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NOTES ON INTEGRATION

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The following result plays a fundamental role in the integration theory:

Theorem. *Let C be the set of all real-valued continuous functions on a compact Hausdorff space S. Let r be a positive linear functional on C and let $\{f_n, n \in N\} \subset C$, $|f_n| \leqq M < \infty$ for all $n \in N$ and* $\lim f_n(s) = 0$ $(s \in S)$. *Then* $\lim r(f_n) = 0$.

One of the "elementary" proofs is given in [1]. More general results from which the above theorem follows can be found in [2] and [3]. In this paper we prove a theorem from which the above theorem immediately follows.

Theorem 1. *Let S be a nonvoid set and let C denote a set of real-valued functions on S which satisfies*

(C1) $o \in C$, *where* $o(s) = 0$ $(s \in S)$;

(C2) *If $f, g \in C$ then*

a) $f + g \in C$

b) $f \vee g = \sup(f, g) \in C$

c) $f \wedge g = \inf(f, g) \in C$.

Let $\{f_n, n \in N\}$ be a sequence of C, $\lim f_n(s) = 0$ $(s \in S)$ *and let r be a real-valued functional on C such that*

(r1) $r(f + g) = r(f) + r(g)$ $(f, g \in C)$;

(r2) $\sup\limits_{n \in N} \sup \{|r(f)|;\ \bigwedge\limits_{i=1}^{n} f_i \wedge o \leqq f \leqq \bigvee\limits_{i=1}^{n} f_i \vee o, f \in C\} < \infty$;

(r3) *if $\{g_n, n \in N\}$ and $\{h_n, n \in N\}$ are sequences of C such that for some $k \in N$*

$$\bigwedge_{i=1}^{k} f_i \wedge o \leqq h_1 \leqq \ldots \leqq h_{n-1} \leqq h_n \leqq \ldots \leqq o \leqq \ldots \leqq g_n \leqq g_{n-1} \leqq \ldots$$
$$\ldots \leqq g_1 \leqq \bigvee_{i=1}^{k} f_i \vee o$$

and $\lim g_n(s) = \lim h_n(s) = 0$ *for all $s \in S$, then* $\lim r(g_n) = \lim r(h_n) = 0$.

Then $\lim r(f_n) = 0$.

Proof. Let $\varepsilon > 0$ and let $\{\varepsilon_n;\, n \in N\}$ be a sequence of positive numbers, $\varepsilon = \sum_{n\in N} \varepsilon_n$. Let $f_n \geqq o$ for all $n \in N$. We set

$$G_1 = \{\bigvee_{i\in K} f_i.\ K \text{ is a finite subset of } N\}\,.$$

Let g_1 be an element of G_1 such that

$$r(g_1) \geqq \sup\{r(f), f \in G_1\} - \varepsilon_1\,.$$

Let $G_1, \ldots, G_n$ and $g_1, \ldots, g_n$ be defined; then we set

$$G_{n+1} = \{\bigvee_{i\in K} (f_i \wedge g_n),\ K \text{ is a finite subset of } N_{n+1}\}$$

where $N_k = \{n \in N,\ n \geqq k\}$ for any $k \in N$; let g_{n+1} be an element of G_{n+1} for which

$$r(g_{n+1}) \geqq \sup\{r(f), f \in G_{n+1}\} - \varepsilon_{n+1}\,.$$

Then we have:

a) $g_n \in C$ for all $n \in N$, $o \leqq \ldots g_n \leqq g_{n-1} \leqq \ldots \leqq g_1 \leqq \bigvee_{i=1}^{k} f_i$ for some $k \in N$, $g_n(s) \leqq \sup_{k\geqq n} f_k(s)$ and hence $\lim g_n(s) = 0$ $(s \in S)$. So, by (r3):

$$\lim r(g_n) = 0\,. \tag{1}$$

b) $f_n \vee g_1 \in G_1$ and consequently

$$r(g_1) \geqq r(f_n \vee g_1) - \varepsilon_1$$

for any $n \in N$. Further we have

$$(f_n \wedge g_1) + (f_n \vee g_1) = f_n + g_1$$

and hence we obtain

$$r(f_n \wedge g_1) = r(f_n) + r(g_1) - r(f_n \vee g_1) \geqq r(f_n) - \varepsilon_1\,. \tag{2}$$

c) For $k \geqq 2$ and $n \geqq k$ we have

$$a_1 = (f_n \wedge g_{k-1}) \wedge g_k = f_n \wedge g_k\,, \quad a_2 = (f_n \wedge g_{k-1}) \vee g_k \in G_k\,.$$

From the equality

$$a_1 + a_2 = (f_n \wedge g_{k-1}) + g_k$$

we obtain that

$$r(f_n \wedge g_k) = r(f_n \wedge g_{k-1}) + r(g_k) - r(a_2) \geqq r(f_n \wedge g_{k-1}) - \varepsilon_k\,. \tag{3}$$

The relation $r(g_k) \geqq r(f_n \wedge g_{k-1}) - \varepsilon_k$ for $n \geqq k$ together with (2) and (3) imply the following assertion:

$$(4) \qquad r(g_k) \geqq r(f_n) - \sum_{i=1}^{k} \varepsilon_i \quad \text{for} \quad k \in N \quad \text{and} \quad n \geqq k\,.$$

Hence, by (1), $\limsup r(f_n) \leqq \varepsilon$ for any $\varepsilon > 0$ and so

$$(5) \qquad \limsup r(f_n) \leqq 0\,.$$

Since the functional $(-r)$ also satisfies the assumptions of Theorem 1, we have

$$(6) \qquad 0 \geqq \limsup\,(-r)\,(f_n) = -\liminf r(f_n)\,.$$

(5) and (6) imply $\lim r(f_n) = 0$.

If $f_n \leqq o$ for $n \in N$, we can prove $\lim r(f_n) = 0$ analogously, by replacing $\vee$ by $\wedge$, sup by inf etc. For arbitrary f_n (which satisfy the assumptions of Theorem 1) we have

$$\lim r(f_n) = \lim r((f_n \vee o) + (f_n \wedge o)) = \lim r(f_n \vee o) + \lim r(f_n \wedge o) = 0$$

because $\{f_n \vee o,\ n \in N\}$ and $\{f_n \wedge o,\ n \in N\}$ also satisfy the assumptions of Theorem 1.

Theorem 2. *Let S be a nonvoid set and let C denote a set of real-valued bounded functions on S such that* (C2) *and*

(C3) *If $f \in C$ then $(-f) \in C$*

are fulfilled.

Let p be a real-valued functional defined on C with the following properties:

(p1) $p(f + g) = p(f) + p(g)$ $(f, g \in C)$

(p2) $|p(f)| \leqq K \sup\{|f(s)|,\ s \in S\}$ *for any* $f \in C$

(p3) $\lim p(g_n) = 0$ *for any sequence* $\{g_n,\ n \in N\}$ *of C such that* $g_{n+1} \leqq g_n$ *for* $n \in N$ *and* $\lim g_n(s) = 0$ $(s \in S)$.

Let $\{f_n,\ n \in N\}$ be a sequence of C such that $|f_n(s)| \leqq M$ $(n \in N,\ s \in S)$ and $\lim f_n(s) = 0$ $(s \in S)$. *Then* $\lim p(f_n) = 0$.

Proof. Theorem 2 is an easy consequence of Theorem 1.

Theorem 3. *Let S be a nonvoid set and let C be a set of real-valued bounded functions on S such that* (C2) *and* (C3) *are fulfilled. Let p be a real-valued functional on C which fulfils the assumptions* (p1), (p2), (p3). *Let $\{f_n,\ n \in N\}$ be a sequence of functions from C such that $|f_n(s)| \leqq M$ $(n \in N,\ s \in S)$ and* $\lim f_n(s) = f(s)$ $(s \in S)$. *Then* $\lim p(f_n)$ *exists and* $\lim p(f_n) = p(f)$ *if* $f \in C$.

Proof. It is very well known that a sequence $\{x_n, n \in N\}$ of real numbers is a Cauchy sequence ($\lim x_n$ exists) if and only if $\lim (x_{k_{n+1}} - x_{k_n}) = 0$ for any sequence $\{k_n, n \in N\}$ of natural numbers with $k_{n+1} > k_n$.

Let $\{k_n, n \in N\}$ be a sequence of N such that $k_{n+1} > k_n$ for any $n \in N$. Then the functions $g_n = f_{k_{n+1}} - f_{k_n}$ satisfy the assumptions of Theorem 2 and so $0 = \lim p(g_n) = \lim (p(f_{k_{n+1}}) - p(f_{k_n}))$. Hence we obtain that $\lim p(f_n)$ exists.

If $f \in C$, then $\lim p(f_n - f) = 0$ by Theorem 2, i.e. $\lim p(f_n) = p(f)$.

Example. Let AP be the set of all continuous almost periodic real-valued functions on the set of real numbers R and let b be a functional defined on AP by

$$b(f) = \lim_{T \to \infty} T^{-1} \int_0^T f(s)\, ds .$$

Then there exists a sequence $\{f_n, n \in N\} \subset AP$ such that $f_n \geqq f_{n+1} \geqq o$ $(n \in N)$, $\lim f_n(s) = 0$ $(s \in R)$ and $b(f_n) \geqq c > 0$.

Proof. If such a sequence does not exist, then the assumptions of Theorem 3 are obviously fulfilled. But for the sequence $\{g_n, n \in N\}$ where $g_n(s) = \cos n^{-1}s$ $(s \in S)$ we have $1 = b(\lim g_n) \neq \lim b(g_n) = 0$.

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