Jiří Veselý
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SEQUENCE SOLUTIONS OF THE DIRICHLET PROBLEM

JIŘÍ VESELÝ, Praha

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Introduction. Many methods for solving the classical Dirichlet problem for an open bounded set $D \subset R^m$, $m \geqq 2$, were invented since 1870 when H. A. Schwarz published his alternating method. They were mostly based on the following procedure: the prescribed continuous boundary condition $f \in C(\partial D)$ is arbitrarily but continuously extended to a function $F \in C(\bar{D})$; then a sequence of functions $\{F_n\}$ is constructed from F in such a way that it is convergent to the required solution. Methods of this type will be termed "sequence methods". Some of them played an important role in the development of the classical potential theory. Thus the balayage is closely related to the method of Poincaré while Wiener's method is connected with the generalized solution (PWB-solution), and Lebesgue's method helped to clear up the importance of Brownian motion for the probabilistic approach to the Dirichlet problem. On the other hand, in axiomatic potential theories only the Perron solution was systematically used. Recently it was shown (see [7], [11]) that a Wiener-type procedure can be defined in the frame of axiomatics of Constantinescu and Cornea [4] and that the procedure leads to the Perron solution. Let us point out that in this context there are possibly more than one reasonable generalized solutions.

In this article we shall present a method which has its origin in a procedure described by Lebesgue in [6]. The method is again studied in the context of axiomatics [4] and leads to the Perron solution as well. We shall prove a theorem on a general sequence procedure; our approach goes back to investigations due to Carathéodory (cf. [3]). The illustrating examples should be of an independent interest. As a by-product we obtain a generalization of a converse of Gauss' Mean Value Theorem.

1. Notation. Symbol X denotes a fixed $\mathfrak{P}$-harmonic space with countable base in the sense of [4]. We suppose that constant functions are harmonic. Symbols $\mathscr{S}(U)$ and $\mathscr{H}(U)$ denote the systems of superharmonic and harmonic functions on an open set $U \subset X$, respectively. As usual, we denote by ∂M, $\bar{M}$ and $\complement M$ the boundary, the closure and the complement of a set M; $C(M)$ will stand for the set of continuous functions on M. In what follows D denotes a fixed open relatively compact subset of X, sometimes for a special choice of X. The set of (positive Radon) measures μ with support $\operatorname{supp} \mu \subset \bar{D}$ for which $\mu(\bar{D}) = 1$ will be denoted by $\mathfrak{M}$; in particular, ε_x stands for the Dirac measure supported by $\{x\}$.

Working mostly with functions defined on D or $\bar{D}$, we denote $\mathscr{S} = \mathscr{S}(D)$, $\mathscr{H} = \mathscr{H}(D)$. We put

$$S = \{s : \bar{D} \to (-\infty, \infty\rangle;\ s \text{ lower semi-continuous, } s|_D \in \mathscr{S}\}$$

(where $s|_D$ is the restriction of s on D) and $H = S \cap (-S)$. Let S_c^+ be the set of positive continuous functions from S. The system of differences of functions from S_c^+ is uniformly dense in $C(\bar{D})$ (cf. [4], Theorem 2.3.1); this fact will be referred to as (1).

We shall deal with positive linear operators A having the property

$$(2) \qquad As \leqq s \quad \text{for all} \quad s \in S\,.$$

The set of such operators will be denoted by $\mathfrak{A}$. Condition (2) together with (1) form the basis for a one-to-one mapping of $\mathfrak{A}$ onto a set of systems of measures: we shall identify $A \in \mathfrak{A}$ with the system of measures $\{\alpha_x;\ x \in \bar{D}\}$ such that

$$Ag(x) = \alpha_x(g)$$

for every $g \in C(\bar{D})$. Operators from $\mathfrak{A}$ may be considered to act on the set of bounded Borel measurable functions on $\bar{D}$, but Ag is well-defined for a larger class of functions on $\bar{D}$. Notice that without any other assumptions on A the function Ag may be non-measurable even for $g \in C(\bar{D})$. Consider functions g which are equal to 0 at a boundary point of D and to 1 elsewhere in $\bar{D}$. Then $Ag \leqq g$ and it is easily seen that for any $A = \{\alpha_x;\ x \in \bar{D}\} \in \mathfrak{A}$ we have $\alpha_x = \varepsilon_x$ for all $x \in \partial D$.

The subset of $\mathfrak{A}$ of such operators A for which

$$(3) \qquad \operatorname{supp} \alpha_x \subset D \quad \text{for all} \quad x \in D$$

will be denoted by $\mathfrak{A}'$. Notice that for $A \in \mathfrak{A}'$ we can neglect α_x for the points x of ∂D and apply A on functions defined on D only. In this case As is well-defined for example for every $s \in \mathscr{S}$.

2. A minimum principle. *Suppose* $A = \{\alpha_x;\ x \in \bar{D}\} \in \mathfrak{A}'$ *and*

$$(4) \qquad \alpha_x \neq \varepsilon_x \quad \textit{for every} \quad x \in D\,.$$

Then for any function u *from the set*

$$\mathscr{S}_A = \{v : D \to (-\infty, \infty\rangle;\ v \text{ l.s.c., } Av \leqq v\}$$

we have

$$(5) \qquad \inf_{z \in \partial D} \big(\liminf_{x \to z,\ x \in D} u(x)\big) \leqq \inf_{x \in D} u(x)\,.$$

Suppose $A = \{\alpha_x;\ x \in \bar{D}\} \in \mathfrak{A}$ *and* (4). *Then for any function* u *from*

$$S_A = \{v : \bar{D} \to (-\infty, \infty\rangle;\ v \text{ l.s.c., } Av \leqq v\}$$

we have

(6) $$\inf_{z\in\partial D} u(z) \leqq \inf_{x\in D} u(x) .$$

Proof. We shall use an argument which is due to H. Bauer; our assertion follows from his general minimum principle (see [2], p. 7). The inequality (5) obviously holds provided the left hand side is equal to $-\infty$. Hence we may suppose that u is lower bounded on D. Consider the extension of u on $\bar{D}$ defined by

(7) $$u(z) = \liminf_{x\to z, x\in D} u(x) ;$$

this extension u is l.s.c. on $\bar{D}$. The resulting situation corresponds to the second part of the assertion and it is enough to prove (6).

Since $S_c^+ \subset S_A$ separates points of $\bar{D}$, it is easily seen (cf. [2]) that every $u \in S_A$ attains its minimal value at such a point, say x', for which the only measure $\mu \in \mathfrak{M}$ with $\mu(v) \leqq v(x')$ for all $v \in S_A$ is the Dirac measure $\varepsilon_{x'}$. But then $\alpha_{x'} = \varepsilon_{x'}$, hence $x' \in \partial D$ and (6) is proved.

3. Proposition. *Suppose* $A \in \mathfrak{A}$ *and* (4). *Denote*

(9) $$H_A = S_A \cap (-S_A) .$$

Then $H = H_A$.

Proof. The inclusion $H \subset H_A$ follows from the definition of $\mathfrak{A}$. Let $h \in H_A$ and let s be an upper function for $f = h|_{\partial D}$, i.e.

$$\liminf_{x\to z, x\in D} s(x) \geqq f(z)$$

for all $z \in \partial D$. Extend $u = s - h$ on $\bar{D}$ by (7). Then u is non-negative on ∂D and by the above minimum principle also on $\bar{D}$. This implies that the upper Perron solution $\bar{H}_f^D$ satisfies $h \leqq \bar{H}_f^D$ on D. Similarly we get $\underline{H}_f^D \leqq h$ and hence $h = H_f^D$ on D. Thus we have $h \in H$.

4. Corollary. *Suppose* $A \in \mathfrak{A}'$ *and* [4]. *Denote the set* $\mathscr{S}_A \cap (-\mathscr{S}_A)$ *by* $\mathscr{H}$. *If for a function* $g \in \mathscr{H}_A$ *there is an* $h \in \mathscr{H}$ *such that* $h - g$ *has finite limits at all boundary points of* D, *then* $g \in \mathscr{H}$.

Proof. It is easily seen that $u = h - g$ can be continuously extended on $\bar{D}$. Then $u \in H$ follows from Proposition 3.

5. Remark. Let $D \subset R^m$, $m \geqq 1$ (notice that R^m for $m = 1$ or 2 is not a $\mathfrak{P}$-harmonic space, but we can put $X = G \subset R^m$, where G is a Green set containing D) and denote for $\Omega_r(x) = \{y; \|x - y\| < r\}$

(10) $$A(u; x, r) = (\operatorname{vol}(\Omega_r(x)))^{-1} \int_{\Omega_r(x)} u(y)\,\mathrm{d}y\,.$$

Similarly, $L(u; x, r)$ denotes the average of u over the sphere $\partial\Omega_r(x)$.

The above two assertions are generalized versions of the following converse of Gauss' Mean Value Theorem:

If for every $x \in D$ and an $r_x \in (0, \operatorname{dist}(x, \complement D))$ we have $u(x) = A(u; x, r_x)$ [or $u(x) = L(u; x, r_x)$] and $u \in C(\bar{D})$, then $u \in H$.

On the other hand, to find some non-trivial conditions which should be imposed on A in order that $h \in \mathscr{H}_A$ may imply $h \in \mathscr{H}$ (even in the classical case and for bounded h) is a difficult problem (see [13] or an expository article on this subject [9]).

Let us turn to the Dirichlet problem. An operator A is said to be iterated with respect to a sequence of operators $\{A_n\}$ (w.r.t. $\{A_n\}$) provided

(11) $$A = A_n \quad \text{for infinitely many} \quad n \in N\,.$$

In other words, given an iterated operator A we have an increasing sequence $\{n_k\}$ of all such $n \in N$ with $A = A_n$. The corresponding sequence $\{n_k\}$ for A will be used in the assumption (iii) of the following theorem.

6. Theorem. *Let us suppose that a sequence of operators $\{A_n\}$ fulfils the following conditions:*

(i) $A_n \in \mathfrak{A}$ *for all* $n \in N$ (cf. (2)).

(ii) *For every continuous function F on $\bar{D}$ the sequence of functions $\{F_n\}$ defined by*

(12) $$F_0 = F\,, \quad F_n = A_n F_{n-1}\,, \quad n \in N$$

is well-defined everywhere in $\bar{D}$; moreover, for $F \in S_c^+$ this sequence is non-increasing.

(iii) *For any $x \in D$ there is an iterated operator A (w.r.t. $\{A_n\}$) and a neighbourhood U of x such that the corresponding $\alpha_x \neq \varepsilon_x$ and all F_{n_k} are continuous on U.*

Then, given $f \in C(\partial D)$ and any $F \in C(\bar{D})$ with $F|_{\partial D} = f$, the sequence $\{F_n\}$ defined by (12) is convergent on $\bar{D}$ and for every $x \in D$ we have

(13) $$\lim_{n \to \infty} F_n(x) = H_f^D(x)\,.$$

This convergence is uniform on compact subsets of D. In case D is a regular set the convergence is uniform on $\bar{D}$.

Let us remark here that conditions (i)–(iii) are chosen to fit in with the study of analogs of standard classical methods in the axiomatics. We do not intend to study their dependence or to formulate them as weak as possible.

Proof. From (1) it is easily seen that it is enough to prove (13) for functions from S_c^+.

Suppose $F \in S_c^+$ and denote $f = F|_{\partial D}$. Lower boundedness of $\{F_n\}$ by a constant function (which is invariant w.r.t. any $A \in \mathfrak{A}$) implies pointwise convergence of $\{F_n\}$. Put $u = H_f^D$ on D and $u = f$ on ∂D. Let $s[v]$ be an upper [lower] function for f extended on $\bar{D}$ by liminf [limsup]. Then for any $A \in \mathfrak{A}$ we have by monotonicity of A

$$v \leqq Av \leqq Au \leqq As \leqq s$$

and consequently $Au = u$ for any $A \in \mathfrak{A}$. Hence $u \leqq F$ implies $H_f^D \leqq h = \lim F_n$ (on D). The limit function h is upper semi-continuous on D; this follows from the continuity in (iii).

Fix for a moment an $x \in D$ and choose by (iii) an iterated operator $A = \{\alpha_x; x \in \bar{D}\}$ with $\alpha_x \neq \varepsilon_x$. Then

$$Ah(x) = \alpha_x(h) = \alpha_x(\lim F_{n_k - 1}) = \lim F_{n_k}(x) = h(x) \tag{14}$$

by the monotone convergence theorem and the system of such measures $\{\alpha_x; x \in D\}$ together with $\alpha_x = \varepsilon_x$ for $x \in \partial D$ determines an operator $A' \in \mathfrak{A}$. Extend now h on $\bar{D}$ by

$$h(z) = \limsup_{x \to z, x \in D} h(x) ;$$

this extended function h is u.s.c. on $\bar{D}$. For an upper function s extended on $\bar{D}$ as above we have

$$A'(s - h) \leqq s - h \in S_{A'}$$

and the minimum principle for $S_{A'}$ implies $h \leqq s$. Hence we have $h \leqq H_f^D$ on D and consequently $h = H_f^D$ on D.

The uniform convergence on compact subsets of D follows from (ii) and (iii) by Dini's theorem. If D is regular, we can subtract u from F and restrict ourselves to the case of those F for which $F|_{\partial D} = 0$. It is easily seen that the convergence in (13) is uniform on $\bar{D}$ for such functions F from S_c^+.

7. Remarks. (1) In the classical case, sequence procedures like Schwarz's alternating method, the balayage method of Poincaré, methods of Lebesgue and Kellog (see below) and Wiener's method can be derived from Theorem 6 (cf. [3]). Let us note that in these cases all F_n are continuous on D.

(2) One should ask what can be expected from generalizations of these methods in axiomatics. For some cases we must overcome principal difficulties (cf. [7], where the case of Wiener's method is studied; let us note that the procedure from [7] cannot be derived from our theorem). We shall concentrate our interest on methods

of Poincaré and Lebesgue — in both cases we shall get again the "right solution", i.e. the Perron solution.

(3) The reasoning which was used in the proof of Theorem 6 is of a certain interest even for cases which are not covered by axiomatics [4]. The method of finite differences offers some discrete analogs of (super-) harmonicity for the Laplace and the heat equation. For these cases we may prove similarly the convergence of the method of successive approximations (for the "discrete harmonicity" for the Laplace equation cf. [1], § 7).

8. Proposition. *Let D_n be a sequence of open subsets of D with $\bar{D}_n \subset D$. Suppose that $\{D_n\}$ has the following property* (Poincaré): *For every $x \in D$ there is a $D' \subset D$ such that $x \in D'$ and*

$$(15) \qquad D' = D_n \quad \text{for infinitely many} \quad n \in N .$$

For any $n \in N$ let us define $A_n = \{\alpha_x;\, x \in \bar{D}\}$ where $\alpha_x = \varepsilon_x^{CD_n}$ for $x \in D$, $\alpha_x = \varepsilon_x$ for $x \in \partial D$. Symbol ε_x^M denotes the balayage of ε_x on M. Then the assumptions of the theorem are satisfied and the corresponding procedure leads thus to the Perron solution.

We shall omit the proof. Notice that (i)–(iii) follow from the properties of balayage of functions (cf. [4]) — in the theorem we work with $F \in S_c^+$ and hence F_n are superharmonic functions. The procedure could be considered being not too far from the "classical" setting where D_n are regular sets. Let us illustrate its use. Recently Watson in [15] worked with a procedure of this type with non-regular sets D_n for the case of the heat equation. The explicit expression of the generalized solution of the Dirichlet problem for the $(m + 1)$-dimensional interval led Friedman [5] to a certain generalization of Poincaré's method for a broad class of parabolic equations. Both mentioned procedures are of the type described in Proposition 8.

The balayage method (for the Laplace equation) may be found in textbooks on potential theory. On the other hand, Lebesgue's method is not so well-known. We shall briefly recall some facts. Lebesgue described this method in 1912 (see [6]). With the help of the notation introduced above we can characterize it as follows: There is a single iterated operator that corresponds to every $x \in D$, i.e. $A_n = A$ for all $n \in N$. For any $F \in C(\bar{D})$ the sequence $\{F_n\}$ is defined by

$$F_0 = F\,, \quad F_n(x) = A(F_{n-1};\, x, r_x)\,,$$

where averages in the recurrent formulae are described in (10) and $r = r_x = \operatorname{dist}(x, \complement D)$ for all $x \in D$. On the boundary we have $F_n|_{\partial D} = F|_{\partial D} = f$.

Lebesgue also mentioned the fact that other more general "averages" may be used (also spherical mean values) and for the sake of simplicity proved the corresponding theorem in R^2. He worked with regular sets D but later Perkins in 1927 (see [10]) remarked that the procedure may be used also for non-regular sets and that it leads to the Perron solution.

An article of Kellog [8] contains a modification of Lebesgue's procedure. Kellog attached to every $x \in D$ a regular set D_x and defined (with the help of the classical solution of the Dirichlet problem for D_x) the operator $A = \{\alpha_x; x \in \bar{D}\}$ by $\alpha_x = \varepsilon_x^{CD_x}$ for $x \in D$ and $\alpha_x = \varepsilon_x$ for $x \in \partial D$. Measures α_x were "tied together" by continuity ($AF \in C(\bar{D})$ for every $F \in C(\bar{D})$).

The next proposition describes a procedure of the same type as Lebesgue's one in the frame of axiomatics. The proof is omitted since the proposition is a consequence of Theorem 6.

10. Proposition. *Let an operator A satisfy the following conditions*:

(i) $A = \{\alpha_x; x \in \bar{D}\} \in \mathfrak{A}$ *and* $\alpha_x(\partial D) = 0$ *for all* $x \in D$.

(ii) *For every* $x \in D$ *we have* $\alpha_x \neq \varepsilon_x$.

(iii) A *is a mapping of* $C(\bar{D})$ *into* $C(\bar{D})$.

Then, given $f \in C(\partial D)$ *and any* $F \in C(\bar{D})$, $F|_{\partial D} = f$, *we have*

$$\lim_{n \to \infty} A^n F(x) = H_f^D(x)$$

for every $x \in D$. *This convergence is uniform on compact subsets of* D *and, for a regular* D, *uniform on* $\bar{D}$.

Any operator having the properties (i)–(iii) from this proposition will be called Lebesgue's operator for D. It remains to show how to construct a Lebesgue's operator for D in a general harmonic space X. We shall show it with the help of the next lemma.

11. Lemma. *Let Ω be an open subset of a metric space (P, ϱ). Let us suppose*:

(i) $\{f_x; x \in \Omega\}$ *is a set of positive non-increasing and uniformly bounded functions defined on* $\langle 0, d\rangle$, $d > 0$.

(ii) *For any* $x \in \Omega$ *and* $s, t, t' \in \langle 0, d\rangle$, $t < s < t'$, *there is a neighbourhood* $U(x)$ *of* x *in* Ω *such that the inequality*

$$f_x(t) \geqq f_y(s) \geqq f_x(t') \tag{16}$$

holds for every $y \in U(x)$.

(iii) *Function* g *is a continuous increasing function on* $\langle 0, d\rangle$, $g(0) = 0$.

Then the average

$$\mathscr{A}(f_x, g, c) = (g(c))^{-1} \int_0^c f_x(t) \, dg(t) \tag{17}$$

is a continuous function of $[x, c]$ *on* $\Omega \times (0, d)$.

Proof. Fix an $x \in \Omega$. Clearly, $c \mapsto \mathscr{A}(f_x, g, c)$ is continuous on $(0, d)$. Then we have for $f_x = f$ and any $c, c' \in (0, d)$, $c < c'$

$$\mathscr{A}(f, g, c') - \mathscr{A}(f, g, c) =$$

$$= (g(c)\, g(c'))^{-1} \left[g(c) \left(\int_0^c f\, \mathrm{d}g + \int_c^{c'} f\, \mathrm{d}g \right) - g(c') \int_0^c f\, \mathrm{d}g \right] =$$

$$= (g(c) - g(c')) \int_0^c f\, \mathrm{d}g + g(c) \int_c^{c'} f\, \mathrm{d}g \leqq$$

$$\leqq (g(c) - g(c')) f(c)\, g(c) + g(c) f(c) (g(c') - g(c)) = 0$$

and hence $c \mapsto \mathscr{A}(f, g, c)$ is non-increasing.

Since there is an $M < \infty$ such that $\|f_x\| \leqq M$ for all $x \in \Omega$ and the functions f_x are monotone, the difference of upper and lower integral sums for the integral in (17) can be estimated for a partition $P = \{0 = t_0 < t_1 < \ldots < t_n = c\}$ by $M\omega(g, |P|)$, where $\omega(g, \Delta) = \sup \{|g(t) - g(t')|;\ |t - t'| < \Delta\}$ and $|P| = \max \{t_i - t_{i-1};\ i = 1, \ldots, n\}$. This estimate does not depend on x. Fix a $c \in (0, d)$ and an $x \in \Omega$. Then for a partition $P = \{0 = t_0 < t_1 < \ldots < t_n = c\}$ and $s_i \in (t_{i-1}, t_i)$ we can choose a neighbourhood $U(x)$ of x such that

(18) $$f_x(t_{i-1}) \geqq f_y(s_i) \geqq f_x(t_i)$$

holds for all $y \in U(x)$ and $i = 1. 2, \ldots, n$. From (18) we obtain easily

$$\left| \sum_{i=1}^n f_x(t_i) (g(t_i) - g(t_{i-1})) - \sum_{i=1}^n f_y(s_i) (g(t_i) - g(t_{i-1})) \right| \leqq M\omega(g, |P|)$$

and hence we conclude

$$\left| \int_0^c f_x(t)\, \mathrm{d}g(t) - \int_0^c f_y(t) \mathrm{d}g(t) \right| \leqq 3M\omega(g, |P|)\,.$$

The last inequality implies that for a fixed c the average $\mathscr{A}(f_x, g, c)$ is a continuous function of x in Ω.

From the separate continuity in x and c and the monotonicity in c we obtain the joint continuity of $\mathscr{A}(f_x, g, c)$ on $\Omega \times (0, d)$ and Lemma 11 is proved.

12. Lebesgue's operators. Now we shall construct a large class of Lebesgue's operators for $D \subset X$. Let us remark that our assumptions on X imply that the harmonic space X is metrisable.

For a chosen metric ϱ compatible with the topology of X and for $x \in D$, $r > 0$ we put

$$D_x(r) = \{y \in X;\ \varrho(x, y) < r\}\,, \quad D^r = \{x \in D;\ r < r_x\}\,,$$

where $r_x = \operatorname{dist}(x, \complement D)$.

It is easily seen that given $x \in D$ and $s, t, t' \in \langle 0, r_x \rangle$, $t < s < t'$, there is a neighbourhood $U(x)$ of x such that

(19) $$D_x(t) \subset D_y(s) \subset D_x(t')$$

for all $y \in U(x)$; for $t = 0$ we put $D_x(0) = \{x\}$. Choose an increasing continuous function g defined on $\langle 0, \sup\{r_x; x \in D\}\rangle$ with $g(0) = 0$ and put for any $F \in C(\bar{D})$ and any continuous function $c(x)$, $0 < c(x) \leqq r_x$

(20) $$\alpha_x(F) = [g(c(x))]^{-1} \int_0^{c(x)} \varepsilon_x^{CD_t}(F)\, \mathrm{d}g(t)$$

for all $x \in D$ and $\alpha_x = \varepsilon_x$ for $x \in \partial D$.

The required Lebesgue's operator is now determined by $A = \{\alpha_x; x \in \bar{D}\}$. We shall show that A has properties (i)–(iii) from Proposition 10.

It is easily seen that (i) and (ii) are satisfied. Since (1) holds it is enough to prove that $AF \in C(\bar{D})$ for any $F \in S_c^+$. Continuity of AF at a point $x \in D$ follows from the continuity of c and Lemma 11, where we put $\Omega = D^r$ with $r < r_x$, $\langle 0, d\rangle = \langle 0, r_x\rangle$, $f_x(t) = \varepsilon_x^{CD_t}(F)$. Since $F \in S_c^+$, f_x is a monotone function on $\langle 0, d\rangle$ by the elementary properties of the balayage (cf. [4], Excercise 7.2.8) and $\|f_x\| \leqq \|F\|$. Inclusions (19) imply (ii), (iii) is satisfied and hence all assumptions of Lemma 11 are fulfilled. Considering the above correspondence we have

$$\alpha_x(F) = \mathscr{A}(f_x, g, c(x)), \quad x \in D^r$$

and the continuity is proved. For an $x \in \partial D$ and $\varepsilon > 0$ we can choose a neighbourhood $U(x)$ such that $|F(x) - F(y)| < \varepsilon$ for all $y \in U(x) \cap D$. Then $D_{c(y)}(y) \subset D_{r_y}(y) \subset U(x) \cap D$ for all y's sufficiently close to x and hence

$$|A\,F(x) - A\,F(y)| = |F(x) - \alpha_y(F)| \leqq \varepsilon.$$

Now it is easily seen that AF is continuous on $\bar{D}$ and so A is a Lebesgue's operator for D.

13. Remarks and examples. (1) If $X = R^m$, $m \geqq 1$ (Standard - Beispiel (1) in [2]; see Remark 5 for $m = 1, 2$), ϱ is the Euclidean metric and $g(t) = t^m/m$, then for $c(x) = r_x$ the operator A coincides with the operator described above at the end of item 9.

(2) Different choices of the function c lead to a class of Lebesgue's operators. A slight modification of the proof should include averages over more general domains. Let us remark that also different metrics ϱ produce different operators. On the other hand, spherical means may be also used to generate other Lebesgue's operators which cannot be obtained by the construction described above for a general harmonic space X.

(3) If $X = R^{m+1}$, $m \geqq 1$, then for the heat equation (Standard - Beispiel (2) in [2] measures $\varepsilon_x^{CD_t}$ have continuous densities with respect to the m-dimensional Hausdorff measure on ∂D_t provided we choose $\varrho(x, y) = \max\{a_i|x_i - y_i|; i = 1, \ldots, m + 1\}$ with all $a_i > 0$ (cf. [4], § 3.3). The corresponding averages of the type

used above were studied for example in [12]. This leads to a sequence procedure which seems to be new. Our approach does not cover averages connected with level sets of the fundamental solution of the heat equation. These averages offer a construction of other different Lebesgue's operators for the heat equation (cf. [14]).

(4) Special averages mentioned in (2) and (3) as $L(F; x, c(x))$ for the Laplace equation or its analog for the level set of the fundamental solution in case of the heat equation offer operators of "Kellog's type". The principal difficulty in generalizations of this procedure to the case of a $\mathfrak{P}$-harmonic space X consists in the problem of existence of a proper choice of sets D_x.

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Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta UK).