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CHARACTERIZATION OF CENTRAL DISPERSIONS OF THE FIRST KIND OF $y'' = Q(t)\,y$

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If the dispersion $\zeta \in (Q, Q)$ and the integral $u \in (Q)$ fulfil the relation (1) there is found a necessary and sufficient condition that ζ be a central dispersion.

1. Introduction. Let $Q(t)$ be a real function of a real variable, continuous in the interval (a, b) of arbitrary type. Every complete solution $\zeta(t)$, i.e. a solution passing from one side to the other in the interval $(a, b) \times (a, b)$, of the differential equation

$$(Q, Q) \qquad \sqrt{\zeta'}\left(\frac{1}{\sqrt{\zeta'}}\right)'' + Q(\zeta)\,\zeta'^2 = Q(t)$$

is an increasing dispersion (of the first kind) of the differential equation

$$(Q) \qquad y'' = Q(t)\,y\,.$$

We denote the set of all increasing dispersions of (Q) by the symbol (Q, Q). Non-trivial solutions of (Q) in (a, b) are called integrals; the set of all solutions of (Q) in (a, b) is also denoted by the symbol (Q).

Dispersions were introduced by O. BORŮVKA in his paper [1]. Here we confine ourselves, without the loss of generality, to increasing dispersions. The domain of a function $f(x)$ will be denoted by $\operatorname{Dom} f$.

In treating equation (Q) we do not suppose anything about the oscillation of integrals. Central dispersions and (increasing) dispersions are, in the general case, defined in a similar manner as in [1]; these definitions are described in [2]. Every dispersion $\zeta \in (Q, Q)$ is defined in a certain sub-interval of the interval (a, b). For every $y \in (Q)$ and every $\zeta \in (Q, Q)$ the function $y[\zeta(t)]/\sqrt{\zeta'(t)}$ is a solution of (Q) in the interval $\operatorname{Dom} \zeta$. Dispersions $\zeta \in (Q, Q)$ fulfilling in $\operatorname{Dom} \zeta$ for every $v \in (Q)$ the relation

$$(1) \qquad \frac{v[\zeta(t)]}{\sqrt{\zeta'(t)}} = \pm v(t)$$

are called central dispersions of (Q); $\varphi \in (Q, Q)$ is a central dispersion if and only if exist two linearly independent integrals $u, v \in (Q)$ satisfying (1); then the sign in both relations is necessarily the same.

2. An arbitrary solution $v \in (Q)$ is determined uniquely at a given $u \in (Q)$, by means of two values: of the Wronskian δ of the ordered pair u, v, and of the value $v(\tau)$ at an arbitrary $\tau \in (a, b)$ with $u(\tau) \neq 0$; the solution $v \in (Q)$ has the initial conditions

$$v(\tau), \quad v'(\tau) = \frac{1}{u(\tau)} [\delta + u'(\tau) v(\tau)] .$$

In this case we shall say that v belongs to the values $v(\tau)$, δ.

Lemma 1. *Let $u \in (Q)$ and let I be an sub-interval of the interval (a, b) such that $u(t) \neq 0$ for $t \in I$. If $\tau \in I$, $v(\tau)$, δ are arbitrary numbers, then the solution $v \in (Q)$ corresponding to the values $v(\tau)$, δ is given in I by the formula*

$$v(t) = \frac{v(\tau)}{u(\tau)} u(t) + \delta u(t) \int_\tau^t \frac{dr}{u^2(r)} . \tag{2}$$

Proof. In determining solutions v of (Q) in the form $v = uz$, we obtain for z the equation $uz'' + 2uz' = 0$. The substitution $w = z'$ yields $w'/w = -2(u'/u)$; thus for an arbitrary fixed $x \in I$ and for every $t \in I$ we obtain $w(t) = w(x) u^2(x)/u^2(t)$. Hence $z'(t) = z'(x) u^2(x)/u^2(t)$ at an arbitrary fixed $\tau \in I$,

$$z(t) - z(\tau) = z'(x) u^2(x) \int_\tau^t \frac{dr}{u^2(r)} ;$$

hence and from relations $z(\tau) = v(\tau)/u(\tau)$, $z'(x) = \delta/u^2(x)$ there follows the formula (2).

Central dispersions make possible an explicit construction of smooth prolongations of the part of a solution v from Lemma 1.

Here we call any part of the solution v in the interval the smooth prolongation of any other part of v in the interval.

Theorem 1.' Let $u \in (Q)$, and let φ be a central dispersion of (Q). Let I be an interval such that $I \subset \mathrm{Dom}\, \varphi$ and $u(t) \neq 0$ for $t \in I$. Set $x = \varphi(t)$, $\xi = \varphi(\tau)$ for $t, \tau \in I$; then the solution $v \in (Q)$ from Lemma 1 is given in the interval $\varphi(I)$ by the formula

$$v(x) = \frac{v(\xi)}{u(\xi)} u(x) + \delta u(x) \int_\xi^x \frac{ds}{u^2(s)} . \tag{3}$$

Proof. For all $t \in \mathrm{Dom}\, \varphi$, at $\zeta = \varphi$ there holds, for the solutions $u, v \in (Q)$, the formula (1) with the same sign. If we use in (2) the substitution $s = \varphi(r)$, $r \in I$, we obtain (3).

3. Let the dispersion $\zeta \in (Q, Q)$ and the integral $u \in (Q)$ fulfil (1); we shall obtain a necessary and sufficient condition that the relation hold for all $v \in (Q)$ in Dom ζ.

A partial solution of this problem is given in the

Lemma 2. *Let $\zeta \in (Q, Q)$ and let the integral $u \in (Q)$ fulfil* (1). *Let the interval $I \subset$ Dom ζ be such that $u(t) \neq 0$ for $t \in I$. If some $v \in (Q)$ fulfils* (1) *in at least one point $t = \tau \in I$ with the same sign as the integral u, then it satisfies this relation for all $t \in I$.*

Proof. The dispersion ζ transfers neighbouring roots of the solution u again into neighbouring roots of u. On the other hand, the dispersion ζ transforms two arbitrary solutions y, z of (Q) into solutions

$$Y(t) = \frac{y[\zeta(t)]}{\sqrt{\zeta'(t)}}, \quad Z(t) = \frac{z[\zeta(t)]}{\sqrt{\zeta'(t)}}$$

of the same equation, where the Wronskian remains the same, i.e. $\Delta = \delta$ for $\delta = yz' - y'z$ and $\Delta = YZ' - Y'Z$.

If δ is the Wronskian of a pair u, v then the Wronskian of the pair $U = u[\zeta]/\sqrt{\zeta'} = \pm u$, $V = v[\zeta]/\sqrt{\zeta'}$ is also δ, and therefore the Wronskian of the pair u, V is $\pm\delta$. According to Lemma 1 with $\tau \in I$, the integral $v \in (Q)$ in the interval I satisfies (2), and the integral V therein has the expression

$$V(t) = \frac{V(\tau)}{u(\tau)}\, u(t) \pm \delta\, u(t) \int_\tau^t \frac{\mathrm{d}r}{u^2(r)}\,. \tag{4}$$

If (1) holds for $v \in (Q)$ and $\tau \in I$, then from the comparison of (2) with (4) there follows $V(t) = \pm v(t)$ for all $t \in I$, which proves Lemma 2.

4. While formula (2) gives an explicit expression of the integral $v \in (Q)$ in an arbitrary interval I between two neighbouring roots of the integral u, formula (3) also gives explicitely the smooth prolongations of this part of the integral in intervals $\varphi(I)$, which are images of the interval I at central dispersions and between neighbouring roots of the integral u.

In a similar manner one may use, instead of the central dispersions, also the dispersions $\zeta \in (Q, Q)$ fulfilling (1) with the integral $u \in (Q)$.

Theorem 1″. *Let $\zeta \in (Q, Q)$ and let the integral $u \in (Q)$ fulfil* (1). *Let I be an interval with $I \subset$ Dom ζ and $u(t) \neq 0$ for $t \in I$. With the notation $x = \zeta(t)$, $\xi = \zeta(\tau)$ for t, $\tau \in I$, the solution $v \in (Q)$ from Lemma* 1 *then satisfies* (3) *in the interval $\zeta(I)$.*

Proof. As the first two suppositions of Theorem 1″ are the same as the first two suppositions of Lemma 2, for $V = v[\zeta]/\sqrt{\zeta'}$ there holds formula (4); applying a substitution $s = \zeta(r)$, $r \in I$ we obtain formula (3).

Theorem 2. *Let $\zeta \in (Q, Q)$ and let the integral $u \in (Q)$ fulfil* (1). *Let $I \subset \mathrm{Dom}\,\zeta$ be an interval such that $u(t) \neq 0$ for $t \in I$. For every integral $v \in (Q)$ and for all τ, $t \in I$ there is*

$$(5) \qquad u(\tau)\left[\pm \frac{v[\zeta(t)]}{\sqrt{\zeta'(t)}} - v(t)\right] = u(t)\left[\pm \frac{v[\zeta(\tau)]}{\sqrt{\zeta'(\tau)}} - v(\tau)\right],$$

where the sign is the same as in (1) *with the integral u.*

Proof. If we substract formula (2), from a ± 1-multiple of formula (4), we obtain formula (5) after a substitution $V = v[\zeta]/\sqrt{\zeta'}$.

Then Lemma 2 is a consequence of Theorem 2. The problem mentioned in section 3 is solved by the

Theorem 3'. *Let $\zeta \in (Q, Q)$ and let the integral $u \in (Q)$ fulfil* (1). *Then* (1) *holds for all integrals $v \in (Q)$ and all $t \in \mathrm{Dom}\,\zeta$ if and only if they holds with the same sign as for the integral u, for at least one integral $v \in (Q)$, linеary independent of u, at least in one point $\tau \in (a, b)$ such that $u(\tau) \neq 0$.*

Proof. Let there exist an integral $v \in (Q)$ with the properties mentioned above; then $\tau \in \mathrm{Dom}\,\zeta$. Let I be an interval such that $\tau \in I$, $I \subset \mathrm{Dom}\,\zeta$ and $u(t) \neq 0$ for $t \in I$.

According to Lemma 2, (1) holds for this integral v and for all $t \in I$. Hence it follows that for all integrals $v \in (Q)$, (1) holds for all $t \in I$, and consequently for any amplitude ϱ of (Q) in I,

$$(6) \qquad \zeta' = \frac{\varrho^2(\zeta)}{\varrho^2(t)}.$$

Let t_0 be an arbitrary point inside the interval I. There exists an integral $y \in (Q)$ such that $y(t_0) = 0$. According to (1), $\zeta(t_0)$ is also a root of y; thus there exists a central dispersion $\varphi \in (Q, Q)$ such that $\varphi(t_0) = \zeta(t_0)$. Both the dispersions φ and ζ satisfy (6) and have the same initial condition; thus $\zeta = \varphi$. Hence it follows that (1) holds for every $v \in (Q)$ and all $t \in \mathrm{Dom}\,\zeta$.

The main result, characterization of central dispersions, is a consequence of Theorem 3'.

Theorem 3. *Let $\zeta \in (Q, Q)$ and let the integral $u \in (Q)$ fulfil* (1). *The dispersion ζ is a central dispersion if and only if there exists a number $\tau \in (a, b)$ such that $u(\tau) \neq 0$ and there exists an integral $v \in (Q)$ lineary independent on u such that v at the point τ fulfils* (1) *with the same sign as u.*

Under the suppositions of Theorem 2, if the closure of I contains a root t_0 of u, then from (5), for $t \to t_0$, $t \in I$,

$$(7) \qquad \zeta'(t_0) = \frac{v^2[\zeta(t_0)]}{v^2(t_0)}$$

follows for every integral $v \in (Q)$ linearly independent on u. The derivative $\zeta'(t_0)$ at a root $t_0 \in \mathrm{Dom}\,\zeta$ of u can also be obtained from the formula

$$\zeta'(t_0) = \frac{u'^2(t_0)}{u'^2[\zeta(t_0)]}, \tag{8}$$

which is a consequence of (1).

References

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[2] *E. Barvínek:* Dispersiones de la ecuacion diferencial $y'' = Q(t)\,y$ en el caso general (to appear in Memorias de la Facultad de Ciencias de la Universidad de la Habana).

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Výtah

CHARAKTERISACE CENTRÁLNÍCH DISPERSÍ 1. DRUHU DIFERENCIÁLNÍ ROVNICE $y'' = Q(t)y$

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Předpokládá se, že $Q(t)$ je spojitá funkce v intervalu (a, b) libovolného typu a že disperse $\zeta \in (Q, Q)$ a integrál $u \in (Q)$ splňují relaci (1); jde o nalezení nutné a postačující podmínky, aby ζ byla centrální disperse.

Lema 1. *Nechť $u \in (Q)$. Nechť I je podinterval intervalu (a, b) takový, že $u(t) \neq 0$ pro $t \in I$. Jestliže $\tau \in I$, $v(\tau)$, δ jsou libovolná čísla, pak řešení $v \in (Q)$ s počátečními podmínkami $v(\tau)$, $v'(\tau) = [\delta + u'(\tau)\,v(\tau)]/u(\tau)$ splňuje v I relaci* (2).

Lokální hladká prodloužení části řešení $v \in (Q)$ z lematu 1 se dostanou pomocí centrálních dispersí φ nebo pomocí dispersí ζ splňujících zmíněný předpoklad.

Věta 1. *Jestliže interval I z lematu* 1 *splňuje navíc inklusi $I \subset \mathrm{Dom}\,\varphi$, resp. $I \subset \mathrm{Dom}\,\zeta$, potom řešení $v \in (Q)$ z lematu* 1 *má v intervalu $\varphi(I)$, resp. $\zeta(I)$ vyjádření* (3), *kde $x = \varphi(t)$, $\xi = \varphi(\tau)$, resp. $x = \zeta(t)$, $\xi = \zeta(\tau)$.*

Věta 2. *Nechť $\zeta \in (Q, Q)$ a integrál $u \in (Q)$ splňují relaci* (1). *Nechť $I \subset \mathrm{Dom}\,\zeta$ je interval takový, že $u(t) \neq 0$ pro $t \in I$. Pro každý integrál $v \in (Q)$ a pro všechna τ, $t \in I$ platí* (5) *se stejným znaménkem jako v* (1) *při integrálu u.*

Hlavním výsledkem je

Věta 3. *Nechť $\zeta \in (Q, Q)$ a integrál $u \in (Q)$ splňují relaci* (1). *Disperse ζ je centrální dispersí tehdy a jen tehdy, jestliže existuje číslo $\tau \in (a, b)$ takové, že $u(\tau) \neq 0$ a existuje integrál $v \in (Q)$ lineárně nezávislý na u tak, že v bodě τ splňuje* (1) *a to se stejným znaménkem jako u.*

Резюме

ХАРАКТЕРИСТИКА ЦЕНТРАЛЬНЫХ ДИСПЕРСИЙ 1-ОГО РОДА ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ $y'' = Q(t)\,y$

ЭРИХ БАРВИНЕК (Erich Barvínek), Брно

Предполагается, что $Q(t)$ непрерывная функция в интервале (a, b) произвольного типа и что дисперсия $\zeta \in (Q, Q)$ и интеграл $u \in (Q)$ выполняют соотношение (1); требуется найти необходимое и достаточное условие для того, чтобы ζ была центральной дисперсией.

Лемма 1. *Пусть $u \in (Q)$. Пусть I подинтервал интервала (a, b) такой, что $u(t) \neq 0$ для $t \in I$. Если $\tau \in I$, $v(\tau)$, δ произвольные числа, то решение $v \in (Q)$ с начальными условиями $v(\tau)$, $v'(\tau) = [\delta + u'(\tau)\,v(\tau)]/u(\tau)$ выполняет в I соотношение* (2).

Локальные гладкие продолжения части решения $v \in (Q)$ из леммы 1 получим при помощи центральных дисперсий φ или при помощи дисперсий ζ удовлетворяющих выксазанному предположению.

Теорема 1. *Если интервал из леммы* 1 *выполняет еще включение $I \subset \operatorname{Dom} \varphi$ или же $I \subset \operatorname{Dom} \zeta$, то решение $v \in (Q)$ из леммы* 1 *имеет в интервале $\varphi(I)$ или же $\zeta(I)$ вид* (3), *где $x = \varphi(t)$, $\xi = \varphi(\tau)$, или же $x = \zeta(t)$, $\xi = \zeta(\tau)$.*

Теорема 2. *Пусть $\zeta \in (Q, Q)$ и интеграл $u \subset (Q)$ выполняют соотношение* (1). *Пусть $I \subset \operatorname{Dom} \zeta$ интервал такой, что $u(t) \neq 0$ для $t \in I$. Для каждого интеграла $v \in (Q)$ и для всех τ, $t \in I$ справедливо* (5) *с тем же знаком, как в* (1) *при интеграле u.*

Главным результатом является

Теорема 3. *Пусть $\zeta \in (Q, Q)$ и интеграл $u \in (Q)$ выполняют соотношение* (1). *Дисперсия ζ является центральной дисперсией тогда и только тогда, если существует число $\tau \in (a, b)$ такое, что $u(\tau) \neq 0$ и существует интеграл $v \in (Q)$ независимый линейно от u так, что в точке τ выполнено* (1), *а именно стем же знаком, как при u.*