \mathscr{H}_1(\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta) \cap \Omega(x, t; r) \cap E) > 0$$

and at the same time

$$(1.5) \qquad \mathscr{H}_1((\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta) \cap \Omega(x, t; r)) - E) > 0\,.$$

For a given $r > 0$ (r is allowed to be $+\infty$) $\tilde{n}_{\xi,\tau}(\theta^*, \eta; r)$ will stand for the number (finite or infinite) of hits of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E which are contained in the set

$$[\Omega^*(\xi; r) \times (\tau, \tau + r)] \cap R_b\,.$$

Note that in the case $r = +\infty$ $\tilde{n}_{\xi,\tau}(\theta^*, \eta; r)$ is the number of all hits $[x, t]$ of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E for which $t \neq b$. In any case all hits of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E lie on the boundary of E. Thus in the case $\tau \geqq a$, $\tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty)$ is the number of all hits $[x, t]$ of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E for which $[x, t] \in B \cap R_b$.

The following assertion is fundamental for our purposes.

1.2. Lemma. *Given* $[\xi, \tau] \in R^{m+1}$, $r > 0$. *The function* $\tilde{n}_{\xi,\tau}(\theta^*, \eta; r)$ *is a measurable function of the variables* (θ^*, η) *on* $\Gamma^* \times (0, \infty)$ *with respect to* $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$. *If we denote*

$$(1.6) \qquad \tilde{v}^r(\xi, \tau) = \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \tilde{n}_{\xi,\tau}(\theta^*, \eta; r)\, \mathrm{d}\eta\,,$$

$$(1.7)\qquad \mathscr{D}^1 = \{\varphi \in \mathscr{D}_b(\xi, \tau);\ \|\varphi\| \leqq 1,\ \operatorname{spt}\varphi \subset [\Omega^*(\xi; r) \times (\tau, \tau + \imath)]\}$$

then

$$(1.8)\qquad \sup\{\tilde{W}\varphi(\xi, \tau);\ \varphi \in \mathscr{D}^1\} = \tfrac{1}{2}\pi^{-m/2}\,\tilde{v}^r(\xi, \tau)\,.$$

Proof of this assertion is quite analogous to the proof of Lemma 1.3 from [21] and we omit it, too.

1.3. Note. Further we shall also write $\tilde{v}(\xi, \tau)$ instead of $\tilde{v}^\infty(\xi, \tau)$. The value $\tilde{v}(\xi, \tau)$ is termed the adjoint parabolic variation of E at $[\xi, \tau]$. As we have already noted, if $\tau \geqq a$ then $\tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty)$ is the number of hits $[x, t]$ of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E for which $[x, t] \in B \cap R_b$ since in this case all hits of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E lie on $\partial E - \bar{R}_a$. This is not valid if $\tau < a$ since then it may happen that there is a hit of $\boldsymbol{H}^{\theta^*}_{\xi,\tau}(\eta)$ on E which lies in $\partial R_a - B$. In the sequel we shall deal with functions on B and with points in B which are hits but not with hits outside B. For this reason we shall restrict, in many cases, our considerations only to the case $\tau \geqq a$. For example the definition of Wf (see below) is this case.

1.4. Given $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$, suppose $\tilde{v}(\xi, \tau) < \infty$. Then

$$(1.9)\qquad \tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty) < \infty$$

for almost all $(\theta^*, \eta) \in \Gamma^* \times (0, \infty)$ (with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$). For $(\theta^*, \eta) \in \Gamma^* \times (0, \infty)$ satisfying (1.9) and for $\varrho > 0$ put

$$s_{\xi,\tau}(\theta^*, \eta, \varrho) = \sigma\ (= \pm 1)$$

provided there is a $\delta > 0$ such that

$$\mathscr{H}_1\left(\left\{\left[\xi + (\varrho + \sigma u)\,\theta^*,\ \tau + \frac{(\varrho + \sigma u)^2}{4\eta}\right];\ u \in (0, \delta)\right\} \cap E\right) = 0\,,$$

$$\mathscr{H}_1\left(\left\{\left[\xi + (\varrho - \sigma u)\,\theta^*,\ \tau + \frac{(\varrho - \sigma u)^2}{4\eta}\right];\ u \in (0, \delta)\right\} - E\right) = 0\,.$$

Further we put (provided (1.9) is still fulfilled)

$$s_{\xi,\tau}(\theta^*, \eta, 0) = -1$$

if there is a $\delta > 0$ such that

$$\mathscr{H}_1\left(\left\{\left[\xi + u\theta^*,\ \tau + \frac{u^2}{4\eta}\right];\ u \in (0, \delta)\right\} - E\right) = 0\,.$$

In all the other cases (for $\theta^* \in \Gamma^*$, $\eta > 0$, $\varrho \geqq 0$) define

$$s_{\xi,\tau}(\theta^*, \eta, \varrho) = 0\,.$$

Let $f \in \mathscr{B}(B)$. For $(\theta^*, \eta) \in \Gamma^* \times (0, \infty)$ we define ($[\xi, \tau]$ is still the given point for which $\tau \geqq a$ and $\tilde{v}(\xi, \tau) < \infty$)

$$(1.10) \qquad \Sigma_f^{\xi,\tau}(\theta^*, \eta) = \sum_\varrho f\left(\xi + \varrho\theta^*, \ \tau + \frac{\varrho^2}{4\eta}\right) s_{\xi,\tau}(\theta^*, \eta, \varrho),$$

where the sum on the right hand side extends over all $\varrho > 0$ for which $s_{\xi,\tau}(\theta^*, \eta, \varrho) \neq 0$ and

$$\tau + \frac{\varrho^2}{4\eta} \neq b$$

(if there is no such ϱ we put $\Sigma_f^{\xi,\tau}(\theta^*, \eta) = 0$). For $\varphi \in \mathscr{D}_b$ put further

$$(1.11) \qquad \tilde{\Sigma}_\varphi^{\xi,\tau}(\theta^*, \eta) = \Sigma_\varphi^{\xi,\tau}(\theta^*, \eta) + \varphi(\xi, \tau)\, s_{\xi,\tau}(\theta^*, \eta, 0).$$

It is directly seen from Lemma 1.1 that the following assertion holds.

1.5. Lemma. *Let $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$, and suppose that*

$$\tilde{v}(\xi, \tau) < \infty .$$

Then for every $\varphi \in \mathscr{D}_b$,

$$(1.12) \qquad \tilde{W}\varphi(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \tilde{\Sigma}_\varphi^{\xi,\tau}(\theta^*, \eta)\, \mathrm{d}\eta .$$

1.6. Lemma 1.5 particularly implies that $\tilde{\Sigma}_\varphi^{\xi,\tau}$ is a measurable function with respect to the measure $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$ on $\Gamma^* \times (0, \infty)$. In the case $\varphi \in \mathscr{D}_b(\xi, \tau)$ the function $\Sigma_\varphi^{\xi,\tau}$ is measurable as well. Since $\Sigma_f^{\xi,\tau}$ does not depend on the values of f ($f \in \mathscr{B}(B)$) on

$$B \cap (\partial R_b \cup \{[\xi, \tau]\})$$

one can obtain by passing to the limit that $\Sigma_f^{\xi,\tau}$ is measurable with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$ on $\Gamma^* \times (0, \infty)$ for any $f \in \mathscr{B}(B)$. Hence $s_{\xi,\tau}(\theta^*, \eta, 0)$ is a measurable function of variables (θ^*, η) with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$ on $\Gamma^* \times (0, \infty)$ (this is seen in the case $a \leqq \tau < b$; but if $\tau \geqq b$ then $s_{\xi,\tau}(\cdot, \cdot, 0) \equiv 0$).

It is seen from the definition of $\Sigma_f^{\xi,\tau}$ and of $\tilde{n}_{\xi,\tau}$ that for $f \in \mathscr{B}(B)$ with $|f| \leqq k$ on B the inequality

$$(1.13) \qquad |\Sigma_f^{\xi,\tau}(\theta^*, \eta)| \leqq k\tilde{n}_{\xi,\tau}(\theta^*, \eta, \infty)$$

is valid. (1.13) makes it possible to define an operator W in the following way. Let $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$ and suppose $\tilde{v}(\xi, \tau) < \infty$. For $f \in \mathscr{B}(B)$ we then define

$$(1.14) \qquad Wf(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \Sigma_f^{\xi,\tau}(\theta^*, \eta)\, \mathrm{d}\eta .$$

From (1.14), (1.13) and (1.6) it follows immediately that

$$(1.15) \qquad |Wf(\xi, \tau)| \leqq k\, \tfrac{1}{2}\pi^{-m/2}\, \tilde{v}(\xi, \tau)$$

provided $f \in \mathscr{B}(B)$, $|f| \leqq k$ on B. Accosding to (1.15) it follows from Lemma 1.2 that

$$(1.16) \qquad \sup\{Wf(\xi, \tau);\, f \in \mathscr{B},\ \|f\| \leqq 1\} = \\ = \sup\{Wf(\xi, \tau);\, f \in \mathscr{C}_0,\ \|f\| \leqq 1\} = \tfrac{1}{2}\pi^{-m/2}\,\tilde{v}(\xi, \tau)\,.$$

As we have noted, if $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$ and $\tilde{v}(\xi, \tau) < \infty$, the function $s_{\xi,\tau}(\cdot, \cdot, 0)$ is measurable on $\Gamma^* \times (0, \infty)$ with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$. In this case we define a value $\tilde{P}_E(\xi, \tau)$, which we call the parabolic density of E, by

$$(1.17) \qquad \tilde{P}_E(\xi, \tau) = -\tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, s_{\xi,\tau}(\theta^*, \eta, 0)\, \mathrm{d}\eta\,.$$

It is easy to see that $0 \leqq \tilde{P}_E(\xi, \tau) \leqq 1$ and from (1.14), (1.12) and (1.11) we obtain the following relation between W and $\tilde{W}$:

$$(1.18) \qquad \tilde{W}\varphi(\xi, \tau) = W\varphi(\xi, \tau) - \varphi(\xi, \tau)\,\tilde{P}_E(\xi, \tau)\,, \quad (\varphi \in \mathscr{D}_b)\,.$$

Let us now find another expression of $W\varphi$ and show some properties of the adjoint parabolic variation $\tilde{v}$.

Since B is supposed to be compact there is a finite $\alpha' > 0$ such that

$$(1.19) \qquad B \subset \Omega^*(0; \alpha') \times R^1\,.$$

Fix this α' and denote

$$(1.20) \qquad E_{\alpha'} = E \cap [\Omega^*(0; \alpha') \times R^1]\,.$$

Recall that the perimeter $\mathscr{P}(M)$ of an open (or Borel) set $M \subset R^{m+1}$ is defined by

$$(1.21) \quad \mathscr{P}(M) = \sup\left\{\iint_M \nabla w(x, t)\,\mathrm{d}x\,\mathrm{d}t;\ w = [w_1, \ldots, w_{m+1}],\ w_j \in \mathscr{D},\ |w| \leqq 1\right\}.$$

One can prove the following assertion in a way quite similar to the proof of Proposition 2.3 from [21].

1.7. Proposition. *Let the points* $[x_i, t_i] \in R^{m+1}$, $i = 1, 2, \ldots, m+2$, *be in general position (that is, not situated on a single hyperplane) and suppose that*

$$\tilde{v}(x_i, t_i) < \infty\,, \quad i = 1, 2, \ldots, m+2\,.$$

Then any $\beta > \max\{t_i;\ i = 1, 2, \ldots, m+2\}$ *satisfies*

$$\mathscr{P}(E_{\alpha'} - \bar{R}_\beta) < \infty$$

(*where* (1.19) *is valid for* α' *and* $E_{\alpha'}$ *is defined by* (1.20)).

1.8. We shall be concerned, in the sequel, only with the case the adjoint parabolic variation $\tilde{v}$ is finite – everywhere or on a "sufficiently large set". This "sufficiently large set" will be R_{ab} or $\bar{R}_{ab}$. If $\tilde{v}$ is finite on R_{ab} (or $\bar{R}_{ab}$) then $\mathscr{P}(E_{\alpha'} - \bar{R}_\beta) < \infty$ for any $\beta > a$ ($E_{\alpha'}$ is defined by (1.20)). Note that it may happen that $\tilde{v}$ is finite on $\bar{R}_{ab}$ but $\mathscr{P}(E_{\alpha'}) = +\infty$. In what follows we shall be concerned with the case when the adjoint parabolic variation $\tilde{v}$ is even bounded on $\bar{R}_{ab}$. The author does not know if in that case $\mathscr{P}(E_{\alpha'}) < \infty$ or if it may happen that $\tilde{v}$ is bounded on $\bar{R}_{ab}$ but $\mathscr{P}(E_{\alpha'}) = +\infty$.

Henceforth we shall usually suppose that there is $\alpha' > 0$ for which (1.19) is valid, such that

$$\mathscr{P}(E_{\alpha'}) < \infty \tag{1.22}$$

holds (where $E_{\alpha'}$ is defined by (1.20)).

Note that quite analogously to the proof of Proposition 2.1 from [21] it can be shown that under the condition (1.22) we have

$$\tilde{v}(\xi, \tau) < \infty$$

for any $[\xi, \tau] \in R^{m+1}$, $[\xi, \tau] \notin B$. Thus we see that in this case Wf is defined for any $f \in \mathscr{B}$ at least on $R^{m+1} - (R_a \cup B)$.

Recall now some facts concerning sets with finite perimeter that we shall need in the following (for further information on sets with finite perimeter see, for example, [1], [2], [7]). A vector $\theta \in \Gamma = \Gamma(0, 0; 1)$ is called the exterior normal of a set $A \subset R^{m+1}$ at a point $[x, t] \in R^{m+1}$ in the sense of Federer provided the symmetric difference of A and the half space

$$M = \{[\xi, \tau] \in R^{m+1};\ ([\xi, \tau] - [x, t])\, \theta < 0\}$$

has the $(m + 1)$-dimensional density 0 at $[x, t]$, that is,

$$\lim_{r \to 0+} \frac{\mathscr{H}_{m+1}(\Omega(x, t; r) \cap [(A - M) \cup (M - A)])}{\mathscr{H}_{m+1}(\Omega(x, t; r))} = 0\,.$$

In what follows we shall write $N = [N_1, \ldots, N_m, N_t] = N(x, t) = N^A(x, t) = \theta$ if there is an exterior normal $\theta \in \Gamma$ at $[x, t]$; in the opposite case we define $N = N(x, t)$ to be the zero vector. It is known that if $\mathscr{P}(A) < \infty$ then

$$\mathscr{H}_m(\{[x, t] \in R^{m+1};\ N^A(x, t) \neq 0\}) < \infty\,.$$

Note that in any case

$$\{[x, t] \in R^{m+1};\ N^A(x, t) \neq 0\} \subset \partial A\,.$$

Further, if $\mathscr{P}(A) < \infty$ and if w is an $(m + 1)$-dimensional vector function, $w = [w_1, \ldots, w_{m+1}]$, $w_j \in \mathscr{D}$ $(j = 1, 2, \ldots, m + 1)$, then the Gauss-Green formula

$$\int_{\partial A} w(x, t)\, N(x, t)\, \mathrm{d}\mathscr{H}_m(x, t) = \iint_A \nabla w(x, t)\, \mathrm{d}x\, \mathrm{d}t$$

is valid.

Let us return to our set E. It is easily seen that if (1.22) is valid for some α' with (1.19) then (1.22) is fulfilled for any $\alpha' > 0$ for which (1.19) holds. Suppose (1.22) is fulfilled and let $w = [w_1, \ldots, w_{m+1}]$ be a vector function such that $w_j \in \mathscr{D}$ $(j = 1, 2, \ldots, m + 1)$ and $w(x, t) = 0$ whenever either $t = a$ or $t = b$. Denoting $N = N^E$ and

$$(1.23) \qquad \hat{B} = \{[x, t] \in B;\ N(x, t) \neq 0\}$$

we see that $\mathscr{H}_m(\hat{B}) < \infty$ and

$$(1.24) \qquad \iint_E \nabla w(x, t)\, dx\, dt = \int_B w(x, t)\, N(x, t)\, d\mathscr{H}_m(x, t) = \\ = \int_{\hat{B}} w(x, t)\, N(x, t)\, d\mathscr{H}_m(x, t)\,.$$

In the rest of the paper $N = N(x, t) = [N_1, \ldots, N_m, N_t]$ will always stand for the exterior normal of E in the sense of Federer.

Now let $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$, $\varphi \in \mathscr{D}_b(\xi, \tau)$ and suppose $\tilde{v}(\xi, \tau) < \infty$. Then it is seen from (1.18) and the definition of $\tilde{W}$ that

$$(1.25) \qquad W\varphi(\xi, \tau) = \tilde{W}\varphi(\xi, \tau) = \\ = - \iint_E \{ \sum_{j=1}^m \partial_j G(x - \xi,\ t - \tau)\, \partial_j \varphi(x, t) - G(x - \xi,\ t - \tau)\, \partial_{m+1}\varphi(x, t)\}\, dx\, dt$$

(note that one could define $W\varphi(\xi, \tau)$ by (1.25) for $\varphi \in \mathscr{D}_b(\xi, \tau)$ without the asumption $\tilde{v}(\xi, \tau) < \infty$ – cf. the definition ot T in [21] – but we shall not need it). Denoting

$$F(x, t) = [-\varphi(x, t)\, \partial_1 G(x - \xi,\ t - \tau), \ldots, -\varphi(x, t)\, \partial_m G(x - \xi,\ t - \tau), \\ \varphi(x, t)\, G(x - \xi,\ t - \tau)]\,,$$

we have, with respect to the assumption $\varphi \in \mathscr{D}_b(\xi, \tau)$, $\tau \geqq a$,

$$(1.26) \qquad W\varphi(\xi, \tau) = \iint_E \nabla F(x, t)\, dx\, dt = \int_B FN\, d\mathscr{H}_m = \\ = \int_B \varphi(x, t)\, \{N_t G(x - \xi,\ t - \tau) - \sum_{j=1}^m N_j\, \partial_j G(x - \xi,\ t - \tau)\}\, d\mathscr{H}_m(x, t)\,.$$

Let $\{f_n\}$ be a sequence of $f_n \in \mathscr{B}$, $|f_n| \leqq k$, which is pointwise convergent to an f on B. Then it follows directly from the definition of W (since $\tilde{v}(\xi, \tau) < \infty$) that $Wf_n(\xi, \tau) \to Wf(\xi, \tau)$ $(n \to \infty)$. As we have noted, under the assumption (1.22) we have $\mathscr{H}_m(\hat{B}) < \infty$ and thus the same assertion on the passage to the limit is valid for the integrals on the right hand side of (1.26) (in this direct way we obtain it in the case $[\xi, \tau] \notin B$; in the case $[\xi, \tau] \in B$ $(\tilde{v}(\xi, \tau) < \infty)$ we use the result of Lemma 1.2). Taking into account that the value $Wf(\xi, \tau)$ does not depend on the values of f at $[\xi, \tau] \in B$ with $\tau = b$ and if $[\xi, \tau] \in B$ the value $Wf(\xi, \tau)$ is independent of $f(\xi, \tau)$

(which follows directly from the definition of W), we conclude that for any $f \in \mathscr{B}(B)$, $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$ (assuming $\tilde{v}(\xi, \tau) < \infty$) we have

(1.27)
$$Wf(\xi, \tau) =$$
$$= \int_{B_0} f(x, t) \{N_t G(x - \xi, t - \tau) - \sum_{j=1}^{m} N_j \, \partial_j G(x - \xi, t - \tau)\} \, d\mathscr{H}_m(x, t)$$

(B_0 is defined by (0.7)). According to the fact that $\mathscr{H}_m(\hat{B}) < \infty$ one can deduce from the expression (1.27) of Wf that Wf as a function of the variables $[\xi, \tau]$ is an adjoint parabolic function (i.e. one solving the adjoint heat equation) on $R^{m+1} - (\bar{R}_a \cup B)$ (if we regarded (1.27) as a definition of Wf then Wf would be adjoint parabolic even on $R^{m+1} - B$).

Note that analogously to Lemma 3.2 from [21] it can be proved that the adjoint parabolic variation $\tilde{v}$ as a function of variables $[\xi, \tau]$ is lower semicontinuous on $R^{m+1} - R_a$ (it is seen from the expression of $W\varphi(\xi, \tau)$ for $\varphi \in \mathscr{D}_b(\xi, \tau)$ that $W\varphi$ is continuous at $[\xi, \tau]$ also if $[\xi, \tau] \in B$).

The following assertion follows easily from Lemma 1.2 and from (1.25), (1.27).

1.9. Proposition. *Let $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$. Then the functional $W.(\xi, \tau)$ is represented by a measure $\nu_{\xi,\tau} \in \mathscr{B}'(B)$, that is,*

(1.28)
$$Wf(\xi, \tau) = \int_B f \, d\nu_{\xi,\tau}$$

for each $f \in \mathscr{B}(B)$, if and only if the condition

(1.29)
$$\tilde{v}(\xi, \tau) < \infty$$

is fulfilled. If (1.29) holds then the measure $\nu_{\xi,\tau} \in \mathscr{B}'(B)$ is uniquely determined and even $\nu_{\xi,\tau} \in \mathscr{B}'_0(B)$. If, in addition, (1.22) holds then for any Borel set M ($M \subset R^{m+1}$ or $M \subset B$)

(1.30)
$$\nu_{\xi,\tau}(M) = \int_{B_0 \cap M} \{N_t G(x - \xi, t - \tau) - \sum_{j=1}^{m} N_j \, \partial_j G(x - \xi, t - \tau)\} \, d\mathscr{H}_m(x, t) =$$
$$= W\chi_M(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} d\mathscr{H}_{m-1}(\theta^*) \int_0^\infty e^{-\eta} \eta^{m/2-1} \, \Sigma_{\chi_M}^{\xi,\tau}(\theta^*, \eta) \, d\eta \,,$$

where χ_M is the characteristic function of M. Further, for $r > 0$,

(1.31)
$$\tfrac{1}{2}\pi^{-m/2} \tilde{v}^r(\xi, \tau) = |\nu_{\xi,\tau}| \left(\Omega^*(\xi; r) \times (\tau, \tau + r)\right) =$$
$$= \int_{B_0 \cap [\Omega^*(\xi; r) \times (\tau, \tau+r)]} |N_t G(x - \xi, t - \tau) - \sum_{j=1}^{m} N_j \, \partial_j G(x - \xi, t - \tau)| \, d\mathscr{H}_m(x, t) \,.$$

1.10. Proposition.

(A) *If* $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$, $\tilde{v}(\xi, \tau) < \infty$, $I \subset R^{m+1}$ *is an interval, then*

$$|v_{\xi,\tau}(B \cap I)| \leqq 1 . \tag{1.32}$$

(B) *If*

$$\tilde{V}_B = \sup \{\tilde{v}(\xi, \tau);\ [\xi, \tau] \in B\} \tag{1.33}$$

then for any $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$,

$$\tilde{v}(\xi, \tau) \leqq \tilde{V}_B + 2\pi^{m/2} . \tag{1.34}$$

Proof. (A) can be proved in the same way as Lemma 3.4 in [21].

Suppose $\tilde{V}_B < \infty$ (there is nothing to prove in the other case). Then for any $\beta > a$

$$\mathscr{P}(E_{\alpha'} \cap R_\beta) < \infty ,$$

where $\alpha' > 0$ is such that (1.19) holds, $E_{\alpha'}$ is defined by (1.20) (as we have already noted, it is not known to the author if under the condition $\tilde{V}_B < \infty$ the condition (1.22) is fulfilled). Similarly to the proof of Proposition 3.5 from [21] we can show that (1.34) holds on any set of the form $R^{m+1} - R_\beta$, where $\beta > a$, that is, (1.34) holds on $R^{m+1} - \bar{R}_a$. Now it suffices to note that $\tilde{v}$ is lower semicontinuous.

1.11. Further we shall investigate the boundary behaviour of the potential Wf. First let us take notice of some simple facts. We shall use, for a while, the following denotation. For $[\xi, \tau] \in R_b$, $\varphi \in \mathscr{D}(\xi, \tau)$ let us define

$$ {}_bW\varphi(\xi, \tau) = \tag{1.35}$$
$$= - \iint_{R_b} \{\hat{\nabla} G(x - \xi,\ t - \tau)\, \hat{\nabla}\varphi(x, t) - G(x - \xi,\ t - \tau)\, \partial_{m+1}\varphi(x, t)\}\, \mathrm{d}x\, \mathrm{d}t .$$

Simple calculation yields

$$ {}_bW\varphi(\xi, \tau) = \iint_{R^m} \varphi(x, b)\, G(x - \xi,\ b - \tau)\, \mathrm{d}x . \tag{1.36}$$

It is seen (as in Lemma 1.1) that for $[\xi, \tau] \in R_b$, $\varphi \in \mathscr{D}(\xi, \tau)$ we have

$$ {}_bW\varphi(\xi, \tau) = \tag{1.37}$$
$$= \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta \int_{R_b(\eta,\theta^*)} \frac{\partial}{\partial \varrho}\, \varphi(\tilde{S}_{\xi,\tau}(\varrho, \eta, \theta^*))\, \mathrm{d}\varrho ,$$

where $\tilde{S}_{\xi,\tau}$ is defined by (1.1) and

$$R_b(\eta, \theta^*) = \{\varrho > 0;\ \tilde{S}_{\xi,\tau}(\varrho, \eta, \theta^*) \in R_b\} = (0, 2\sqrt{[\eta(b - \tau)]}) .$$

For a bounded Baire function f on ∂R_b and for $[\xi, \tau] \in R_b$, $(\theta^*, \eta) \in \Gamma^* \times (0, \infty)$ let us denote

$$f_{\xi,\tau}(\theta^*, \eta) = f\left(\xi + \varrho\theta^*, \tau + \frac{\varrho^2}{4\eta}\right),$$

where $\varrho = 2\sqrt{[\eta(b - \tau)]}$, that is,

$$f_{\xi,\tau}(\theta^*, \eta) = f(\xi + 2\sqrt{[\eta(b - \tau)]}\,\theta^*, b).$$

For $[\xi, \tau] \in R_b$, $\varphi \in \mathscr{D}(\xi, \tau)$, we get immediately by (1.37) that

$$(1.38) \qquad {}_bW\varphi(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \varphi_{\xi,\tau}(\theta^*, \eta)\, \mathrm{d}\eta\,.$$

Now we may define ${}_bWf$ for any bounded Baire function f on ∂R_b by

$$(1.39) \qquad {}_bWf(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1} f_{\xi,\tau}(\theta^*, \eta)\, \mathrm{d}\eta\,.$$

By passing to the limits in (1.38) as well as in (1.36) we obtain that for any bounded Baire function f on ∂R_b,

$$(1.40) \qquad {}_bWf(\xi, \tau) = \int_{R^m} f(x, b)\, G(x - \xi,\, b - \tau)\, \mathrm{d}x$$

(and that, in fact, the function $f_{\xi,\tau}$ in (1.39) is measurable with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$ on $\Gamma^* \times (0, \infty)$ and the definition of ${}_bWf$ by (1.39) is correct). Thus we find that ${}_bWf$ is the adjoint Weierstrass integral. Many facts concerning the (adjoint) Weierstrass integral are known. We shall need only one of them – the fact that ${}_bWf$ as a function on R_b is continuous there.

Let f_1 stand for the function from $\mathscr{B}(B)$ with $f_1 \equiv 1$ on B and let g be a function on ∂R_b such that $g(x, b) = 1$ if $[x, b] \in (\partial E - B) \cap \partial R_b$ and $g(x, b) = 0$ elsewhere on ∂R_b. Consider now the sum $Wf_1 + {}_bWg$. For $[\xi, \tau] \in R_b$, $\tau \geqq a$, we have, provided $\tilde{v}(\xi, \tau) < \infty$,

$$(1.41) \qquad Wf_1(\xi, \tau) + {}_bWg(\xi, \tau) =$$

$$= \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\, \eta^{m/2-1}\{\Sigma_{f_1}^{\xi,\tau}(\theta^*, \eta) + g_{\xi,\tau}(\theta^*, \eta)\}\, \mathrm{d}\eta\,.$$

Let $[\xi, \tau] \in R_b$, $\tau \geqq a$ and suppose $\tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty) < \infty$. It is easy to see that if $[\xi, \tau] \in E$ then

$$\Sigma_{f_1}^{\xi,\tau}(\theta^*, \eta) + g_{\xi,\tau}(\theta^*, \eta) = 1$$

and if $[\xi, \tau] \notin \bar{E}$ then

$$\Sigma_{f_1}^{\xi,\tau}(\theta^*, \eta) + g_{\xi,\tau}(\theta^*, \eta) = 0\,.$$

Hence it follows that for $[\xi, \tau] \notin \bar{E}$, $\tau \geqq a$ (provided $\tilde{v}(\xi, \tau) < \infty$)

$$(1.42) \qquad Wf_1(\xi, \tau) + {}_bWg(\xi, \tau) = 0$$

and for $[\xi, \tau] \in E$,

$$(1.43) \quad Wf_1(\xi, \tau) + {}_bWg(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{\Gamma^*} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta} \eta^{m/2-1} \,\mathrm{d}\eta = 1\,.$$

Now let $[\xi, \tau] \in B$, $\tau < b$, $\tilde{v}(\xi, \tau) < \infty$. One can easily verify that if $\tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty) < < \infty$ then

$$\Sigma_{f_1}^{\xi,\tau}(\theta^*, \eta) + g_{\xi,\tau}(\theta^*, \eta) = -s(\theta^*, \eta, 0)$$

($s(\theta^*, \eta, 0)$ is defined in Subsection 1.4). Hence, for such $[\xi, \tau]$ we get

$$(1.44) \qquad Wf_1(\xi, \tau) + {}_bWg(\xi, \tau) = \tilde{P}_E(\xi, \tau)\,.$$

According to (1.44) and (1.43) we have, for $[\xi_0, \tau_0] \in B$, $\tau_0 < b$,

$$(1.45) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\in E}} \{Wf_1(\xi, \tau) + {}_bWg(\xi, \tau)\} =$$

$$= (Wf_1(\xi_0, \tau_0) + {}_bWg(\xi_0, \tau_0)) + (1 - \tilde{P}_E(\xi, \tau))$$

and, since ${}_bWg$ is continuous on R_b,

$$(1.46) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\in E}} Wf_1(\xi, \tau) = Wf_1(\xi_0, \tau_0) + (1 - \tilde{P}_E(\xi_0, \tau_0))\,.$$

In a similar way we obtain, using (1.42) and supposing, in addition, $[\xi_0, \tau_0] \in \in \overline{(R^{m+1} - \bar{E})}$,

$$(1.47) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\notin E, a\leqq\tau<b}} Wf_1(\xi, \tau) = Wf_1(\xi_0, \tau_0) - \tilde{P}_E(\xi_0, \tau_0)\,.$$

Now we are in position to prove the following assertion.

1.12. Proposition. *Suppose that the condition* (1.22) *is fulfilled and that*

$$(1.48) \qquad \partial(R^{m+1} - \bar{E}) \supset B\,.$$

Then there are finite limits

$$(1.49) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\in E}} Wf(\xi, \tau)\,, \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\notin E, a\leqq\tau<b}} Wf(\xi, \tau)$$

for every $f \in \mathscr{C}_0(B)$, $[\xi_0, \tau_0] \in B$, *if and only if*

$$(1.50) \qquad \tilde{V}_B = \sup \{\tilde{v}(\xi, \tau);\ [\xi, \tau] \in B\} < \infty\,.$$

If the condition (1.50) *is fulfilled then for any* $[\xi_0, \tau_0] \in B$ *and any* $f \in \mathscr{B}(B)$ *which is continuous at* $[\xi_0, \tau_0]$ *and* $f(\xi_0, \tau_0) = 0$ *in the case* $\tau_0 = b$ *we obtain*

$$(1.51) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\in E}} Wf(\xi, \tau) = Wf(\xi_0, \tau_0) + f(\xi_0, \tau_0)\left(1 - \tilde{P}_E(\xi_0, \tau_0)\right),$$

$$(1.52) \qquad \lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ [\xi,\tau]\notin E, a\leqq\tau<b}} Wf(\xi, \tau) = Wf(\xi_0, \tau_0) - f(\xi_0, \tau_0)\, \tilde{P}_E(\xi_0, \tau_0).$$

Proof. According to the Banach-Steinhaus theorem the condition (1.50) follows from the existence of finite limits (1.49) (also in virtue of the lower semicontinuity of $\tilde{v}$ on $R^{m+1} - R_a$).

With respect to the linearity of W and to (1.46), (1.47) it suffices to show that if $\tilde{V}_B < \infty$, $[\xi_0, \tau_0] \in B$, $f \in \mathscr{B}(B)$, $f(\xi_0, \tau_0) = 0$ and f is continuous at $[\xi_0, \tau_0]$ then there is a limit

$$\lim_{\substack{[\xi,\tau]\to[\xi_0,\tau_0] \\ \tau\geqq a}} Wf(\xi, \tau) = Wf(\xi_0, \tau_0).$$

However, this can be proved in a way quite similar to the proof of Theorem 3.10 in [21] (that is, by a decomposition of f into a sum $f_n + g_n$, where f_n is a function which vanishes on a neighbourhood of $[\xi_0, \tau_0]$ and $\|g_n\| \leqq 1/n$).

1.13. Now we are in position to state some properties of the operator H and to define an operator $\overline{W}$ which, as we shall see, is adjoint to H.

For $\mu \in \mathscr{B}'(B)$ the functional H_μ is defined on $\mathscr{D}_b$ by (0.11). According to the fact that for $[\xi, \tau] \in B$, $\varphi \in \mathscr{D}_b(\xi, \tau)$

$$(1.53) \qquad \langle \varphi, H_{\delta_{\xi,\tau}} \rangle = \tilde{W}\varphi(\xi, \tau)$$

it is easy to show by Lemma 1.2 that $H_{\delta_{\xi,\tau}}$ can be represented by a measure in R^{m+1} if and only if

$$(1.54) \qquad \tilde{v}(\xi, \tau) < \infty.$$

Suppose now that (1.54) is fulfilled and regard $H_{\delta_{\xi,\tau}}$ as a measure. This measure is uniquely determined by the condition

$$|H_{\delta_{\xi,\tau}}|\,(R^{m+1} - R_b) = 0.$$

It is easy to see (suppose still $[\xi, \tau] \in B$) that the support of $H_{\delta_{\xi,\tau}}$ is contained in B and, by the preceding,

$$|H_{\delta_{\xi,\tau}}|\,(B \cap \partial R_b) = 0.$$

Thus we see that for $[\xi, \tau] \in B$ the functional $H_{\delta_{\xi,\tau}}$ can be identified with a unique measure from $\mathscr{B}'_0(B)$ provided (1.54) is fulfilled. Furthermore, it is evident that if $\tau = b$ then $H_{\delta_{\xi,\tau}}$ is the zero measure. If (1.54) holds then for $\varphi \in \mathscr{D}_b$

$$\tilde{W}\varphi(\xi, \tau) = W\varphi(\xi, \tau) - \varphi(\xi, \tau)\, \tilde{P}_E(\xi, \tau)$$

(see (1.18)) and $W.(\xi, \tau)$ is also a measure (see Proposition 1.9), $W.(\xi, \tau)$ does not charge $[\xi, \tau]$ and the norm of $W.(\xi, \tau)$ is equal to $\frac{1}{2}\pi^{-m/2}\tilde{v}(\xi, \tau)$. This together with (1.53) gives

$$(1.55) \qquad \|H_{\delta_{\xi,\tau}}\| = \tfrac{1}{2}\pi^{-m/2}\,\tilde{v}(\xi, \tau) + \tilde{P}_E(\xi, \tau)\,.$$

We also know that for $\mu \in \mathscr{B}'(B)$, $\varphi \in \mathscr{D}_b$ (see (0.12)),

$$\langle \varphi, H_\mu \rangle = \int_B \langle \varphi, H_{\delta_{\xi,\tau}} \rangle \,\mathrm{d}\mu(\xi, \tau)\,.$$

By means of this equality (and of the preceding one) one can prove the following assertion. Its proof is analogous to the proof of Theorem 1.11 in [11] and we omit it.

1.14. Proposition. *H_μ can be represented by a measure for every $\mu \in \mathscr{B}'(B)$ if and only if*

$$(1.56) \qquad \tilde{V}_B = \sup\{\tilde{v}(\xi, \tau);\ [\xi, \tau] \in B\} < \infty\,.$$

If (1.56) is fulfilled then that measure is uniquely determined by the condition

$$(1.57) \qquad |H_\mu|\,(R^{m+1} - R_b) = 0$$

and then $H_\mu \in \mathscr{B}'_0(B)$. H regarded as an operator on $\mathscr{B}'_0$ ($H : \mu \mapsto H_\mu$, $H : \mathscr{B}'_0 \to \mathscr{B}'_0$) is then a bounded operator and

$$(1.58) \qquad \|H\| = \sup\{\tfrac{1}{2}\pi^{-m/2}\,\tilde{v}(\xi, \tau) + \tilde{P}_E(\xi, \tau);\ [\xi, \tau] \in B\}\,.$$

Note. In the last proposition H is regarded as an operator on $\mathscr{B}'_0$ (provided (1.56) holds) while H_μ is defined for any $\mu \in \mathscr{B}'$. But it is easily seen that if $\mu \in \mathscr{B}'$ is such that $|\mu|\,(B \cap R_b) = 0$ then $H_\mu = 0$ and the restriction of H from $\mathscr{B}'$ to $\mathscr{B}'_0$ is natural. Note also that for any $\xi \in R^m$ we have $\tilde{v}(\xi, b) = 0$ and $\tilde{P}_E(\xi, b) = 0$ and thus the supremum in (1.58) can be considered as the same supremum but taken over B_0.

1.15. Let us now suppose that the conditions (1.22) and (1.56) are fulfilled and that also the condition (1.48) is fulfilled. Then for $[\xi, \tau] \in B$, $f \in \mathscr{B}(B)$

$$(1.59) \qquad \langle f, H_{\delta_{\xi,\tau}} \rangle = Wf(\xi, \tau) - f(\xi, \tau)\,\tilde{P}_E(\xi, \tau)$$

(as $H_\mu \in \mathscr{B}'_0$ for any $\mu \in \mathscr{B}'$ under the condition $\tilde{V}_B < \infty$, $\langle f, H_\mu \rangle$ is then defined for each $f \in \mathscr{B}(B)$). Keeping the notation of Proposition 1.9 we have

$$Wf(\xi, \tau) = \int_B f \,\mathrm{d}\nu_{\xi,\tau}$$

and the equality (1.59) can be written in the form

$$(1.60) \qquad H_{\delta_{\xi,\tau}} = \nu_{\xi,\tau} - \tilde{P}_E(\xi, \tau)\,\delta_{\xi,\tau}\,.$$

Define now an operator $\overline{W}$ on $\mathscr{B}(B)$ by

$$\overline{W}f(\xi, \tau) = Wf(\xi, \tau) - f(\xi, \tau)\, \tilde{P}_E(\xi, \tau) . \tag{1.61}$$

Note, first, that for $f \in \mathscr{B}_0$, $\mu \in \mathscr{B}'_0$ we have

$$\langle f, H_\mu \rangle = \int_B \langle f, H_{\delta_{\xi,\tau}} \rangle \, d\mu(\xi, \tau) = \langle \overline{W}f, \mu \rangle , \tag{1.62}$$

that is, the operator H and $\overline{W}$ are ajoint to each other. Further, if $f \in \mathscr{C}_0(B)$ then by the above

$$\overline{W}f(\xi, \tau) = \lim_{\substack{[x,t] \to [\xi,\tau] \\ [x,t] \notin E, a \leqq t < b}} Wf(x, t) . \tag{1.63}$$

Hence it follows, among others, that for any $f \in \mathscr{C}_0(B)$ also $\overline{W}f \in \mathscr{C}_0(B)$. $\overline{W}$ can be regarded then either as an operator on $\mathscr{B}_0$ $(\overline{W} : \mathscr{B}_0 \to \mathscr{B}_0)$ or as an operator on $\mathscr{C}_0$ $(\overline{W} : \mathscr{C}_0 \to \mathscr{C}_0)$. Consider the equations

$$H_\mu = v , \tag{1.64}$$

$$\overline{W}f = g . \tag{1.65}$$

In (1.64), $v \in \mathscr{B}'_0$ is a given measure and the measure μ is unknown and assumed to belong also to $\mathscr{B}'_0$. If the equation (1.64) has a solution $\mu \in \mathscr{B}'_0$ then the heat potential U_μ considered on E can be regarded as a solution of the third boundary value problem on E with the boundary condition of the form

$$\Big(\sum_{j=1}^{m} n_j \, \partial_j U_\mu\Big) \mathscr{H}_{m-1} \otimes \mathscr{H}_1 + N_t U_\mu \mathscr{H}_m = v \tag{1.66}$$

(that is,

$$\Big(\sum_{j=1}^{m} n_j \, \partial_j U_\mu\Big) \, d\mathscr{H}_{m-1}(x) \, dt + N_t U_\mu \, d\mathscr{H}_m(x, t) = dv(x, t))$$

prescribed on B – see (0.11) in the introduction. This boundary characterization is, of course, weak.

In (1.65) $g \in \mathscr{C}_0(B)$ will be given and $f \in \mathscr{C}_0(B)$ unknown. The equations (1.64), (1.65) are adjoint to each other. If the equation (1.65) has a solution $f \in \mathscr{C}_0(B)$ then it is seen from (1.63) that the potential Wf considered on

$$R_{ab} - \bar{E}$$

is the classical solution of the first boundary value problem for the adjoint heat equation on the mentioned set with the boundary function g on B (and vanishing on $\partial(R_{ab} - \bar{E}) \cap \partial R_b$).

In the following section we shall investigate the equations (1.64), (1.65) together with the integral equations corresponding to the third boundary value problem for the heat equation on E with a little more general boundary condition then (1.66).

Keep all the notations from the preceding sections. Throughout this section we shall always suppose that the conditions (1.22), (1.56) and (1.48) are fulfilled. For $\alpha \in R^1$ let us define operators H_α, W_α acting on $\mathscr{B}'_0$, $\mathscr{C}_0$, respectively, by

$$H_\alpha = H + \alpha I\,, \quad W_\alpha = \overline{W} + \alpha I\,, \tag{2.1}$$

where I stands for the identity operator (on $\mathscr{B}'_0$ and $\mathscr{C}_0$, respectively). In what follows we shall consider the equations (1.64), (1.65) in the form

$$\alpha\left(\frac{1}{\alpha} H_\alpha - I\right)(\mu) = \nu\,, \quad \alpha\left(\frac{1}{\alpha} W_\alpha - I\right)(f) = g \tag{2.2}$$

($\alpha \neq 0$) and in this connection it will be useful to investigate the operators H_α, W_α.

It follows from (1.61) and the definition of W_α that for $f \in \mathscr{C}_0$ (or $f \in \mathscr{B}_0$), $[\xi, \tau] \in B$,

$$W_\alpha f(\xi, \tau) = Wf(\xi, \tau) + f(\xi, \tau)\,(\alpha - \tilde{P}_E(\xi, \tau))\,. \tag{2.3}$$

Among other it follows from (2.3), Proposition 1.9 and the assumption (1.56) that W_α is a bounded operator and that

$$\|W_\alpha\| = \sup\,\{\tfrac{1}{2}\pi^{-m/2}\tilde{v}(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|;\ [\xi, \tau] \in \mathscr{B}_0\}\,. \tag{2.4}$$

We shall evaluate the Fredholm radius of the operator W_α in this paragraph. For this purpose let us introduce the following notation. For $r > 0$, $[\xi, \tau] \in B$ let

$$B_r(\xi, \tau) = \hat{B}_0 - \Omega(\xi, \tau; r)\,, \tag{2.5}$$

where $\hat{B}_0 = \hat{B} \cap B_0 = \hat{B} \cap R_b$ ($\hat{B}$ is defined by (1.23)). Define an operator Z_r on $\mathscr{C}_0(B)$ (Z_r can be considered also on $\mathscr{B}$) by putting for $f \in \mathscr{C}_0$ ($f \in \mathscr{B}$), $[\xi, \tau] \in B$

$$Z_r f(\xi, \tau) = \int_{B_r(\xi,\tau)} f\, \mathrm{d}\nu_{\xi,\tau} = \tag{2.6}$$

$$= \int_{B_r(\xi,\tau)} f(x, t)\,\{N_t G(x - \xi,\, t - \tau) - \sum_{j=1}^{m} N_j\, \partial_j G(x - \xi, t - \tau)\}\, \mathrm{d}\mathscr{H}_m(x, t)$$

(for the definition of $\nu_{\xi,\tau}$ see Proposition 1.9).

2.1. Lemma. *Given $r > 0$, define for $0 < \delta < r$*

$$q_r(\delta) = \sup_{[\xi,\tau]\in B}\,\{\mathscr{H}_m(\hat{B}_0 \cap [\Omega(\xi, \tau; r + \delta) - \Omega(\xi, \tau; r - \delta)])\}\,. \tag{2.7}$$

Then there is a constant c such that for every $f \in \mathscr{B}(B)$ with $\|f\| \leqq 1$, $[\xi, \tau]$, $[\xi', \tau'] \in B$ with $0 < |[\xi, \tau] - [\xi', \tau']| < r$ the inequality

$$|Z_r f(\xi, \tau) - Z_r(\xi', \tau')| \leqq c[q_r(|[\xi, \tau] - [\xi', \tau']|) + |[\xi, \tau] - [\xi', \tau']|] \tag{2.8}$$

is valid.

Proof. It is easily seen that there is a constant $c_1 \in R^1$ such that for any $[x, t]$, $[x', t'] \in R^{m+1}$ with $|[x, t]| \geqq r$, $|[x', t']| \geqq r$ the following two inequalities are valid:

$$G(x, t) + \sum_{j=1}^{m} |\partial_j G(x, t)| \leqq c_1 , \tag{2.9}$$

$$|G(x, t) - G(x', t')| + \sum_{j=1}^{m} |\partial_j G(x, t) - \partial_j G(x', t')| \leqq c_1 |[x, t] - [x', t']| . \tag{2.10}$$

Let $[\xi, \tau]$, $[\xi', \tau'] \in B$ and denote

$$M_1 = B_r(\xi, \tau) \cap B_r(\xi', \tau') , \quad M_2 = B_r(\xi, \tau) - B_r(\xi', \tau') ,$$
$$M_3 = B_r(\xi', \tau') - B_r(\xi, \tau) .$$

Then for $f \in \mathscr{B}(B)$ we have

$$\begin{aligned} Z_r f(\xi, \tau) - Z_r f(\xi', \tau') = \\ = \int_{M_1} f \,\mathrm{d}\nu_{\xi,\tau} - \int_{M_1} f \,\mathrm{d}\nu_{\xi',\tau'} + \int_{M_2} f \,\mathrm{d}\nu_{\xi,\tau} - \int_{M_3} f \,\mathrm{d}\nu_{\xi',\tau'} = \\ = \int_{M_1} f(x, t) \{N_t(G(x - \xi, t - \tau) - G(x - \xi', t - \tau')) - \\ - \sum_{j=1}^{m} N_j(\partial_j G(x - \xi, t - \tau) - \partial_j G(x - \xi', t - \tau'))\} \,\mathrm{d}\mathscr{H}_m(x, t) + \\ + \int_{M_2} f(x, t) \{N_t G(x - \xi, t - \tau) - \sum_{j=1}^{m} N_j \,\partial_j G(x - \xi, t - \tau)\} \,\mathrm{d}\mathscr{H}_m(x, t) - \\ - \int_{M_3} f(x, t) \{N_t G(x - \xi', t - \tau') - \sum_{j=1}^{m} N_j \,\partial_j G(x - \xi', t - \tau')\} \,\mathrm{d}\mathscr{H}_m(x, t) = \\ = I_1 + I_2 + I_3 . \end{aligned} \tag{2.11}$$

Suppose now that $\|f\| \leqq 1$. Then it follows from (2.10) that

$$|I_1| \leqq c_1 \,\mathscr{H}_m(\hat{B}_0) \,|[\xi, \tau] - [\xi', \tau']| . \tag{2.12}$$

Writing

$$\delta = |[\xi, \tau] - [\xi', \tau']| ,$$

let us suppose that $\delta < r$. Then

$$M_2 \cup M_3 \subset [\Omega(\xi, \tau;\ r + \delta) - \Omega(\xi, \tau;\ r - \delta)] \cap \hat{B}_0$$

and from (2.9) we get

$$|I_2| + |I_3| \leqq c_1 \mathscr{H}_m(M_2 \cup M_3) \leqq c_1 \, q_r(\delta) . \tag{2.13}$$

Now it suffices to put

$$c = c_1 \max \{1, \mathscr{H}_m(\hat{B}_0)\}$$

and the assertion follows from (2.11), (2.12) and (2.13).

2.2. Corollary. *Let* $r > 0$ *be such that*

$$\mathscr{H}_m(\hat{B}_0 \cap \Gamma(\xi, \tau; r)) = 0 \tag{2.14}$$

for each $[\xi, \tau] \in B$. *Then*

$$Z_r : \mathscr{C}_0(B) \to \mathscr{C}_0(B)$$

and Z_r *(as an operator on* $\mathscr{C}_0$*) is a compact operator.*

Proof. It suffices to take notice of the fact that (2.14) implies

$$\lim_{\delta \to 0+} q_r(\delta) = 0 .$$

It follows then from Lemma 2.1 that the set

$$\{Z_r f;\ f \in \mathscr{C}_0,\ \|f\| \leqq 1\}$$

is a set of equicontinuous functions (on B and belonging to $\mathscr{C}_0(B)$). It is also easy to see that functions from this set are bounded by a common constant. Consequently, Z_r is a compact operator on $\mathscr{C}_0$.

2.3. Notation. In what follows let ωA stand for the reciprocal value of the Fredholm radius of an operator A, that is, if $A : \mathscr{C}_0(B) \to \mathscr{C}_0(B)$ is a linear operator then

$$\omega A = \inf_Q \|A - Q\| , \tag{2.15}$$

where the infimum is taken over all compact operators Q on $\mathscr{C}_0(B)$. In the sequel it will be important to know the value ωW_α for the above defined operators W_α.

2.4. Lemma. *For any* $\alpha > 0$,

$$\omega W_\alpha \leqq \lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2}\pi^{-m/2} \tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\Big) . \tag{2.16}$$

Proof. As $\mathscr{H}_m(\hat{B}_0) < \infty$ (under the assumption (1.22)) there are at most countably many $r > 0$ with the property that there is a $[\xi, \tau] \in B$ such that

$$\mathscr{H}_m(\hat{B}_0 \cap \Gamma(\xi, \tau; r)) \neq 0 .$$

Thus there is a sequence $\{r_i\}$, $r_i > 0$, $r_i \to 0$ for $i \to +\infty$, such that for each $i = 1, 2, \ldots$ and any $[\xi, \tau] \in B$,

$$\mathscr{H}_m(\hat{B}_0 \cap \Gamma(\xi, \tau; r_i)) = 0 .$$

According to Corollary 2.2 the operators Z_{r_i} are compact and hence

$$\omega W_\alpha \leqq \inf_i \|W_\alpha - Z_{r_i}\| . \tag{2.17}$$

Since

$$[\Omega(\xi, \tau; r) - \bar{R}_\tau] \subset [\Omega^*(\xi; r) \times (\tau, \tau + r)] ,$$

we have (in accordance with the definitions of W_α and Z_r)

$$\|W_\alpha - Z_{r_i}\| = \tag{2.18}$$

$$= \sup_{[\xi,\tau]\in B_0} \left\{ \int_{B_0 \cap \Omega(\xi,\tau;r_i)} \left| N_t G(x - \xi, t - \tau) - \sum_{j=1}^m N_j \partial_j G(x - \xi, t - \tau) \right| d\mathscr{H}_m(x, t) + \right.$$

$$\left. + \left|\alpha - \tilde{P}_E(\xi, \tau)\right| \right\} \leqq \sup_{[\xi,\tau]\in B_0} \{\tfrac{1}{2}\pi^{-m/2}\, \tilde{v}^{r_i}(\xi, \tau) + \left|\alpha - \tilde{P}_E(\xi, \tau)\right|\}$$

(see also (1.31)). For $0 < r \leqq r'$ we have

$$\tilde{v}^r(\xi, \tau) \leqq \tilde{v}^{r'}(\xi, \tau)$$

and now it is seen that the assertion immediately follows from (2.18), (2.17).

One can prove the following assertion in the same way as Lemma 3.4 from [11]

2.5. Lemma. *Let Q be a compact operator on $\mathscr{C}_0(B)$. Then for any $\varepsilon > 0$ there are $f_1, \ldots, f_n \in \mathscr{C}_0(B)$, $\mu_1, \ldots, \mu_n \in \mathscr{B}'_0$ such that for the operator Q_ε,*

$$Q_\varepsilon f = \sum_{j=1}^n \langle f, \mu_j \rangle f_j , \quad (f \in \mathscr{C}_0) , \tag{2.19}$$

the inequality

$$\|Q - Q_\varepsilon\| \leqq \varepsilon \tag{2.20}$$

is valid.

The following assertion is an analogue of Lemma 3.5 from [11].

2.6. Lemma. *For any $\alpha > 0$,*

$$\omega W_\alpha \geqq \lim_{r\to 0+} \left(\sup_{[\xi,\tau]\in B_0} \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, \tau) + \left|\alpha - \tilde{P}_E(\xi, \tau)\right|\}\right) . \tag{2.21}$$

Proof. For $r > 0$, $[\xi, \tau] \in B$ let us define

$$w^r(\xi, \tau) = \left|\nu_{\xi,\tau}\right| (\Omega(\xi, \tau; r)) . \tag{2.22}$$

Let $r > 0$ be such that for each $[\xi, \tau] \in B$,

$$\mathscr{H}_m(\hat{B}_0 \cap \Gamma(\xi, \tau; r)) = 0 .$$

For $f \in \mathscr{C}_0(B)$, $[\xi, \tau] \in B$ we have

$$(2.23) \qquad (W_\alpha - Z_r) f(\xi, \tau) = \int_{B \cap \Omega(\xi,\tau;r)} f \, \mathrm{d}v_{\xi,\tau} + f(\xi, \tau)\,(\alpha - \tilde{P}_E(\xi, \tau))$$

and for a fixed $f (\in \mathscr{C}_0)$ the term in (2.23) is a continuous function of the variables $[\xi, \tau]$ on B. Hence it is seen that

$$(2.24) \quad \sup \{(W_\alpha - Z_r) f(\xi, \tau);\ f \in \mathscr{C}_0,\ \|f\| \leqq 1\} = w^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|$$

and that the term on the right hand side of (2.24) is a lower semicontinuous function of the variables $[\xi, \tau]$ on B.

Now it can be proved in the same way as in the proof of Lemma 3.5 from [11] (by means of Lemma 2.5) that for every $k > \omega W_\alpha$ the inequality

$$(2.25) \qquad \lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{w^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\Big) \leqq k$$

is valid. For $r_1 = \frac{1}{2} r \sqrt{2}$ we have

$$\Omega^*(\xi; r_1) \times (\tau, \tau + r_1) \subset \Omega(\xi, \tau; r)$$

which yields

$$(2.26) \qquad \tfrac{1}{2} \pi^{-m/2} \tilde{v}^{r_1}(\xi, \tau) \leqq w^r(\xi, \tau).$$

According to (2.26) and (2.25)

$$\lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2} \pi^{-m/2} \tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\Big) \leqq k$$

and the assertion follows.

The following assertion is an immediate consequence of Lemmas 2.4 and 2.6.

2.7. Proposition. *For any $\alpha > 0$ the equality*

$$(2.27) \qquad \omega W_\alpha = \lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2} \pi^{-m/2} \tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\Big)$$

is valid.

3. OPERATORS L, A, V

In connection with the third boundary value problem for the heat equation on E we shall introduce and study operators L, A, V in this section. In the case of cylindrical sets in R^{m+1} these operators have been studied in [15]. The case of time moving boundary but with $m = 1$ has been investigated in [5], II.

Throughout this section let $\lambda \in \mathscr{B}_0'(B)$ be fixed. For this fixed λ let us define operators L, A in the following way.

For $\mu \in \mathscr{B}'(B)$ denote

$$\mathscr{D}(\mu) = \left\{\varphi \in \mathscr{D}_b;\ \int_B |\varphi(x, t)|\ U_{|\mu|}(x, t)\ \mathrm{d}|\lambda|\ (x, t) < \infty\right\}$$

(here $U_{|\mu|}$ is the heat potential of the total variation $|\mu|$ of μ). For a given $\mu \in \mathscr{B}'(B)$ let L_μ stand for the functional on $\mathscr{D}(\mu)$ defined by

$$\langle \varphi, L_\mu \rangle = \int_B \varphi(x, t)\ U_\mu(x, t)\ \mathrm{d}\lambda(x, t)\,, \quad \varphi \in \mathscr{D}(\mu)\,. \tag{3.1}$$

Further, put

$$A_\mu = H_\mu + L_\mu\,, \tag{3.2}$$

that is, for $\varphi \in \mathscr{D}(\mu)$,

$$\langle \varphi, A_\mu \rangle = \langle \varphi, H_\mu \rangle + \langle \varphi, L_\mu \rangle\,.$$

By the introductory remarks (see (0.11)) A_μ can be regarded as a weak characterization of the term

$$(-\sum_{j=1}^{m} n_j\ \partial_j U_\mu)\ \mathscr{H}_{m-1} \otimes \mathscr{H}_1 + N_t U_\mu \mathscr{H}_m + U_\mu \lambda \tag{3.3}$$

on B. Note that in the terms

$$(-\sum_{j=1}^{m} n_j\ \partial_j U_\mu)\ \mathscr{H}_{m-1} \otimes \mathscr{H}_1 \quad \text{and} \quad N_t U_\mu \mathscr{H}_m$$

in (3.3) we mean by $\partial_j U_\mu$ and U_μ on B the "boundary limits" of $\partial_j U_\mu$ and U_μ from within E, while in the term $U_\mu \lambda$ in (3.3) U_μ means the actual values of U_μ on B. Our aim is to find some conditions under which the equation

$$A_\mu = \nu \tag{3.4}$$

has a solution $\mu \in \mathscr{B}'_0$ for any given $\nu \in \mathscr{B}'_0$. If $\mu \in \mathscr{B}'_0$ is a solution of (3.4) and if U_μ is in a sense continuous on $E \cup B$ (that is, if the values of U_μ on B are equal, in a sense, to the boundary limits of U_μ from within E) then the heat potential U_μ on E can be considered a solution of the third boundary value problem for the heat equation on E with the boundary condition

$$(-\sum_{j=1}^{m} n_j\ \partial_j U_\mu)\ \mathscr{H}_{m-1} \otimes \mathscr{H}_1 + U_\mu(N_t \mathscr{H}_m + \lambda) = \nu \tag{3.5}$$

(that is,

$$(-\sum_{j=1}^{m} n_j\ \partial_j U_\mu)\ \mathrm{d}\mathscr{H}_{m-1}(x)\ \mathrm{d}t + U_\mu[N_t\ \mathrm{d}\mathscr{H}_m(x, t) + \mathrm{d}\lambda(x, t)] = \mathrm{d}\nu(x, t))$$

considered on B.

Let us take notice of the following simple fact we shall frequently need in the sequel. If $\mu, \lambda \in \mathscr{B}'(B)$ are not negative or such that

$$\int_B U_{|\mu|}(x, t)\, \mathrm{d}|\lambda|\,(x, t) < \infty$$

then it follows from the Fubini theorem that

$$\int_B U_\mu(x, t)\, \mathrm{d}\lambda(x, t) = \int_B U^*_\lambda(\xi, \tau)\, \mathrm{d}\mu(\xi, \tau)\,. \tag{3.6}$$

First we shall find some conditions under which the functional A_μ can be represented by a unique measure from $\mathscr{B}'_0(B)$. If A_μ can be represented by a unique measure from $\mathscr{B}'_0(B)$ then, particularly, A_μ has a unique linear extension from $\mathscr{D}(\mu)$ to $\mathscr{D}_b$. The proof of the following assertion is quite analogous to the proof of Proposition 3.1 from [5] and we omit it.

3.1. Proposition. *The following two conditions are equivalent to each other:*

(i) *For any $\mu \in \mathscr{B}'_0$ there is a unique linear extension of A_μ from $\mathscr{D}(\mu)$ to $\mathscr{D}_b$.*

(ii) *The potential $U^*_{|\lambda|}$ is bounded on any compact set contained in B_0 (B_0 is defined by (0.7)).*

3.2. Lemma. *There is a number $\gamma > 0$ with the following property: For any $\tau_0 < b$ there is a $\varphi_{\tau_0} \in \mathscr{D}_b$ such that $0 \leqq \varphi_{\tau_0} \leqq 1$ in R^{m+1}, $\varphi_{\tau_0} = 1$ on $B \cap R_{\tau_0}$ and*

$$|\langle \varphi_{\tau_0}, H_{\delta_{\xi,\tau}} \rangle| < \gamma \tag{3.7}$$

for every $[\xi, \tau] \in B_0$.

Proof of this lemma is analogous to the proof of Lemma 3.2 from [5] but we present it here for completion.

Let $\psi_1 : R^1 \to R^1$ be an infinitely differentiable function with compact support such that $0 \leqq \psi_1 \leqq 1$ on R^1, $\operatorname{spt} \psi_1 \subset (-\infty, b)$, $\psi_1 = 1$ on $\langle a, \tau_0 \rangle$, $\psi'_1 \leqq 0$ on (τ_0, b). Let $\varrho > \sup \{|x|; [x, t] \in B\}$ and let $\psi_2 : R^m \to R^1$ be infinitely differentiable with compact support in R^m, such that $0 \leqq \psi_2 \leqq 1$ on R^m, $|\partial_j \psi_2| \leqq 1$ $(j = 1, 2, \ldots$ $\ldots, m)$ on R^m, $\psi_2 = 1$ on $\Omega^*(0; \varrho)$. Define φ_{τ_0} by

$$\varphi_{\tau_0}(x, t) = \psi_1(t)\, \psi_2(x)\,, \quad ([x, t] \in R^{m+1})\,.$$

Then $\varphi_{\tau_0} \in \mathscr{D}_b$, $\varphi_{\tau_0} = 1$ on $B \cap R_{\tau_0}$, $0 \leqq \varphi_{\tau_0} \leqq 1$ on R^{m+1}.

Given $[\xi, \tau] \in B_0$ then

$$|\langle \varphi_{\tau_0}, H_{\delta_{\xi,\tau}} \rangle| \leqq \iint_E \psi_1(t) \sum_{j=1}^m |\partial_j G(x - \xi, t - \tau)|\, |\partial_j \psi_2(x)|\, \mathrm{d}x\, \mathrm{d}t + \tag{3.8}$$

$$+ \iint_E \psi_2(x)\, |\psi'_1(t)|\, G(x - \xi, t - \tau)\, \mathrm{d}x\, \mathrm{d}t = I_1 + I_2\,.$$

Since $|\partial_j \psi_2| \leqq 1$ $(j = 1, \ldots, m)$, $|\psi_1| \leqq 1$ then (0.9) yields

(3.9) $$I_1 \leqq \frac{2m}{\sqrt{\pi}} \sqrt{(b - a)}\,.$$

Put, for a while, $E_\tau = E - \bar{R}_\tau$. Then

$$I_2 = \iint_{E_\tau} [4\pi(t - \tau)]^{-m/2} \exp\left(-\frac{|x - \xi|^2}{4(t - \tau)}\right) \psi_2(x) \, |\psi_1'(t)| \, dx \, dt\,.$$

Denote $\tau_1 = \sup\{t; \psi_1(t) \neq 0\}$. If $\tau \geqq \tau_1$ then $I_2 = 0$. Suppose that $\tau < \tau_1$. Then

$$I_2 \leqq \int_\tau^{\tau_1} |\psi_1'(t)| \, [4\pi(t - \tau)]^{-m/2} \int_{R^m} \exp\left(-\frac{|x - \xi|^2}{4(t - \tau)}\right) dx \Big\} \, dt\,.$$

Since

$$\int_{R^m} \exp\left(-\frac{|x - \xi|^2}{4(t - \tau)}\right) dx = [4\pi(t - \tau)]^{m/2}$$

and since $\psi_1' \leqq 0$ on $\langle a, b\rangle$ (and at the same time $0 \leqq \psi_1 \leqq 1$), we have

(3.10) $$I_2 \leqq \int_\tau^{\tau_1} |\psi_1'(t)| \, dt \leqq 1\,.$$

By (3.8), (3.9) and (3.10) it suffices to put

$$\gamma = 1 + \frac{2m}{\sqrt{\pi}} \sqrt{(b - a)}\,.$$

3.3. In what follows we shall denote

$$m_\lambda^* = \sup\{U_{|\lambda|}^*(x, t); \; [x, t] \in B_0\}\,,$$
$$\tilde{V}_B = \sup\{\tilde{v}(x, t); \; [x, t] \in B\}\,.$$

Using Lemma 3.2 one can prove the following assertion in the same way as Theorem 3.4 from [5].

3.4. Theorem. *Suppose that λ $(\in \mathscr{B}_0')$ is non-negative. Then the following two conditions are equivalent to each other:*

(i) *For any $\mu \in \mathscr{B}_0'$ there is a unique $\nu_\mu \in \mathscr{B}_0'$ representing A_μ on $\mathscr{D}_b$, that is,*

(3.11) $$\langle \varphi, A_\mu \rangle = \int_B \varphi \, d\nu_\mu\,, \quad \varphi \in \mathscr{D}_b\,,$$

(ii) $$\tilde{V}_B + m_\lambda^* < \infty\,.$$

Let us now define an operator V in the following way. For $f \in \mathscr{B}(B)$, $[x, t] \in B$ put

$$(3.12) \qquad Vf(x, t) = U^*_{f\lambda}(x, t) = \int_B f(\xi, \tau)\, G^*(x - \xi,\ t - \tau)\, \mathrm{d}\lambda(\xi, \tau) =$$
$$= \int_{B - R_t} f(\xi, \tau)\, [2\pi(\tau - t)]^{-m/2} \exp\left(-\frac{|x - \xi|^2}{4(\tau - t)}\right) \mathrm{d}\lambda(\xi, \tau),$$

provided the integrals in (3.12) exist ($f\lambda$ stands for the product of the function f and the measure λ).

Note that if $m^*_\lambda < \infty$ then for $f \in \mathscr{B}(B)$ the function Vf is bounded on B. If even $U^*_\lambda|_B$ is continuous on B then $Vf \in \mathscr{C}_0(B)$ for any $f \in \mathscr{C}(B)$ and hence V can be regarded as an operator on $\mathscr{C}(B)$ or on $\mathscr{C}_0(B)$ (that is, $V: \mathscr{C} \to \mathscr{C}$ or $V: \mathscr{C}_0 \to \mathscr{C}_0$).

3.5. Proposition. *Suppose $\lambda \in \mathscr{B}'_0$ is non-negative. Then the following two conditions are equivalent to each other:*

(i) *$Vf \in \mathscr{C}_0(B)$ for each $f \in \mathscr{C}_0(B)$.*

(ii) *$U^*_\lambda|_{B_0}$ is continuous and bounded on B_0.*

Proof of this assertion is quite analogous to the proof of Proposition 4.1 from [5] and we omit it.

3.6. Remark. Suppose $\lambda \in \mathscr{B}'_0$ is such that the restriction $U^*_{|\lambda|}|_{B_0}$ is continuous and bounded on B_0. Then $U^*_{\lambda^+}|_{B_0}$, $U^*_{\lambda^-}|_{B_0}$ are continuous and bounded on B_0, too (see Corollary 0.2). Since $m^*_\lambda < \infty$, for any $\mu \in \mathscr{B}'_0$ the functional L_μ can be represented by a unique measure from $\mathscr{B}'_0$. In this case we identify L_μ with this representing measure and L can be then regarded as an operator on $\mathscr{B}'_0(B)$ ($\mu \mapsto L_\mu$). For $\mu \in \mathscr{B}'_0$, $f \in \mathscr{C}_0(B)$ the following equality follows from the Fubini theorem:

$$(3.13) \qquad \langle Vf, \mu \rangle = \int_B f(\xi, \tau)\, U_\mu(\xi, \tau)\, \mathrm{d}\lambda(\xi, \tau) = \langle f, L_\mu \rangle\,.$$

This means that the operators L and V (on $\mathscr{B}'_0$ and on $\mathscr{C}_0$, respectively; $f \mapsto Vf$, $V: \mathscr{C}_0 \to \mathscr{C}_0$) are adjoint to each other.

The following assertion can be proved in the same way as Proposition 4.3 in [5]; we omit the proof here.

3.7. Proposition. *Suppose $\lambda \in \mathscr{B}'_0$ is non-negative and such that $U^*_\lambda|_{B_0}$ is continuous and bounded on B_0. Then the operator V is compact (operator on $\mathscr{C}_0$) if and only if $U^*_\lambda|_B$ is continuous on B.*

The following auxiliary assertion is an analogue of Lemma 4.4 from [5]; we present its proof here for completion.

3.8. Lemma. *Let $\lambda \in \mathscr{B}'_0(B)$ be non-negative and such that $U^*_\lambda/_B$ is continuous on B. Suppose in addition that for any $t \in R^1$, $\lambda(\partial R_t) = 0$. For each $\delta > 0$, $t \in R^1$ put*

$$\lambda_{t,\delta} = \lambda|_{(R_{t+\delta} - R_t)}$$

and define a function S_δ on B by

$$S_\delta(x, t) = U^*_{\lambda_{t,\delta}}(x, t) \quad ([x, t] \in B).$$

Then for any $\delta > 0$, S_δ is continuous on B and

$$\lim_{\delta \to 0+} (\sup \{S_\delta(x, t);\ [x, t] \in B\}) = 0. \tag{3.14}$$

Proof. Since we suppose $\lambda(\partial R_t) = 0$ for any $t \in R^1$ and since U^*_λ is finite on B, we have for each $[x, t] \in B$

$$S_\delta(x, t) \to 0$$

for $\delta \to 0+$ and this convergence is monotonous. B is compact and thus it suffices, according to the Dini theorem, to show that for any fixed $\delta > 0$ S_δ is continuous on B.

If $[x, t]$, $[x_1, t_1] \in B$, $t_1 \geqq t$, then

$$|S_\delta(x, t) - S_\delta(x_1, t_1)| = |U^*_{\lambda_{t,\delta}}(x, t) - U_{\lambda_{t_1,\delta}}(x_1, t_1)| = \tag{3.15}$$

$$= \left| U^*_{\lambda_{t,\delta}}(x, t) - U^*_{\lambda_{t,\delta}}(x_1, t_1) - \int_{R_{t_1+\delta} - R_{t+\delta}} G^*(x_1 - \xi,\ t_1 - \tau)\, \mathrm{d}\lambda(\xi, \tau) \right|$$

$(G^*(x_1 - \xi,\ t_1 - \tau) = 0$ for $\tau \leqq t_1$, $\xi \in R^m)$. $U^*_\lambda|_B$ is supposed to be continuous, so $U^*_{\lambda_{t,\delta}}$ is continuous (t fixed). It means that for a given $\varepsilon > 0$ there is a $\delta' > 0$, $\delta' < \delta$, such that

$$|U^*_{\lambda_{t,\delta}}(x, t) - U^*_{\lambda_{t,\delta}}(x_1, t_1)| < \tfrac{1}{2}\varepsilon \tag{3.16}$$

for any $[x_1, t_1] \in B \cap \Omega(x, t; \delta')$. δ' can be chosen such that

$$[4\pi(\delta - \delta')]^{-m/2}\, \lambda(R_{t+\delta+\delta'} - R_{t+\delta}) < \tfrac{1}{2}\varepsilon$$

(for $\lambda(\partial R_{t+\delta}) = 0$). For $[x_1, t_1] \in R^{m+1}$, $t \leqq t_1 < t + \delta'$, $[\xi, \tau] \in R_{t_1+\delta} - R_{t+\delta}$ we have

$$G^*(x_1 - \xi,\ t_1 - \tau) \leqq [4\pi(t_1 - \tau)]^{-m/2} \leqq [4\pi(\delta - \delta')]^{-m/2}.$$

Hence

$$\int_{R_{t_1+\delta} - R_{t+\delta}} G^*(x_1 - \xi,\ t_1 - \tau)\, \mathrm{d}\lambda(\xi, \tau) \leqq \tag{3.17}$$

$$\leqq [4\pi(\delta - \delta')]^{-m/2}\, \lambda(R_{t_1+\delta} - R_{t+\delta}) \leqq [4\pi(\delta - \delta')]^{-m/2}\, \lambda(R_{t+\delta+\delta'} - R_{t+\delta}) < \tfrac{1}{2}\varepsilon.$$

From (3.15), (3.16) and (3.17) it follows that

$$|S_\delta(x, t) - S_\delta(x_1, t_1)| < \varepsilon$$

for $[x_1, t_1] \in B \cap \Omega(x, t; \delta')$, $t_1 \geqq t$ ($[x, t]$ fixed).

Similarly in the case $t_1 \leqq t$; let $t - \delta < t_1 \leqq t$. As $G^*(x - \xi,\ t - \tau) = 0$ if $\tau \leqq t$ and $G^*(x_1 - \xi,\ t_1 - \tau) = 0$ if $\tau \leqq t_1$ ($\xi \in R^m$), we have

$$U^*_{\lambda_{t,\delta}}(x, t) = U^*_{\lambda_{t-\delta,2\delta}}(x, t),$$

$$U^*_{\lambda_{t_1,\delta}}(x_1, t_1) = U^*_{\lambda_{t-\delta,2\delta}}(x_1, t_1) - \int_{R_{t+\delta} - R_{t_1+\delta}} G^*(x_1 - \xi, t_1 - \tau)\, d\lambda(\xi, \tau)$$

and hence

$$|S_\delta(x, t) - S_\delta(x_1, t_1)| \leqq \tag{3.18}$$

$$\leqq |U^*_{\lambda_{t-\delta,2\delta}}(x, t) - U^*_{\lambda_{t-\delta,2\delta}}(x_1, t_1)| + \int_{R_{t+\delta} - R_{t_1+\delta}} G^*(x_1 - \xi, t_1 - \tau)\, d\lambda(\xi, \tau).$$

$U^*_{\lambda_{t-\delta,2\delta}}|_B$ is continuous on B (t fixed) and the last integral in (3.18) can be estimated in a way similar to the preceding one. Consequently, S_δ is continuous on B.

4. THE EQUATION $A_\mu = v$

4.1. Henceforth we shall suppose that the condition (1.22) is fulfilled for some α' with (1.19). Note once more that then

$$\mathscr{H}_m(\hat{B}) < \infty$$

($\hat{B}$ is defined by (1.23)).

Further, we shall suppose henceforth that the conditions (1.48) and (1.50) are fulfilled, that the restriction $U^*_{|\lambda|}|_B$ is continuous on B ($\lambda \in \mathscr{B}'_0$ fixed) and that $|\lambda| (\partial R_t) = 0$ for each $t \in R^1$. Then $A_\mu \in \mathscr{B}'_0$ for any $\mu \in \mathscr{B}'_0$ by Theorem 3.4 (write $A_\mu = v_\mu$ – see (3.11)) and A is an operator acting on $\mathscr{B}'_0$ ($A : \mu \mapsto A_\mu$, $A : \mathscr{B}'_0 \to \mathscr{B}'_0$). For any $f \in \mathscr{C}_0$ we have $Vf \in \mathscr{C}_0$ (see Remark 3.6). Note that (1.62) and (3.13) imply that for $f \in \mathscr{C}_0(B)$, $\mu \in \mathscr{B}'_0(B)$,

$$\langle f, A_\mu \rangle = \langle \overline{W}f, \mu \rangle + \langle Lf, \mu \rangle, \tag{4.1}$$

that is, the operators A and $(\overline{W} + V)$ are adjoint to each other.

We shall investigate the following two equations which are adjoint to each other:

$$A_\mu = v$$

($v \in \mathscr{B}'_0$ is given, $\mu \in \mathscr{B}'_0$ unknown),

$$(\overline{W} + V) f = g$$

($g \in \mathscr{C}_0$ is given, $f \in \mathscr{C}_0$ unknown).

4.2. Lemma. *Suppose that the assumptions from* 4.1 *concerning E and λ are fulfilled and suppose that there is an $\alpha > 0$ such that*

$$\frac{1}{\alpha} \lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{ \tfrac{1}{2}\pi^{-m/2}\, \tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)| \} \Big) < 1. \tag{4.2}$$

Then the equation

$$(\overline{W} + V)f = 0$$

has in $\mathscr{C}_0$ only the trivial solution.

Proof. For $r > 0$, $\tau \in R^1$ let us denote

$$\lambda_{\tau,r} = |\lambda| |_{(R_{\tau+r} - R_\tau)}$$

and define a function S_r on B by

$$S_r(\xi, \tau) = U^*_{\lambda_{\tau,r}}(\xi, \tau), \quad ([\xi, \tau] \in B).$$

According to the assumption (4.2) and to Lemma 3.8 there is an $r_0 > 0$ such that

$$(4.3) \qquad \gamma = \frac{1}{\alpha} \sup_{[\xi,\tau]\in B_0} \{\tfrac{1}{2}\pi^{-m/2}\, \tilde{v}_{r_0}(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau) + S_{r_0}(\xi, \tau)\} < 1 .$$

We can write W in the form

$$(4.4) \qquad \overline{W} = \alpha\left(\frac{1}{\alpha} W_\alpha - I\right),$$

where I is the identity operator, $W_\alpha = \overline{W} + \alpha I$, that is, for $f \in \mathscr{C}_0$, $[\xi, \tau] \in B$,

$$(4.5) \qquad W_\alpha f(\xi, \tau) = Wf(\xi, \tau) + f(\xi, \tau)(\alpha - \tilde{P}_E(\xi, \tau)) .$$

Suppose there is an $f \in \mathscr{C}_0(B)$ not vanishing identically and such that

$$(\overline{W} + V)f = 0 .$$

Put

$$\beta = \sup\{\tau; [\xi, \tau] \in B, f(\xi, \tau) \neq 0\} .$$

Let us take notice first of the fact that $f(\xi, \tau) = 0$ for any $[\xi, \tau] \in B$ with $\tau = \beta$. If $\beta = b$ then there is nothing to prove since $f \in \mathscr{C}_0$. Suppose $\beta < b$ and let $[\xi, \tau] \in B$, $\tau = \beta$. Then $f(x, t) = 0$ whenever $[x, t] \in B$, $t > \beta$ and thus $Wf(\xi, \tau) = 0$ as well as $Vf(\xi, \tau) = 0$. Hence

$$0 = \left[\alpha\left(\frac{1}{\alpha} W_\alpha - I\right) + V\right] f(\xi, \tau) = \alpha\left(\frac{1}{\alpha}(\alpha - \tilde{P}_E(\xi, \tau)) f(\xi, \tau) - f(\xi, \tau)\right).$$

However, according to the assumption (4.2)

$$\frac{1}{\alpha}(\alpha - \tilde{P}_E(\xi, \tau)) < 1$$

and we find that $f(\xi, \tau) = 0$, indeed.

It is easy to see that there is a $k < \infty$ such that for $t \in R^1$, $x \in R^m$, $|x| > r$

$$G(x, t) + \sum_{j=1}^{m} |\partial_j G(x, t)| \leqq k .$$

Since $\mathscr{H}_m(\hat{B}) < \infty$ there is a $\delta > 0$, $\delta \leqq r_0$ such that

$$\frac{1}{\alpha} k\mathscr{H}_m(\hat{B} \cap (R_\beta - \bar{R}_{\beta-\delta})) + \gamma < 1$$

(where γ is defined by (4.3)). B is compact, f is continuous on B and hence there is a $[\xi_0, \tau_0] \in B \cap (\bar{R}_\beta - R_{\beta-\delta})$ such that

$$|f(\xi_0, \tau_0)| = \sup \{|f(\xi, \tau)|;\ [\xi, \tau] \in B,\ \tau \geqq \beta - \delta\}\,.$$

According to the definition of β we see that $|f(\xi_0, \tau_0)| > 0$ and by the above, $\tau_0 < \beta$. Let us denote

$$M_1 = \hat{B} \cap (R_\beta - \bar{R}_{\beta-\delta}) \cap [\Omega^*(\xi_0; r_0) \times R^1]\,,$$
$$M_2 = [\hat{B} \cap (R_\beta - \bar{R}_{\beta-\delta})] - [\Omega^*(\xi_0; r_0) \times R^1]\,.$$

For $[\xi, \tau] \in M_1 \cup M_2$ we have $|f(\xi, \tau)| \leqq |f(\xi_0, \tau_0)|$ which yields

$$|W_\alpha f(\xi_0, \tau_0) + Vf(\xi_0, \tau_0)| \leqq$$
$$\leqq \left| \int_{B_0} f(x, t) \{N_t G(x - \xi_0, t - \tau_0) - \sum_{j=1}^m N_j\, \partial_j G(x - \xi_0, t - \tau_0)\}\, d\mathscr{H}_m(x, t) \right| +$$
$$+ |f(\xi_0, \tau_0)|\, |\alpha - \tilde{P}_E(\xi_0, \tau_0)| + U^*_{|f|\,|\lambda|}(\xi_0, \tau_0) \leqq$$
$$\leqq \left| \int_{M_1} f(x, t) \{N_t G(x - \xi_0, t - \tau_0) - \sum_{j=1}^m N_j\, \partial_j G(x - \xi_0, t - \tau_0)\}\, d\mathscr{H}_m(x, t) \right| +$$
$$+ |f(\xi_0, \tau_0)|\, |\alpha - \tilde{P}_E(\xi_0, \tau_0)| + |f(\xi_0, \tau_0)|\, S_{r_0}(\xi_0, \tau_0) +$$
$$+ \left| \int_{M_2} f(x, t) \{N_t G(x - \xi_0, t - \tau_0) - \sum_{j=1}^m N_j\, \partial_j G(x - \xi_0, t - \tau_0)\}\, d\mathscr{H}_m(x, t) \right| \leqq$$
$$\leqq |f(\xi_0, \tau_0)|\, [\tfrac{1}{2}\pi^{-m/2} \tilde{v}^{r_0}(\xi_0, \tau_0) + |\alpha - \tilde{P}_E(\xi_0, \tau_0)| +$$
$$+ S_{r_0}(\xi_0, \tau_0) + k\mathscr{H}_m(\hat{B} \cap (R_\beta - \bar{R}_{\beta-\delta}))] \leqq$$
$$\leqq \alpha |f(\xi_0, \tau_0)| \left(\gamma + \frac{1}{\alpha} k\mathscr{H}_m(\hat{B} \cap (R_\beta - \bar{R}_{\beta-\delta})) \right) < \alpha |f(\xi_0, \tau_0)|\,.$$

However, this is a contradiction since, by the assumption,

$$0 = (\bar{W} + V) f(\xi_0, \tau_0) = W_\alpha f(\xi_0, \tau_0) + Vf(\xi_0, \tau_0) - \alpha f(\xi_0, \tau_0)\,.$$

4.3. Theorem. *Suppose that the assumptions from* 4.1 *concerning E and λ are fulfilled and suppose that there is an $\alpha > 0$ such that*

$$\frac{1}{\alpha} \lim_{r \to 0+} \Big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2}\pi^{-m/2} \tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\} \Big) < 1\,.$$

Then for any $v \in \mathscr{B}_0'(B)$ *the equation*

(4.6) $$A_\mu = v$$

has in $\mathscr{B}_0'(B)$ *a unique solution* μ, *and for any* $g \in \mathscr{C}_0(B)$ *the equation*

(4.7) $$(\overline{W} + V) f = g$$

has in $\mathscr{C}_0(B)$ *a unique solution* f.

Proof. Proposition 2.7 asserts that

$$\omega W_\alpha = \lim_{r \to 0+} \big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\big),$$

where ωW_α stands for the reciprocal value of the Fredholm radius of W_α. Under the assumptions of the theorem V is compact (see Proposition 3.7) and thus

$$\omega(W_\alpha + V) = \omega W_\alpha .$$

Writing the equation (4.7) in the form

$$\alpha \left[\frac{1}{\alpha}(W_\alpha + V) - I\right] f = g$$

and taking into account that $\omega[1/\alpha\,(W_\alpha + V)] < 1$ by the assumption and that the subspace

$$\{f \in \mathscr{C}_0(B);\ (\overline{W} + V)f = 0\}$$

is trivial according to Lemma 4.2, we conclude by the Riesz-Schauder theory (see [19]) that the equation (4.7) has a unique solution for any $g \in \mathscr{C}_0$. As the operators A, $(\overline{W} + V)$ are adjoint to each other, it follows from the Riesz-Schauder theory that the equation (4.6) has a unique solution for any $v \in \mathscr{B}_0'$ as well.

Taking λ to be the zero measure we obtain the following immediate consequence of Theorem 4.3.

4.4. Theorem. *Suppose that the conditions* (1.22), (1.48), (1.50) *are fulfilled and let*

$$\frac{1}{\alpha} \lim_{r \to 0+} \big(\sup_{[\xi,\tau] \in B_0} \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, \tau) + |\alpha - \tilde{P}_E(\xi, \tau)|\}\big) < 1$$

for some $\alpha > 0$. *Then for any* $v \in \mathscr{B}_0'$ *the equation*

(4.8) $$H_\mu = v$$

has in $\mathscr{B}_0'$ *a unique solution* μ *and for any* $g \in \mathscr{C}_0$ *the equation*

(4.9) $$\overline{W} f = g$$

has in $\mathscr{C}_0$ *a unique solution* f.

4.5. Remark. Given $\nu \in \mathscr{B}'_0$, suppose that the conditions of Theorem 4.3 are fulfilled and let $\mu \in \mathscr{B}'_0$ be the solution of the equation (4.6). If the heat potential U_μ is, in a sense, continuous on $E \cup B$ (it suffices, for instance, if

$$\lim_{\substack{[\xi,\tau]\to[x,t] \\ [\xi,\tau]\in E}} U_\mu(\xi, \tau) = U_\mu(x, t)$$

for almost all $[x, t] \in B$ with respect to λ – compare introductory remarks of Section 3) then U_μ is an integral expression of the solution of the third boundary value problem for the heat equation on E with a boundary condition of the form (3.5) prescribed on B.

If the conditions of Theorem 4.4 are fulfilled and if $f \in \mathscr{C}_0$ is the solution of the equation (4.9) for a given $g \in \mathscr{C}_0$, then the (generalized double-layer adjoint heat) potential Wf is an integral expression of the solution of the first boundary value problem for the adjoint heat equation on $R_{ab} - \bar{E}$ with the boundary condition g prescribed on B (and vanishing on $\partial(R_{ab} - \bar{E}) \cap \partial R_b$).

4.6. Remark. Consider the case that E is of the form

$$E = D \times (a, b), \tag{4.10}$$

where $D \subset R^m$ is an open set with compact boundary $C \neq \emptyset$. Then

$$B = C \times \langle a, b \rangle .$$

In [11] J. Král has stated conditions under which the equations (4.8), (4.9) are solvable provided E is of the form (4.10), in terms of the so-called cyclic variation of D. Recall here the definition of the cyclic variation of $D \subset R^m$. For $\xi \in R^m$, $\theta^* \in \Gamma^*$ let

$$\mathbf{L}_\xi^{\theta^*} = \{\xi + \varrho\theta^*;\ \varrho > 0\} .$$

An $x \in \mathbf{L}_\xi^{\theta^*}$ is called a hit of $\mathbf{L}_\xi^{\theta^*}$ on D if for any $r > 0$,

$$\mathscr{H}_1(\mathbf{L}_\xi^{\theta^*} \cap \Omega^*(x; r) \cap D) > 0 , \quad \mathscr{H}_1((\mathbf{L}_\xi^{\theta^*} \cap \Omega^*(x; r)) - D) > 0 .$$

If $n_\xi(\theta^*; r)$ is the number of all hits of $\mathbf{L}_\xi^{\theta^*}$ on D belonging to $\Omega^*(\xi; r)$ then we define

$$v_D^r(\xi) = \int_{\Gamma^*} n_\xi(\theta^*; r)\, d\mathscr{H}_{m-1}(\theta^*) .$$

For $\eta > 0$, $\theta^* \in \Gamma^*$, $\varrho > 0$, $[\xi, \tau] \in R^{m+1}$, $\tau \geqq a$ we have

$$\left[\xi + \varrho\theta^*,\ \tau + \frac{\varrho^2}{4\eta}\right] \in E$$

if and only if

$$\xi + \varrho\theta^* \in D , \quad \tau + \frac{\varrho^2}{4\eta} \in (a, b) .$$

Hence $[x, t] \in H^{\theta^*}_{\xi,\tau}(\eta)$ with $t \neq b$ is a hit of $H^{\theta^*}_{\xi,\tau}(\eta)$ on E if and only if $t < b$ (that is, $\varrho < 2\sqrt{[\eta(b-\tau)]}$) and x is a hit of $L^{\theta^*}_{\xi}$ on D. It follows that

$$\tilde{n}_{\xi,\tau}(\theta^*, \eta; \infty) = n_\xi(\theta^*; 2\sqrt{[\eta(b-\tau)]}) .$$

We see further that for $[\xi, \tau] \in R^{m+1}$, $\tau < b$, $\theta^* \in \Gamma^*$, $\eta > 0$, $\varrho > 0$, $0 < r < \infty$,

$$\left[\xi + \varrho\theta^*, \tau + \frac{4\eta}{\varrho^2}\right] \in [\Omega^*(\xi; r) \times (\tau, \tau + r)] \cap R_t$$

if and only if

$$\varrho < \min\{r, 2\sqrt{[\eta \min\{r, b-\tau\}]}\} = r(\tau, \eta)$$

and thus

$$\tilde{n}_{\xi,\tau}(\theta^*, \eta; r) = n_\xi(\theta^*; r(\tau, \eta)) .$$

Consequently,

$$\tilde{v}^r(\xi, \tau) = \int_0^\infty e^{-\eta}\eta^{m/2-1} \int_{\Gamma^*} n_\xi(\theta^*; r(\tau, \eta))\, d\mathscr{H}_{m-1}(\theta^*) = \tag{4.11}$$

$$= \int_0^\infty v_D^{r(\tau,\eta)}(\xi)\, e^{-\eta}\eta^{m/2-1}\, d\eta .$$

Thus we see that the adjoint parabolic variation is, provided E is cylindrical, equal to the term used by J. Král in [11] (see Proposition 1.8 from [11]).

Further, it is seen that for $[\xi, \tau] \in R^{m+1}$, $a \leqq \tau < b$, $\theta^* \in \Gamma^*$, $\eta > 0$ there is a $\delta > 0$ such that

$$\mathscr{H}_1\left(\left\{\left[\xi + u\theta^*, \tau + \frac{u^2}{4\eta}\right];\ u \in (0, \delta)\right\} - E\right) = 0$$

if and only if there is a $\delta > 0$ with

$$\mathscr{H}_1(\{\xi + u\theta^*;\ u \in (0, \delta)\} - D) = 0 ; \tag{4.12}$$

the term in (4.12) is independent of $\eta > 0$. Let L_ξ stand for the set of all $\theta^* \in \Gamma^*$ for which there is a $\delta > 0$ such that (4.12) is valid. Supposing $v_D^\infty(\xi) < \infty$, then for almost all $(\theta^*, \eta) \in \Gamma^* \times (0, \infty)$ (with respect to $\mathscr{H}_{m-1} \otimes \mathscr{H}_1$), $s(\theta^*, \eta, 0) = -1$ if and only if $\theta^* \in L_\xi$. If $v_D^\infty(\xi) < \infty$ then moreover (see [8], Lemma 2.7)

$$\mathscr{H}_{m-1}(L_\xi) = A\, d_D(\xi) ,$$

where

$$A = \mathscr{H}_{m-1}(\Gamma^*) \tag{4.13}$$

and $d_D(\xi)$ is the m-dimensional density of D at ξ. Now we have (supposing still $\tau \in \langle a, b))$

$$(4.14)\qquad \tilde{P}_E(\xi, \tau) = \tfrac{1}{2}\pi^{-m/2} \int_{L_\xi} \mathrm{d}\mathscr{H}_{m-1}(\theta^*) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta = d_D(\xi)\,.$$

Let us show that if for $K \subset C$, $K \neq \emptyset$,

$$V_0(K) = \lim_{r\to 0+} (\sup \{v_D^r(\xi);\ \xi \in K\})\,,$$

then

$$(4.15)\qquad \lim_{r\to 0+} (\sup \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, \tau);\ [\xi, \tau] \in B_0,\ \xi \in K\}) = \frac{V_0(K)}{A}\,.$$

We proceed similarly as in the proof of Lemma 3.7 in [11].

For $[\xi, \tau_1], [\xi, \tau_2] \in B_0$, $\tau_1 \leqq \tau_2$, clearly

$$\tilde{v}^r(\xi, \tau_1) \geqq \tilde{v}^r(\xi, \tau_2)\,;$$

hence

$$\sup \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, \tau);\ [\xi, \tau] \in B_0,\ \xi \in K\} = \sup \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, a);\ \xi \in K\}\,.$$

If $r > 0$, $r < b - a$ then for any $\eta > 0$,

$$r(a, \eta) = \min \{r, 2\sqrt{(\eta r)}\}\,.$$

If $\eta < r/4$ then $r(a, \eta) = 2\sqrt{(\eta r)}$ and if $\eta > r/4$ then $r(a, \eta) = r$. Thus we obtain

$$\tilde{v}^r(\xi, a) = \int_0^{r/4} v_D^{2\sqrt{(\eta r)}}(\xi)\, \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta + \int_{r/4}^\infty \tilde{v}_D^r(\xi)\, \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta\,.$$

As $v_D^{2\sqrt{(\eta r)}}(\xi) \leqq v_D^r(\xi)$ for $\eta < r/4$, we have

$$\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, a) \leqq \tfrac{1}{2}\pi^{-m/2}\, v_D^r(\xi) \int_0^\infty \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta = \frac{1}{A}\, v_D^r(\xi)\,.$$

Further,

$$\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, a) \geqq \frac{1}{A}\, v_D^r(\xi) - \tfrac{1}{2}\pi^{-m/2}\, v_D^r(\xi) \int_0^{r/4} \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta\,,$$

hence

$$\sup \{\tfrac{1}{2}\pi^{-m/2}\tilde{v}^r(\xi, a);\ \xi \in K\} \geqq$$
$$\geqq \sup \{v_D^r(\xi);\ \xi \in K\} \left(\frac{1}{A} - \tfrac{1}{2}\pi^{-m/2} \int_0^{r/4} \mathrm{e}^{-\eta}\eta^{m/2-1}\, \mathrm{d}\eta\right)$$

and (4.15) follows.

Now, by means of (4.14), (4.15) one can, provided E is cylindrical, express the condition under which the equations (4.8), (4.9) are solvable in terms of the cyclic variation of D and the m-dimensional density of D. We see that these conditions are the same as in [11] (compare [11], Theorem 3.10, Theorem 3.9).

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