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THE INNER PRODUCT OF S-FUNCTIONS

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The Kronecker product of two irreducible representations of a group is equivilant to a direct sum of irreducible representations. This problem is equivilant to the expression of the product of two characters of a group as a sum of simple characters. For the full linear group, the characters are S-functions and a method has been formed for expressing the product of two S-functions as the sum of S-functions[1]). For the symmetric group, if (λ) & (μ) are partitions of the same integer n and χ^λ_ϱ, χ^μ_β are respectively the group characters, the problem of expressing the Kronecker product as a direct sum of irreducible representations is equivalent to the evaluation of $\chi^\lambda_\varrho \chi^\mu_\varrho$ in the form $\sum g_{\lambda\mu\nu}\chi^\nu_\varrho$. LITTLEWOOD[2]) denoted this product by the symbol $\{\lambda\} \otimes \{\mu\}$ and called it the inner product of S-functions such that $\{\lambda\} \otimes \{\mu\} = \sum g_{\lambda\mu\nu}\{\nu\}$ whenever $\chi^\lambda_\varrho \chi^\mu_\varrho = \sum g_{\lambda\mu\nu}\chi^\nu_\varrho$.

This problem has been attacked 1938 by MURNAGNAN[3]). He assumed the inner products $\{n - r, \lambda\} \otimes \{n - s, \mu\}$ where λ, μ are partitions of r & s respectively. He tabulated the results for all cases when $r = 1$, $s = 2, 3, 4$ and $r = 3$, $s = 3, 4$ by availing himself of 2 central facts:

(a) that the coefficients of the analysis is independent of n,

(b) that the analysis of $\{n - r, \lambda\} \otimes \{n - s, \mu\}$ does not go deeper than the term $\{n - s - r, \ldots\}$.

MAKAR[4]) in 1947 gave six formulae for the inner products of $\{\lambda\}$ with either $\{n - 1, 1\}, \{n - 2, 2\}, \{n - 2, 1^2\}, \{n - 3, 3\}, \{n - 3, 2, 1\}$ or $\{n - 3, 1^3\}$. The formulae actually turn the problem to one involving the multiplication of S-functions. Later Makar[5]) transformed these six formulae into other six formulae which he applied to obtain $\{n - r, \lambda\} \otimes \{n - s, \mu\}$ where $r = 5, 6$; $s = 2$; $r = 5$, $s = 3$ and (λ), (μ) are partitions of r & s.

[1]) Littlewood [3].
[2]) Littlewood [2].
[3]) Murnaghan [6, 7].
[4]) Makar [4].
[5]) Makar [5].

Recent papers gave particular care for the evaluation of $\{\lambda\} \otimes \{\mu\}$. In 1954 ROBINSON & TAULBEE[6]) gave a direct method of analysing the general case $\{\lambda\} \otimes \{\mu\}$. Then in 1955 Littlewood[7]) modified Robinson-Taulbee method. With his simplified formulae he evaluated the difficult inner product $\{321\} \otimes \{321\}$. KAKAR[8]) in 1956 applied Littlewood's method to obtain $\{n - r, \lambda\} \otimes \{n - s, \mu\}$ where $r = s = 4$.

Very recently Murnaghan[9]) gave a master formula for the evaluation of $\{n - r, \lambda\} \otimes \{n - s, \mu\}$ which again involved multiplication of S-functions.

In this paper some formulae, very easy to apply, are given for the analysis of $\{n - r, \lambda\} \otimes \{n - s, \mu\}$. To use these formulae we have to follow three rules:

(i) In any S function two consecutive parts may be interchanged provided that the preceeding part is decreased by unity & the succeeding part increased by unity, the S function thereby changed in sign, i.e.

$$\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_p\} = -\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_p\} .$$

(ii) In any S function if any part exceeds by unity the preceeding part the value of the S function is zero i.e. if $\lambda_{i+1} = \lambda_i + 1$ then $\{\lambda\} = 0$.

(iii) The value of any S function is zero if the last part is a negative number.

Formula 1) $\{n - 1, 1\} \otimes \{n - m, m\} = \sum_{r=m-1}^{m+1} \{n - r, r\} + \sum_{r=m}^{m+1} \{n - r, r - 1, 1\}$.

For $m = 1, 2, 3, 4$ or 5 etc. we get

$$\{n - 1, 1\} \otimes \{n - 1, 1\} = n + \{n - 1, 1\} + \{n - 2, 2\} + \{n - 2, 1^2\} ,$$

$$\{n - 1, 1\} \otimes \{n - 2, 2\} = \{n - 1, 1\} + \{n - 2, 2\} + \{n - 2, 1^2\} + \{n - 3, 3\} + \{n - 3, 2, 1\} ,$$

$$\{n - 1, 1\} \otimes \{n - 3, 3\} = \{n - 2, 2\} + \{n - 3, 3\} + \{n - 3, 2, 1\} + \{n - 4, 4\} + \{n - 4, 3, 1\} ,$$

$$\{n - 1, 1\} \otimes \{n - 4, 4\} = \{n - 3, 3\} + \{n - 4, 4\} + \{n - 4, 3, 1\} + \{n - 5, 5\} + \{n - 5, 4, 1\} ,$$

$$\{n - 1, 1\} \otimes \{n - 5, 5\} = \{n - 4, 4\} + \{n - 5, 5\} + \{n - 5, 4, 1\} + \{n - 6, 6\} + \{n - 6, 5, 1\} \quad \text{etc.}$$

Formula 2) $\{n - 1, 1\} \otimes \{n - m, 1^m\} = \sum_{r=m-1}^{m+1} \{n - r, 1^r\} + \sum_{r=m}^{m+1} \{n - r, 2, 1^{r-2}\}$.

For $m = 2, 3, 4, 5$ etc. we get

$$\{n - 1, 1\} \otimes \{n - 2, 1^2\} = \{n - 1, 1\} + \{n - 2, 1^2\} + \{n - 3, 1^3\} + \{n - 2, 2\} + \{n - 3, 2, 1\} ,$$

6) Robinson [11].

7) Littlewood [2].

8) Kakar [1].

9) Murnaghan [8].

$$\{n-1,1\} \otimes \{n-3,1^3\} = \{n-2,1^2\} + \{n-3,1^3\} + \{n-4,1^4\} + $$
$$+ \{n-3,2,1\} + \{n-4,2,1^2\},$$

$$\{n-1,1\} \otimes \{n-4,1^4\} = \{n-3,1^3\} + \{n-4,1^4\} + \{n-5,1^5\} + $$
$$+ \{n-4,2,1^2] + \{n-5,2,1^3\},$$

$$\{n-1,1\} \otimes \{n-5,1^5\} = \{n-4,1^4\} + \{n-5,1^5\} + \{n-6,1^6\} + $$
$$+ \{n-5,2,1^3\} + \{n-6,2,1^4\} \quad \text{etc.}$$

Formula 3) $\{n-1,1\} \otimes \{n-m, m-1, 1\} = \{n-m, m-1, 1\} +$
$$+ \sum_{r=m-1}^{m} \{n-r, r\} + \sum_{r=m-1}^{m+1} \{n-r, r-1, 1\} + \sum_{r=m}^{m+1} \{n-r, r-2, 2\} +$$
$$+ \sum_{r=m}^{m+1} \{n-r, r-2, 1^2\}.$$

For $m = 2, 3, 4$ or 5 we get

$$\{n-1,1\} \otimes \{n-2,1^2\} = \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\} +$$
$$+ \{n-3,2,1\} + \{n-3,1^3\},$$

$$\{n-1,1\} \otimes \{n-3,2,1\} = \{n-2,2\} + \{n-3,3\} + 2\{n-3,2,1\} +$$
$$+ \{n-4,2,1^2\} + \{n-4,3,1^2\} + \{n-4,2^2\} +$$
$$+ \{n-2,1^2\} + \{n-3,1^3\},$$

$$\{n-1,1\} \otimes \{n-4,3,1\} = \{n-3,3\} + \{n-3,2,1\} + \{n-4,4\} +$$
$$+ 2\{n-4,3,1\} + \{n-4,2^2\} + \{n-4,2,1^2\} +$$
$$+ \{n-5,4,1\} + \{n-5,3,2\} + \{n-5,3,1^2\},$$

$$\{n-1,1\} \otimes \{n-5,4,1\} = \{n-4,4\} + \{n-4,3,1\} + \{n-5,5\} +$$
$$+ 2\{n-5,4,1\} + \{n-5,3,2\} + \{n-5,3,1^2\} +$$
$$+ \{n-6,5,1\} + \{n-6,4,2\} + \{n-6,4,1^2\}.$$

Formula 4) $\{n-1,1\} \otimes \{n-m, 2, 1^{m-2}\} = \{n-m, 2, 1^{m-2}\} +$
$$+ \sum_{r=m-1}^{m} \{n-r, 1^r\} + \sum_{r=m-1}^{m+1} \{n-r, 2, 1^{r-2}\} + \sum_{r=m}^{r=m+1} \{n-r, 2^2, 1^{r-4}\} +$$
$$+ \sum_{r=m}^{m+1} \{n-r, 3, 1^{r-3}\}, \; m > 2.$$

For $m = 3, 4$, or 5 it gives

$$\{n-1,1\} \otimes \{n-3,2,1\} = \{n-2,2\} + \{n-2,1^2\} + \{n-3,3\} +$$
$$+ 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,3,1\} +$$
$$+ \{n-4,2^2\} + \{n-4,2,1^2\},$$

$$\{n-1,1\} \otimes \{n-4,2,1^2\} = \{n-3,2,1\} + \{n-3,1^3\} + \{n-4,3,1\} + \{n-4,2^2\} + 2\{n-4,2,1^2\} + \{n-4,1^4\} + \{n-5,3,1^2\} + \{n-5,2^2,1\} + \{n-5,2,1^3\}$$

$$\{n-1,1\} \otimes \{n-5,2,1^3\} = \{n-4,2,1^2\} + \{n-4,1^4\} + \{n-5,3,1^2\} + \{n-5,2^2,1\} + 2\{n-5,2,1^3\} + \{n-5,1^5\} + \{n-6,3,1^3\} + \{n-6,2^2,1^2\} + \{n-6,2,1^4\}.$$

It is appropriate to mention here that formulae 1, 2, 3 & 4 satisfy the following rule

$$\text{If} \quad \{n-1,1\} \otimes \{n-m,\nu\} = \sum\{n-r,\lambda\}$$

where ν is a partition of m and λ a partition of r then

$$\{n-1,1\} \otimes \{n-m,\nu^*\} = \sum\{n-r,\lambda^*\}$$

where ν^* & λ^* are the conjugate partitions of ν & λ. This rule reduces the four previous formulae to two formulae only.

Formula 5) $\{n-2,2\} \otimes \{n-m,m\} = \{n-m,m\} + \{n-m,m-1,1\} + \{n-m-1,m,1\} + \{n-m-1,m-1,1^2\} + \sum_{r=m-2}^{m+2}\{n-r,r\} + \sum_{m-1}^{m+2}\{n-r,r-1,1\} + \sum_{m}^{m+2}\{n-r,r-2,2\}$, $m > 1$.

For $m = 2, 3, 4$ & 5 we get

$$\{n-2,2\} \otimes \{n-2,2\} = \{n\} + \{n-1,1\} + 2\{n-2,2\} + \{n-2,1^2\} + \{n-3,3\} + 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,3,1\} + \{n-4,2^2\} + \{n-4,4\},$$

$$\{n-2,2\} \otimes \{n-3,3\} = \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\} + 2\{n-3,3\} + 2\{n-3,2,1\} + \{n-4,4\} + 2\{n-4,3,1\} + \{n-4,2^2\} + \{n-4,2,1^2\} + \{n-5,5\} + \{n-5,4,1\} + \{n-5,3,2\},$$

$$\{n-2,2\} \otimes \{n-4,4\} = \{n-2,2\} + \{n-3,3\} + \{n-3,2,1\} + 2\{n-4,4\} + 2\{n-4,3,1\} + \{n-4,2^2\} + \{n-5,5\} + \{n-5,3,1^2\} + \{n-5,3,2\} + 2\{n-5,4,1\} + \{n-6,6\} + \{n-6,5,1\} + \{n-6,4,2\},$$

$$\{n-2,2\}\otimes\{n-5,5\}=\{n-3,3\}+\{n-4,4\}+\{n-4,3,1\}+2\{n-5,5\}+$$
$$+2\{n-5,4,1\}+\{n-5,3,2\}+\{n-6,6\}+$$
$$+2\{n-6,5,1\}+\{n-6,4,2\}+\{n-6,4,1^2\}+$$
$$+\{n-7,7\}+\{n-7,6,1\}+\{n-7,5,2\}.$$

Formula 6) $\{n-2,2\}\otimes\{n-m,1^m\}=\{n-m,1^m\}+\sum_{r=m-1}^{m+1}\{n-r,1^r\}+$
$+\sum_{m-1}^{m+2}\{n-r,2,1^{r-2}\}+\sum_{m}^{m+1}\{n-r,2,1^{r-2}\}+\sum_{m-1}^{m+2}\{n-r,3,1^{r-3}\}+$
$+\sum_{m}^{m+1}\{n-r,2^2,1^{r-4}\},\ m>2.$

For $m=3,4,5$ & 6 we get

$$\{n-2,2\}\otimes\{n-3,1^3\}=\{n-2,2\}+\{n-2,1^2\}+2\{n-3,2,1\}+$$
$$+2\{n-3,1^3\}+\{n-4,3,1\}+\{n-4,2^2\}+$$
$$+2\{n-4,2,1^2\}+\{n-4,1^4\}+\{n-5,3,1^2\}+$$
$$+\{n-5,2,1^3\},$$

$$\{n-2,2\}\otimes\{n-4,1^4\}=\{n-3,2,1\}+\{n-3,1^3\}+\{n-4,2^2\}+$$
$$+2\{n-4,2,1^2\}+2\{n-4,1^4\}+\{n-5,3,1^2\}+$$
$$+\{n-5,2^2,1\}+2\{n-5,2,1^3\}+\{n-5,1^5\}+$$
$$+\{n-6,3,1^3\}+\{n-6,2,1^4\}.$$

$$\{n-2,2\}\otimes\{n-5,1^5\}=\{n-4,2,1^2\}+\{n-4,1^4\}+\{n-5,2^2,1\}+$$
$$+2\{n-5,2,1^3\}+2\{n-5,1^5\}+\{n-6,3,1^3\}+$$
$$+\{n-6,2^2,1^2\}+2\{n-6,2,1^4\}+\{n-6,1^6\}+$$
$$+\{n-7,3,1^4\}+\{n-7,2,1^5\}.$$

$$\{n-2,2\}\otimes\{n-6,1^6\}=\{n-5,2,1^3\}+\{n-5,1^5\}+\{n-6,2^2,1^2\}+$$
$$+2\{n-6,2,1^4\}+2\{n-6,1^6\}+\{n-7,3,1^4\}+$$
$$+\{n-7,2^2,1^3\}+2\{n-7,2,1^5\}+\{n-7,1^7\}+$$
$$+\{n-8,3,1^5\}+\{n-8,2,1^6\}.$$

Formula 7) $\{n-2,1^2\}\otimes\{n-m,m\}=\{n-m-1,m-1,2\}+$
$+\sum_{m-1}^{m+1}\{n-r,r\}+\sum_{m-1}^{m+2}\{n-r,r-1,1\}+\sum_{r=m}^{m+1}\{n-r,r-1,1\}+$
$+\sum_{r=m}^{m+2}\{n-r,r-2,1^2\},\ m>1.$

For $m = 2, 3, 4$ & 5 we get

$$\{n-2,1^2\} \otimes \{n-2,2\} = \{n-1,1\} + \{n-2,2\} + 2\{n-2,1^2\} + \{n-3,3\} + 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,3,1\} + \{n-4,2,1^2\},$$

$$\{n-2,1^2\} \otimes \{n-3,3\} = \{n-2,2\} + \{n-2,1^2\} + \{n-3,3\} + 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,4\} + 2\{n-4,3,1\} + \{n-4,2^2\} + \{n-4,2,1^2\} + \{n-5,4,1\} + \{n-5,3,1^2\},$$

$$\{n-2,1^2\} \otimes \{n-4,4\} = \{n-3,3\} + \{n-3,2,1\} + \{n-4,4\} + 2\{n-4,3,1\} + \{n-4,2,1^2\} + \{n-5,5\} + \{n-5,4,1\} + \{n-5,3,2\} + \{n-5,3,1^2\} + \{n-6,5,1\} + \{n-6,4,1^2\},$$

$$\{n-2,1^2\} \otimes \{n-5,5\} = \{n-4,4\} + \{n-4,3,1\} + \{n-5,5\} + 2\{n-5,4,1\} + \{n-5,3,1^2\} + \{n-6,6\} + 2\{n-6,5,1\} + \{n-6,4,2\} + \{n-6,4,1^2\} + \{n-7,6,1\} + \{n-7,5,1^2\}.$$

Formula 8) $\{n-2,1^2\} \otimes \{n-m,1^m\} = \sum_{m-2}^{m+2} \{n-r,1^r\} + \sum_{r=m-1}^{m+2} \{n-r,2,1^{r-2}\} + \sum_{m+1}^{m+2} \{n-r,2^2,1^{r-4}\} + \sum_{r=m}^{m+1} \{n-r,2,1^{r-2}\} + \sum_{r=m}^{m+1} \{n-r,3,1^{r-3}\}.$

For $m = 2, 3, 4$ & 5 we get

$$\{n-2,1^2\} \otimes \{n-2,1^2\} = \{n\} + \{n-1,1\} + 2\{n-2,2\} + \{n-2,1^2\} + \{n-3,3\} + 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,2^2\} + \{n-4,2,1^2\} + \{n-4,1^4\},$$

$$\{n-2,1^2\} \otimes \{n-3,1^3\} = \{n-1,1\} + \{n-2,2\} + \{n-2,1^2\} + \{n-3,3\} + 2\{n-3,2,1\} + \{n-3,1^3\} + \{n-4,3,1\} + \{n-4,2^2\} + 2\{n-4,2,1^2\} + \{n-4,1^4\} + \{n-5,2^2,1\} + \{n-5,2,1^3\} + \{n-5,1^5\},$$

$$\{n-2, 1^2\} \otimes \{n-4, 1^4\} = \{n-2, 1^2\} + \{n-3, 2, 1\} + \{n-3, 1^3\} + \\ + \{n-4, 3, 1\} + 2\{n-4, 2, 1^2\} + \{n-4, 1^4\} + \\ + \{n-5, 3, 1^2\} + \{n-5, 2^2. 1\} + 2\{n-5, 2, 1^3\} + \\ + \{n-5, 1^5\} + \{n-6, 2^2, 1^2\} + \{n-6, 2, 1^4\} + \\ + \{n-6, 1^6\},$$

$$\{n-2, 1^2\} \otimes \{n-5, 1^5\} = \{n-3, 1^3\} + \{n-4, 2, 1^2\} + \{n-4, 1^4\} + \\ + \{n-5, 3, 1^2\} + 2\{n-5, 2, 1^3\} + \{n-5, 1^5\} + \\ + \{n-6, 3, 1^3\} + \{n-6, 2^2, 1^2\} + 2\{n-6, 2, 1^4\} + \\ + \{n-6, 1^6\} + \{n-7, 2^2, 1^2\} + \{n-7, 2, 1^4\} + \\ + \{n-7, 1^7\}.$$

The author is looking forward for more formulae that will give $\{n - r, \lambda\} \otimes \otimes \{n - s, \mu\}$ in a systematic way.

References

[1] *Kakar A. G.:* On the inner product of *S*-functions. J.L.M.S. (31) 1956.

[2] *Littlewood D. E.:* The Kronecker product of symmetric group representations. J.L.M.S. 1956.

[3] *Littlewood D. E.:* The theory of group characters and matrix representation. Oxford, 1940.

[4] *Makar R. H.:* On the analysis of the Kronecher product of irreducible representations of symmetric groups. Edinb. Math. Soc. 1949.

[5] *Makar R. H.:* On the Kronecker product of irreducible representations of symmetric group. Ain shams Science Bulliten 1957.

[6] *Murnaghan F. D.:* The analysis of the direct product of irreducible representation of symmetric group. Amer. J. Math. 1938.

[7] *Murnaghan F. D.:* On the analysis of the Kronecker product of irreducible representations of symmetric group. Amer. J. Math. 1938.

[8] *Murnaghan F. D.:* On the analysis of the Kronecker product of the irreducible representation of S_n. Proc. Nat. Acad. Sci. U.S.A. 1955.

[9] *Murnaghan F. D.:* Proc. Nat. Acad. Sci. U.S.A. 1955.

[10] *Murnaghan F. D.:* Proc. Nat. Acad. Sci. U.S.A. 1956.

[11] *Robinson & Taulbee:* Proc. Nat. Acad. Sci. U.S.A. 1954.

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