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# MAL'CEV-TYPE THEOREMS FOR PARTIAL CONGRUENCES

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*Summary.* It is shown that some properties of partial congruences (= congruences which do not satisfy the axiom of reflexivity) are definable by Mal'cev conditions.

*Keywords:* Partial congruence, variety of algebras, Mal'cev condition.

*AMS classification:* 08A40, 08B05.

## 1. BASIC CONCEPTS

Binary relations which need not be reflexive on the whole base set were studied by O. Borůvka [1], I. Chajda [2], H. Draškovičová [4], B. M. Schein [7], F. Šik [8], and others. From [7] we adopt the following

**Definition 1.** Let $\varrho$ be a binary relation in a set $A$. We say that $\varrho$ is *partly reflexive* in $A$ whenever the implication $\langle a, b\rangle \in \varrho \Rightarrow \langle a, a\rangle \in \varrho$ and $\langle b, b\rangle \in \varrho$ holds for any $a, b \in A$.

**Lemma 1.** *Let $\vartheta$ be a symmetric and transitive binary relation in a set $A$. Then $\vartheta$ is partly reflexive in $A$.*

Proof. Immediate.

With the aid of Lemma 1 we introduce

**Definition 2.** Let $A$ be a set. A symmetric and transitive binary relation in $A$ is called a *partial equivalence* in $A$.

A partly reflexive and symmetric binary relation in $A$ is called a *partial tolerance* in $A$.

**Definition 3.** Let $\mathfrak{A} = \langle A, F\rangle$ be an algebra. A partial equivalence in $A$ which is compatible with the set of all fundamental operations $F$ is called a *partial congruence* in $\mathfrak{A}$.

A partly reflexive, symmetric and compatible binary relation in $\mathfrak{A}$ is called a *partial compatible tolerance* in $\mathfrak{A}$.

**Lemma 2.** *Let $\varrho$ be a partly reflexive binary relation in a set $A$. Then*
(a) *$\varrho^m \subseteq \varrho^n$ holds for any integers $m < n$;*
(b) *$\bigcup_{k<\omega} \varrho^k$ is the transitive closure of $\varrho$.*

Proof. (a) Let $\langle x, y\rangle \in \varrho^m$ and $m < n$. Then $\langle a, y\rangle \in \varrho$ for some element $a \in A$. Hence $\langle y, y\rangle \in \varrho$ and so $\langle y, y\rangle \in \varrho^{n-m}$. This yields $\langle x, y\rangle \in \varrho^m \circ \varrho^{n-m} = \varrho^n$, as required.
(b) Evident.

## 2. COMPACT PARTIAL CONGRUENCES

One can easily verify that partial congruences as well as partial compatible tolerances in a given algebra $\mathfrak{A}$ form algebraic lattices. As usual, *compact elements* of these two lattices play the crucial role. The least partial congruence (partial compatible tolerance) containing a subset $S \subseteq \mathfrak{A} \times \mathfrak{A}$ is denoted by $\vartheta(S)$ ($\tau(S)$, respectively). Further, the symbol $Sg_{\mathfrak{A}\times\mathfrak{A}}(S)$ stands for the subalgebra of $\mathfrak{A} \times \mathfrak{A}$ generated by $S$.

**Lemma 3.** *Let $a, b$ be elements of an algebra $\mathfrak{A}$. Then $\tau(a, b) = Sg_{\mathfrak{A}\times\mathfrak{A}}(\langle a, b\rangle, \langle b, a\rangle, \langle a, a\rangle, \langle b, b\rangle)$.*

Proof. For the sake of brevity denote $\sigma = Sg_{\mathfrak{A}\times\mathfrak{A}}(\langle a, b\rangle, \langle b, a\rangle, \langle a, a\rangle, \langle b, b\rangle)$. Then clearly $\sigma = \{\langle \boldsymbol{p}(a, b, a, b), \boldsymbol{p}(b, a, a, b)\rangle;\ \boldsymbol{p}$ is a quaternary term of $\mathfrak{A}\}$. We want to prove that $\sigma$ is a partial compatible tolerance containing the pair $\langle a, b\rangle$:

(i) Choosing $\boldsymbol{p} = \boldsymbol{e}_0^4$ (the symbol $\boldsymbol{e}_0^4$ denotes the trivial operation $\boldsymbol{e}_0^4(x_0, x_1, x_2, x_3) = x_0$) we infer that $\langle a, b\rangle \in \sigma$.

(ii) Partial reflexivity: Let $\langle x, y\rangle \in \sigma$. This means that $x = \boldsymbol{p}(a, b, a, b)$ and $y = \boldsymbol{p}(b, a, a, b)$ for some quaternary term $\boldsymbol{p}$. Let us introduce a quaternary term $\boldsymbol{q}$ via $\boldsymbol{q}(x_0, x_1, x_2, x_3) = \boldsymbol{p}(x_2, x_3, x_2, x_3)$. Then $\boldsymbol{q}(a, b, a, b) = \boldsymbol{p}(a, b, a, b) = x$ and $\boldsymbol{q}(b, a, a, b) = \boldsymbol{p}(a, b, a, b) = x$ which means that $\langle x, x\rangle \in \sigma$. Analogously we obtain $\langle y, y\rangle \in \sigma$.

(iii) Symmetry: Suppose that $\langle x, y\rangle \in \sigma$. Thus $x = \boldsymbol{p}(a, b, a, b)$ and $y = \boldsymbol{p}(b, a, a, b)$ for some quaternary term $\boldsymbol{p}$.

Define another quaternary term $\boldsymbol{r}$ by the rule $\boldsymbol{r}(x_0, x_1, x_2, x_3) = \boldsymbol{p}(x_1, x_0, x_2, x_3)$. Then $\boldsymbol{r}(a, b, a, b) = \boldsymbol{p}(b, a, a, b) = y$ and $\boldsymbol{r}(b, a, a, b) = \boldsymbol{p}(a, b, a, b) = x$ or, equivalently, $\langle y, x\rangle \in \sigma$.

(iv) Compatibility of $\sigma$ follows directly from the definition of $\sigma$.

Now the inclusion $\sigma \supseteq \tau(a, b)$ is a consequence of the properties (i), …, (iv). The opposite inclusion is trivial.

**Lemma 4.** *Let $a, b$ be elements of an algebra $\mathfrak{A}$. Then $\vartheta(a, b) = \bigcup_{n<\omega} \tau^n(a, b)$.*

Proof. Evidently $\langle a, b\rangle \in \bigcup_{n<\omega} \tau^n(a, b)$. Further, one can easily verify that the set-union $\bigcup_{n<\omega} \tau^n(a, b)$ is a symmetric, transitive and compatible binary relation in $\mathfrak{A}$, see Lemma 2. Consequently $\vartheta(a, b) \subseteq \bigcup_{n<\omega} \tau^n(a, b)$.

On the other hand, the inclusion $\tau(a, b) \subseteq \vartheta(a, b)$ holds. Since $\vartheta(a, b)$ is transitive we have also $\tau^n(a, b) \subseteq \vartheta(a, b)$ for any $n < \omega$. Hence the remaining inclusion $\bigcup_{n<\omega} \tau^n(a, b) \subseteq \vartheta(a, b)$ follows.

**Lemma 5.** (Mal'cev lemma for principal partial congruences). *Let $x, y, a, b$ be elements of an algebra $\mathfrak{A}$. The following conditions are equivalent:*

(1) $\langle x, y\rangle \in \vartheta(a, b)$;

(2) *there exist an integer $n$ and quaternary terms $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ such that*

$$x = \boldsymbol{q}_1(a, b, a, b)\,,$$
$$\boldsymbol{q}_i(b, a, a, b) = \boldsymbol{q}_{i+1}(a, b, a, b)\,, \quad 1 \leqq i < n\,,$$
$$y = \boldsymbol{q}_n(b, a, a, b)\,.$$

Proof. (1) $\Rightarrow$ (2). By Lemma 4 we have $\vartheta(a, b) = \bigcup_{n<\omega} \tau^n(a, b)$. Then the assumption $\langle x, y\rangle \in \vartheta(a, b)$ yields $\langle x, y\rangle \in \tau^n(a, b)$ for some $n < \omega$. This means that $x = c_1$, $\langle c_i, c_{i+1}\rangle \in \tau(a, b)$, $1 \leqq i \leqq n$, and $c_{n+1} = y$ for some elements $c_1, \ldots, c_{n+1} \in \mathfrak{A}$. Applying Lemma 3 we get $c_i = \boldsymbol{q}_i(a, b, a, b)$ and $c_{i+1} = \boldsymbol{q}_i(b, a, a, b)$, $1 \leqq i \leqq n$, for suitable quaternary terms $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. The equalities (2) follow.

(2) $\Rightarrow$ (1). Since $\langle a, b\rangle, \langle b, a\rangle, \langle a, a\rangle, \langle b, b\rangle \in \vartheta(a, b)$ we have also $\langle \boldsymbol{q}_i(a, b, a, b), \boldsymbol{q}_i(b, a, a, b)\rangle \in \vartheta(a, b)$ for any $1 \leqq i \leqq n$. Now the transitivity of $\vartheta(a, b)$ together with the equations (2) give the required result $\langle x, y\rangle \in \vartheta(a, b)$. The proof is complete.

## 3. APPLICATIONS: MAL'CEV CONDITIONS FOR PARTIAL CONGRUENCES

In this section we show that some properties of partial congruences in algebras from a variety are definable by Mal'cev conditions. In particular, we give here identities characterizing the partial principality and partial regularity.

Varieties with principal compact congruences were investigated in [10]; for partial congruences we introduce

**Definition 4.** An algebra $\mathfrak{A}$ has *principal compact partial congruences* whenever any compact partial congruence in $\mathfrak{A}$ is of the form $\vartheta(p, q)$ for some elements $p, q \in \mathfrak{A}$.

A variety $\boldsymbol{V}$ has *principal compact partial congruences* whenever each $\boldsymbol{V}$-algebra has this property.

**Theorem 1.** *For a variety $V$ the following conditions are equivalent:*

(1) *$V$ has principal compact partial congruences;*

(2) *there exist integers $m, n$ and quaternary terms $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{s}_1, \ldots \boldsymbol{s}_m, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_n$ such that the identities*

$$\boldsymbol{p}(x, x, u, u) = \boldsymbol{q}(x, x, u, u),$$
$$x = \boldsymbol{s}_1(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$\boldsymbol{s}_i(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)) =$$
$$= \boldsymbol{s}_{i+1}(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$1 \leqq i < m,$$
$$y = \boldsymbol{s}_m(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$u = \boldsymbol{t}_1(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$\boldsymbol{t}_i(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)) =$$
$$= \boldsymbol{t}_{i+1}(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$1 \leqq i < n,$$
$$v = \boldsymbol{t}_n(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$

*hold in $V$.*

Proof. (1) ⇒ (2). Let $\mathfrak{A} = \mathfrak{F}_V(x, y, u, v)$ be the $V$-free algebra with free generators $x, y, u, v$. Then $\vartheta(x, y) \vee \vartheta(u, v) = \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$, by hypothesis. The identity $\boldsymbol{p}(x, x, u, u) = \boldsymbol{q}(x, x, u, u)$ follows directly from the inclusion $\vartheta(x, y) \vee \vee\, \vartheta(u, v) \supseteq \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$. Further, $\langle x, y\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$ yields

$$x = \boldsymbol{s}_1(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$\boldsymbol{s}_i(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)) =$$
$$= \boldsymbol{s}_{i+1}(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$1 \leqq i < m,$$
$$y = \boldsymbol{s}_m(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$$

for some quaternary terms $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m$, see Lemma 5.

Finally, applying Lemma 5 to the relation

$$\langle u, v\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$$

we get the remaining identities

$$u = \boldsymbol{t}_1(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$
$$\boldsymbol{t}_i(\boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)) =$$
$$= \boldsymbol{t}_{i+1}(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v), \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)),$$

$$1 \leqq i < n \,,$$

$$v = \boldsymbol{t}_n(\boldsymbol{q}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))\,.$$

(2) ⇒ (1). Let $\mathfrak{A}$ be an arbitrary $\boldsymbol{V}$-algebra with elements $x, y, u, v$. We want to prove the equality $\vartheta(x, y) \vee \vartheta(u, v) = \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v))$.

Since evidently $\langle x, y\rangle \in \vartheta(x, y) \vee \vartheta(u, v)$ and $\langle u, v\rangle \in \vartheta(x, y) \vee \vartheta(u, v)$ we have also $\langle x, x\rangle \in \vartheta(x, y) \vee \vartheta(u, v)$ and $\langle u, u\rangle \in \vartheta(x, y) \vee \vartheta(u, v)$, see Lemma 1. Then compatibility implies

$$\langle \boldsymbol{p}(x, x, u, u),\ \boldsymbol{p}(x, y, u, v)\rangle \in \vartheta(x, y) \vee \vartheta(u, v) \quad \text{and}$$

$$\langle \boldsymbol{q}(x, x, u, u),\ \boldsymbol{q}(x, y, u, v)\rangle \in \vartheta(x, y) \vee \vartheta(u, v)\,.$$

The hypothesis $\boldsymbol{p}(x, x, u, u) = \boldsymbol{q}(x, x, u, u)$ and the transitivity of partial congruences yield $\langle \boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)\rangle \in \vartheta(x, y) \vee \vartheta(u, v)$, which means that $\vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)) \subseteq \vartheta(x, y) \vee \vartheta(u, v)$.

Conversely, $\langle \boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v)\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$ gives $\langle \boldsymbol{q}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v)\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$, by symmetry, and $\langle \boldsymbol{p}(x, y, u, v), \boldsymbol{p}(x, y, u, v)\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v), \boldsymbol{q}(x, y, u, v)), \langle \boldsymbol{q}(x, y, u, v), \boldsymbol{q}(x, y, u, v)\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$, by Lemma 1. Now applying the quaternary terms $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_m$ we find that

$$\langle \boldsymbol{s}_i(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))\,,$$
$$\boldsymbol{s}_i(\boldsymbol{q}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v),\ \boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))\rangle \in$$
$$\in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))\,, \quad 1 \leqq i \leqq m\,.$$

Using the identities from (2) and the transitivity of the partial congruence $\vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$ we conclude that $\langle x, y\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$. The relation $\langle u, v\rangle \in \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$ can be verified in a similar way. Altogether we have $\vartheta(x, y) \vee \vartheta(u, v) = \vartheta(\boldsymbol{p}(x, y, u, v),\ \boldsymbol{q}(x, y, u, v))$ which was to be proved.

Mal'cev classes of congruence regular varieties were studied by B. Csákány [3], G. Grätzer [5] and R. Wille [9]. Analogously we introduce the concept of regular partial congruences.

**Definition 5.** An algebra $\mathfrak{A}$ has *regular partial congruences* whenever any partial congruence in $\mathfrak{A}$ is uniquely determined by any of its blocks.

A variety $\boldsymbol{V}$ has *regular partial congruences* whenever every $\boldsymbol{V}$-algebra has this property.

**Theorem 2.** *For a variety* $\boldsymbol{V}$ *the following conditions are equivalent*:

(1) $\boldsymbol{V}$ *has regular partial congruences*;

(2) *there exist an integer* $n$, *ternary terms* $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$, *and quaternary terms* $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ *such that the identities*

$$x = \boldsymbol{r}_1(z, \boldsymbol{p}_1(x, y, z), z, \boldsymbol{p}_1(x, y, z)),$$

$$\boldsymbol{r}_i(\boldsymbol{p}_i(x, y, z), z, z, \boldsymbol{p}_i(x, y, z)) =$$

$$= \boldsymbol{r}_{i+1}(z, \boldsymbol{p}_{i+1}(x, y, z), z, \boldsymbol{p}_{i+1}(x, y, z)), \quad 1 \leqq i < n,$$

$$y = \boldsymbol{r}_n(\boldsymbol{p}_n(x, y, z), z, z, \boldsymbol{p}_n(x, y, z)),$$

$$z = \boldsymbol{p}_i(x, x, z), \quad 1 \leqq i \leqq n,$$

*hold in* $\boldsymbol{V}$.

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{A} = \mathfrak{F}_{\boldsymbol{V}}(x, y, z)$ be the $\boldsymbol{V}$-free algebra over the free generating set $\{x, y, z\}$. Denote by $\gamma$ the partial congruence $\vartheta(\{\langle x, y\rangle, \langle z, z\rangle\})$. Then $[z]\,\gamma$ is nonvoid. We claim that the partial congruence $\vartheta([z]\,\gamma \times [z]\,\gamma)$ has the same $z$-block as the original partial congruence $\gamma$:

(i) $[z]\,\gamma \supseteq [z]\,\vartheta([z]\,\gamma \times [z]\,\gamma)$ is a consequence of $\gamma \supseteq \vartheta([z]\,\gamma \times [z]\,\gamma)$;

(ii) $[z]\,\gamma \subseteq [z]\,\vartheta([z]\,\gamma \times [z]\,\gamma)$ follows from the inclusion $[z]\,\gamma \times [z]\,\gamma \subseteq \subseteq \vartheta([z]\,\gamma \times [z]\,\gamma)$.

By hypothesis the equality of blocks implies the equality of partial congruences $\vartheta(\{\langle x, y\rangle, \langle z, z\rangle\}) = \vartheta([z]\,\gamma \times [z]\,\gamma)$. Since the partial congruence on the left-hand side is compact we have $\vartheta(\{\langle x, y\rangle, \langle z, z\rangle\}) = \vartheta(\{\langle z, \boldsymbol{p}_1\rangle, \ldots, \langle z, \boldsymbol{p}_m\rangle\})$ for some $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m \in \mathfrak{A} = \mathfrak{F}_{\boldsymbol{V}}(x, y, z)$. This fact immediately gives the identities $z = \boldsymbol{p}_i(x, x, z)$, $1 \leqq i \leqq m$.

Further, from $\langle x, y\rangle \in \vartheta(\{\langle z, \boldsymbol{p}_1\rangle, \ldots, \langle z, \boldsymbol{p}_m\rangle\})$ we find

$$x = \boldsymbol{r}_1(z, \boldsymbol{p}_1(x, y, z), z, \boldsymbol{p}_1(x, y, z)),$$

$$\boldsymbol{r}_i(\boldsymbol{p}_i(x, y, z), z, z, \boldsymbol{p}_i(x, y, z)) =$$

$$= \boldsymbol{r}_{i+1}(z, \boldsymbol{p}_{i+1}(x, y, z), z, \boldsymbol{p}_{i+1}(x, y, z)), \quad 1 \leqq i < n,$$

$$y = \boldsymbol{r}_n(\boldsymbol{p}_n(x, y, z), z, z, \boldsymbol{p}_n(x, y, z))$$

where $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ are suitable quaternary terms and $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m\}$.

(2) $\Rightarrow$ (1). Let $\alpha$ be a partial congruence in an algebra $\mathfrak{A} \in \boldsymbol{V}$ and let $\langle a, a\rangle \in \alpha$. We want to prove that the block $[a]\,\alpha$ determines the original partial congruence $\alpha$. To do this it suffices to verify the equality $\vartheta([a]\,\alpha \times [a]\,\alpha) = \alpha$.

The inclusion $\vartheta([a]\,\alpha \times [a]\,\alpha) \subseteq \alpha$ being trivial we take $\langle x, y\rangle \in \alpha$. Then $\langle x, x\rangle$, $\langle x, y\rangle$, $\langle a, a\rangle \in \alpha$ and so $\langle a, \boldsymbol{p}_i(x, y, a)\rangle \in \alpha$, $1 \leqq i \leqq n$, by compatibility and (2). Consequently $\langle a, \boldsymbol{p}_i(x, y, a)\rangle \in [a]\,\alpha \times [a]\,\alpha$ and, further, $\langle a, \boldsymbol{p}_i(x, y, a)\rangle \in \in \vartheta([a]\,\alpha \times [a]\,\alpha)$ for $1 \leqq i \leqq n$. Since also $\langle a, a\rangle \in \vartheta([a]\,\alpha \times [a]\,\alpha)$ and $\langle \boldsymbol{p}_i(x, y, a), \boldsymbol{p}_i(x, y, a)\rangle \in \vartheta([a]\,\alpha \times [a]\,\alpha)$, $1 \leqq i \leqq n$, the identities (2) imply $\langle x, y\rangle \in \vartheta([a]\,\alpha \times [a]\,\alpha)$. The inclusion $\alpha \subseteq \vartheta([a]\,\alpha \times [a]\,\alpha)$ follows. The proof is complete.

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## Souhrn

### VĚTY MAL'CEVOVA TYPU PRO PARCIÁLNÍ KONGRUENCE V ALGEBRÁCH

JAROMÍR DUDA

Jsou odvozeny dvě Mal'cevovy podmínky charakterizující vlastnosti parciálních kongruencí v algebrách tvořících varietu.



## Резюме

### УСЛОВИЯ МАЛЬЦЕВА ДЛЯ КОНГРУЭНЦИЙ В АЛГЕБРАХ

JAROMÍR DUDA

Выведены условия Мальцева для частичных конгруэнций в алгебрах, образующих многообразие.



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