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Časopis pro pěstování matematiky, Vol. 91 (1966), No. 2, 121--124

Persistent URL: `http://dml.cz/dmlcz/108106`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 91 * PRAHA 10. 5. 1966 * ČÍSLO 2

A NOTE TO THE APPROXIMATION OF ONE MATRIX BY A MATRIX OF ANOTHER RANK

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(Received March 14, 1964)

Let $\mathbf{A}$ be a matrix of order (m, n). Let us define the norm of the matrix $\mathbf{A}$ by the relation

$$\|\mathbf{A}\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| ,$$

where $\mathbf{x}$ is a column vector of order n with elements $x_1, \ldots, x_n$ and where $\|\mathbf{x}\|$ is the quadratic norm of the vector $\mathbf{x}$ $[\|\mathbf{x}\| = \sqrt{(\sum_{i=1}^{n} |x_i|^2)}]$. It is known that

$$\|\mathbf{A}\| = \|\mathbf{A}^*\| = \|\mathbf{U}\mathbf{A}\mathbf{V}\| ,$$

where $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$ and where $\mathbf{U}$ and $\mathbf{V}$ are arbitrary unitary matrices of order (m, m) and (n, n).

On using these notations we shall prove following three assertions:

Lemma 1. *Let $\mathbf{C}$ be a matrix of order (m, n) and of rank r; let $r < n$. Then there exists a vector $\mathbf{x} = (x_1, \ldots, x_n)$ of order n such that*

1. $\|\mathbf{x}\| = 1$;
2. $\mathbf{C}\mathbf{x} = \mathbf{o}$, *where $\mathbf{o}$ is a zero vector of order m*;
3. $x_i = 0$ *for $r + 2 \leqq i \leqq n$.*

Proof. Let $\tilde{\mathbf{C}}$ be a matrix of order $(m, r + 1)$, whose all columns are by turns equal to first $r + 1$ columns of the matrix $\mathbf{C}$. As it is known by the theory of linear homogeneous equations, there exists (in regard to the fact, that the rank of matrix $\tilde{\mathbf{C}}$ is at most equal to r) a non zero vector $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{r+1})$ such that $\tilde{\mathbf{C}}\tilde{\mathbf{x}} = \tilde{\mathbf{o}}$, where $\tilde{\mathbf{o}}$ is a zero vector of order m. Then the vector $\mathbf{x} = (x_1, \ldots, x_n)$ defined by relations

$x_i = 0$ for $r + 2 \leqq i \leqq n$ and

$$x_i = \frac{\tilde{x}_i}{\sqrt{\left(\sum_{j=1}^{r} |\tilde{x}_j|^2\right)}} \quad \text{for} \quad 1 \leqq i \leqq r + 1$$

fulfils obviously properties 1, 2 and 3 of the lemma.

Theorem 1. *Let* $\mathbf{A}$ *be a matrix of order* (m, n) *and of rank* h; *let* $\mathbf{B}$ *be a matrix of order* (m, n) *and of rank* r. *Further let* $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n$ *be square roots (taken with positive signs) of eigenvalues of the Hermitian matrix* $\mathbf{A}^*\mathbf{A}$. *Then* $\|\mathbf{A} - \mathbf{B}\| \geqq \geqq \alpha_{r+1}$, *where we put* $\alpha_{r+1} = 0$ *in case* $r = n$.

Proof. The assertion of the theorem is obvious if $r = n$. For that reason let us assume that $r < n$. As stated in [1], there exist two unitary matrices $\mathbf{U}$ and $\mathbf{V}$ of order (m, m) and (n, n) such that $\mathbf{UAV} = \mathbf{D}$, where $\mathbf{D}$ is the diagonal matrix of order (m, n) whose elements d_{ik} are defined by relations $d_{ii} = \alpha_i$ for $i = 1, \ldots, h$ and $d_{ik} = 0$ otherwise.

Since the matrix $\mathbf{UBV}$ is of rank r there exists according to the lemma 1 a column vector $\mathbf{x}$ of order n such that $\|\mathbf{x}\| = 1$, $\mathbf{UBVx} = \mathbf{o}$ (where $\mathbf{o}$ is the zero vector of order m) and whose elements $x_1, \ldots, x_n$ satisfy relations $x_i = 0$ for $r + 2 \leqq i \leqq n$. Then we can write

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{U}(\mathbf{A} - \mathbf{B})\,\mathbf{V}\| \geqq \|\mathbf{Dx}\| = \left(\sum_{i=1}^{r+1} \alpha_i^2 |x_i|^2\right)^{\frac{1}{2}} \geqq \alpha_{r+1}\,.$$

Theorem 2. *Let* $\mathbf{A}$ *be a matrix of order* (m, n) *and of rank* h. *Let* $\alpha_1 \geqq \ldots \geqq \alpha_n$ *be square roots (taken with positive signs) of eigenvalues of the Hermitian matrix* $\mathbf{A}^*\mathbf{A}$. *Let* r *be a non-negative integer such that* $r \leqq h$. *Then there exists a matrix* $\mathbf{B}$ *of order* (m, n) *and of rank* r *such that* $\|\mathbf{A} - \mathbf{B}\| = \alpha_{r+1}$.

Proof. In the case of $r = h$ the theorem is obvious. Let us therefore assume that $r < h$. Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{D}$ be matrices of the same meaning with respect to the matrix $\mathbf{A}$ as in the proof of the theorem 1.

Further let us define a matrix $\mathbf{B}$ by relation $\mathbf{B} = \mathbf{U}^*\mathbf{D}_r\mathbf{V}^*$, where $\mathbf{D}_r$ is the diagonal matrix of order (m, n) with elements $d_{ik}^{(r)}$ satisfying relations $d_{ii}^{(r)} = \alpha_i$ for $1 \leqq i \leqq r$ and $d_{ik}^{(r)} = 0$ otherwise. Then the rank of the matrix $\mathbf{B}$ is obviously r and further it follows that

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{UAV} - \mathbf{UBV}\| = \|\mathbf{D} - \mathbf{D}_r\| = \alpha_{r+1}\,.$$

A note to the theorem 2. In terms of theorem 2 there can exist in general infinitely many matrices $\mathbf{B}$ of rank r satisfying the relation $\|\mathbf{A} - \mathbf{B}\| = \alpha_{r+1}$ even in the case that all square roots of the eigenvalues of the matrix $\mathbf{A}^*\mathbf{A}$ are different from each other. For instance if $A = \begin{pmatrix} \alpha_1, & 0 \\ 0, & \alpha_2 \end{pmatrix}$ where $\alpha_1 > \alpha_2 \geqq 0$, then α_1 and α_2 are two different square roots of the eigenvalues of the matrix A^*A; but every matrix $B = \begin{pmatrix} \beta, & 0 \\ 0, & 0 \end{pmatrix}$, where $|\beta - \alpha_1| < \alpha_2$, is the matrix of rank 1 and such that $\|A - B\| = \alpha_2$.

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Výtah

POZNÁMKA K APROXIMACI MATICE MATICÍ S JINOU HODNOSTÍ

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Nechť A jest obdélníková matice typu (m, n) s hodností h. Normu matice A definujme vztahem $\|A\| = \sup_{\|x\|=1} \|Ax\|$, kde x jest n-členný vektor a $\|x\|$ jest norma vektoru x daná vztahem $\|x\| = \sqrt{\left(\sum_{i=1}^{n} |x_i|^2\right)}$. Nechť dále $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n$ jsou druhé odmocniny charakteristických čísel Hermiteovsky symetrické matice A^*A. (Znakem A^* značíme matici Hermiteovsky sdruženou s maticí A.) Potom platí následující dvě věty:

Věta 1. *Nechť B jest matice typu (m, n) s hodností r. Pak jest $\|A - B\| \geqq \alpha_{r+1}$, kde klademe $\alpha_{r+1} = 0$ v případě, že $r = n$.*

Věta 2. *Nechť r jest nezáporné celé číslo, $r \leqq h$. Potom existuje matice B typu (m, n) s hodností r taková, že $\|A - B\| = \alpha_{r+1}$.*

V poznámce ke větě 2 je ukázáno, že může existovati nekonečně mnoho matic s hodností r, které splňují vztah $\|A - B\| = \alpha_{r+1}$ i když všechna charakteristická čísla matice $\mathbf{A}^*\mathbf{A}$ jsou vesměs různá. Kupř. je-li $\mathbf{A} = \begin{pmatrix} \alpha_1, & 0 \\ 0, & \alpha_2 \end{pmatrix}$, kde $\alpha_1 > \alpha_2 \geqq 0$, pak α_1 a α_2 jsou různé druhé odmocniny charakteristických čísel matice $\mathbf{A}^*\mathbf{A}$; při tom však každá matice $\mathbf{B} = \begin{pmatrix} \beta, & 0 \\ 0, & 0 \end{pmatrix}$, kde $|\beta - \alpha_1| < \alpha_2$, je matice s hodností 1 a zároveň jest $\|\mathbf{A} - \mathbf{B}\| = \alpha_2$.

Резюме

ЗАМЕТКА К АППРОКСИМАЦИИ МАТРИЦЫ МАТРИЦЕЙ ДРУГОГО РАНГА

ЙИРЖИ СЕЙЦ (Jiří Seitz), Прага

Пусть $\mathbf{A}$ – прямоугольная матрица ранга h с размерами $m \times n$, норма которой определеяется по равенству $\|\mathbf{A}\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$, где $\mathbf{x}$ – n-мерный вектор и $\|\mathbf{x}\|$ – норма $\sqrt{\left(\sum_{i=1}^{n} |x_i|^2\right)}$ вектора $\mathbf{x}$. Пусть, далее, $\alpha_1, \geqq \ldots \geqq \alpha_n$ – квадратные корни (с положительным знаком) собственных значений эрмитовей матрицы $\mathbf{A}^*\mathbf{A}$ (здесь $\mathbf{A}^*$ – эрмитова сопряженная матрица по отношению к матрице $\mathbf{A}$).

Тогда справедливы следующие две теоремы:

Теорема 1. *Пусть* $\mathbf{B}$*-матрица ранга* r *с размерами* $m \times n$. *Тогда* $\|\mathbf{A} - \mathbf{B}\| \geqq \alpha_{r+1}$, *причем* $\alpha_{r+1} = 0$ *в случае* $r = n$.

Теорема 2. *Пусть* r *– неотрицательное целое число,* $r \leqq h$. *Тогда существует матрица* $\mathbf{B}$ *ранга* r *с размерами* $m \times n$, *для которой* $\|\mathbf{A} - \mathbf{B}\| = \alpha_{r+1}$.

В заметке к теореме 2 доказывается возможность существования бесконечного числа матриц $\mathbf{B}$ ранга r, удовлетворяющих соотношению $\|\mathbf{A} - \mathbf{B}\| = \alpha_{r+1}$, даже если все собственные значения матрицы $\mathbf{A}^*\mathbf{A}$ различны. Например, если $\mathbf{A} = \begin{pmatrix} \alpha_1, & 0 \\ 0, & \alpha_2 \end{pmatrix}$, где $\alpha_1 > \alpha_2 \geqq 0$, то α_1 и α_2 являются различными квадратными корнями собственных значений матрицы $\mathbf{A}^*\mathbf{A}$; тогда свякая матрица $\mathbf{B} = \begin{pmatrix} \beta, & 0 \\ 0, & 0 \end{pmatrix}$, где $|\beta - \alpha_1| < \alpha_2$, является матрицей ранга 1, и одновременно $\|\mathbf{A} - \mathbf{B}\| = \alpha_2$.