

Abhijit Banerjee; Sonali Mukherjee

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NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

ABHIJIT BANERJEE AND SONALI MUKHERJEE

ABSTRACT. Dealing with a question of Lahiri [6] we study the uniqueness problem of meromorphic functions concerning two nonlinear differential polynomials sharing a small function. Our results will not only improve and supplement the results of Lin-Yi [16], Lahiri Sarkar [12] but also improve and supplement a very recent result of the first author [1].

1. INTRODUCTION DEFINITIONS AND RESULTS

Let f and g be two nonconstant meromorphic functions defined in the open complex plane \mathbb{C} . A meromorphic function α is said to be a small function of f provided that $T(r, \alpha) = S(r, f)$, that is $T(r, \alpha) = o(T(r, f))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure. Clearly if f is rational then α is a constant and if f is transcendental then α is a nonconstant meromorphic function. We denote by $S(f)$ the set of all small functions of f .

If for some $\alpha \in S(f) \cap S(g)$, $f - \alpha$ and $g - \alpha$ have the same set of zeros with the same multiplicities, we say that f and g share α CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share α IM (ignoring multiplicities).

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.

Let $N_E(r, \alpha; f, g)$ ($\overline{N}_E(r, \alpha; f, g)$) be the counting function (reduced counting function) of all common zeros of $f - \alpha$ and $g - \alpha$ with the same multiplicities and $N_0(r, \alpha; f, g)$ ($\overline{N}_0(r, \alpha; f, g)$) be the counting function (reduced counting function) of all common zeros of $f - \alpha$ and $g - \alpha$ ignoring multiplicities.

If

$$\overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) - 2\overline{N}_E(r, \alpha; f, g) = S(r, f) + S(r, g)$$

then we say that f and g share α “CM”.

On the other hand if

$$\overline{N}(r, \alpha; f) + \overline{N}(r, \alpha; g) - 2\overline{N}_0(r, \alpha; f, g) = S(r, f) + S(r, g)$$

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then we say that f and g share α “IM”.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

In [6] Lahiri studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper [6] regarding the nonlinear differential polynomials Lahiri asked the following question. *What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?*

Naturally several authors investigate the possible answer to the above question and continuous efforts are being carried out to relax the hypothesis of the results. (cf. [1], [2], [3], [11], [12], [14], [15], [16]).

In 2002 Fang and Fang [2] and in 2004 Lin-Yi [15] independently proved the following result.

Theorem A. *Let f and g be two nonconstant meromorphic functions and $n (\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share 1 CM, then $f \equiv g$.*

In 2004 Lin-Yi [16] improved Theorem A by generalizing it in view of fixed point. Lin-Yi [16] proved the following result.

Theorem B. *Let f and g be two transcendental meromorphic functions and $n (\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share z CM, then $f \equiv g$.*

In the same paper Lin-Yi [16] mentioned that in Theorem B z can be replaced by $\alpha(z)$.

In 2001 an idea of gradation of sharing of values was introduced in ([8], [9]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 ([8, 9]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

The definition implies that if f, g share a value a with weight k then z_0 is an a -point of f with multiplicity $m (\leq k)$ if and only if it is an a -point of g with multiplicity $m (\leq k)$ and z_0 is an a -point of f with multiplicity $m (> k)$ if and only if it is an a -point of g with multiplicity $n (> k)$, where m is not necessarily equal to n .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) , then f, g share (a, p) for any integer $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of value recently the first author [1] improved Theorem A as follows.

Theorem C ([1]). *Let f and g be two nonconstant meromorphic functions and $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$, is an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $(1, 2)$ then $f \equiv g$.*

In the mean time Lahiri and Sarkar [12] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from that of previously mentioned and proved the following.

Theorem D ([12]). *Let f and g be two nonconstant meromorphic functions such that $f^n(f^2 - 1)f'$ and $g^n(g^2 - 1)g'$ share $(1, 2)$, where $n (\geq 13)$ is an integer then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

From the above discussion it will be a natural query to investigate the uniqueness of meromorphic functions when two non linear differential polynomials of more general form namely $f^n(af^2 + bf + c)f'$ and $g^n(ag^2 + bg + c)g'$ where $a \neq 0$ and $|b| + |c| \neq 0$ share a small function.

In this paper we will study the above problem with the notion of weakly weighted sharing which has recently been introduced by Lin and Lin [13] generalizing the idea of weighted sharing of values. We are now giving the definition.

Definition 1.2 ([13]). Let f, g share α “IM” for $\alpha \in S(f) \cap S(g)$ and k is a positive integer or ∞ .

- (i) $\overline{N}^E(r, \alpha; f, g | \leq k)$ denotes the reduced counting function of those α -points of f whose multiplicities are equal to the corresponding α -points of g , both of their multiplicities are not greater than k .
- (ii) $\overline{N}^0(r, \alpha; f, g | > k)$ denotes the reduced counting function of those α -points of f which are α -points of g , both of their multiplicities are not less than k .

Definition 1.3 ([13]). For $\alpha \in S(f) \cap S(g)$, if k is a positive integer or ∞ and

$$\begin{aligned} \overline{N}(r, \alpha; f | \leq k) - \overline{N}^E(r, \alpha; f, g | \leq k) &= S(r, f), \\ \overline{N}(r, \alpha; g | \leq k) - \overline{N}^E(r, \alpha; f, g | \leq k) &= S(r, g), \\ \overline{N}(r, \alpha; f | \geq k + 1) - \overline{N}^0(r, \alpha; f, g | \geq k + 1) &= S(r, f), \\ \overline{N}(r, \alpha; g | \geq k + 1) - \overline{N}^0(r, \alpha; f, g | \geq k + 1) &= S(r, g) \end{aligned}$$

or if $k = 0$ and

$$\begin{aligned} \overline{N}(r, \alpha; f) - \overline{N}_0(r, \alpha; f, g) &= S(r, f), \\ \overline{N}(r, \alpha; g) - \overline{N}_0(r, \alpha; f, g) &= S(r, g), \end{aligned}$$

then we say f, g weakly share α with weight k . Here we write f, g share “ (α, k) ” to mean that f, g weakly share α with weight k .

Obviously if f, g share “ (α, k) ”, then f, g share “ (α, p) ” for any integer $p, 0 \leq p < k$. Also we note that f, g share α “IM” or “CM” if and only if f, g share “ $(\alpha, 0)$ ” or “ (α, ∞) ” respectively.

We now state the following theorem which is the main result of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions such that $f^n(af^2 + bf + c)f'$ and $g^n(ag^2 + bg + c)g'$ where $a \neq 0$ and $|b| + |c| \neq 0$ share “ $(\alpha, 2)$ ”. Then the following holds.*

- (i) If $b \neq 0$, $c = 0$ and $n > \max [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}, \frac{4}{\Theta(\infty; f) + \Theta(\infty; g)} - 2]$, be an integer, where $\Theta(\infty; f) + \Theta(\infty; g) > 0$, then $f \equiv g$.
- (ii) If $b \neq 0$, $c \neq 0$, $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$, the roots of the equation $az^2 + bz + c = 0$ are distinct and one of f and g is non entire meromorphic function having only multiple poles, then $f \equiv g$.
- (iii) If $b \neq 0$, $c \neq 0$, $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$ and the roots of the equation $az^2 + bz + c = 0$ coincides, then $f \equiv g$.
- (iv) $b = 0$, $c \neq 0$, $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$, then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility $f \equiv -g$ does not arise.

From Theorem 1.1 we can immediately deduce the following corollaries.

Corollary 1.1. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$, and $n (\geq 13)$ be an integer. If $f^n(af^2 + bf)f'$ and $g^n(ag^2 + bg)g'$ share “ $(\alpha, 2)$ ” then $f \equiv g$.*

Corollary 1.2. *Let f and g be two transcendental meromorphic functions and one of f and g is non entire meromorphic function having only multiple poles, such that $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$ be an integer. If $af^n(f - \beta_1)(f - \beta_2)f'$ and $ag^n(g - \beta_1)(g - \beta_2)g'$ share “ $(\alpha, 2)$ ”, where β_1 and β_2 are the distinct roots of the equation $az^2 + bz + c = 0$ with $|\beta_1| \neq |\beta_2|$, then $f \equiv g$.*

Corollary 1.3. *Let f and g be two transcendental meromorphic functions such that $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$ be an integer. If $af^n(f + k)^2f'$ and $ag^n(g + k)^2g'$ share “ $(\alpha, 2)$ ” where k is a nonzero constant then $f \equiv g$.*

Corollary 1.4. *Let f and g be two transcendental meromorphic functions such that $n > [12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}]$ be an integer. If $f^n(af^2 + c)f'$ and $g^n(ag^2 + c)g'$ share “ $(\alpha, 2)$ ” then $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility $f \equiv -g$ does not arise.*

Though we use the standard notations and definitions of the value distribution theory available in [5], we explain some definitions and notations which are used in the paper.

Definition 1.4 ([7]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f | = 1)$ the counting function of simple a points of f . For a positive integer m we denote by $N(r, a; f | \leq m)$ ($N(r, a; f | \geq m)$) the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\bar{N}(r, a; f | \leq m)$ ($\bar{N}(r, a; f | \geq m)$) are defined similarly, where in counting the a -points of f we ignore the multiplicities.

Also $N(r, a; f | < m)$, $N(r, a; f | > m)$, $\bar{N}(r, a; f | < m)$ and $\bar{N}(r, a; f | > m)$ are defined analogously.

Definition 1.5 ([9], cf.[20]). We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2)$.

Definition 1.6 ([9]). Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p , a 1-point of g with multiplicity q . We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g for which $p > q$, each point in this counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$.

Definition 1.7 ([10]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | g = b)$ the counting function of those a -points of f , counted according to multiplicity, which are b -points of g .

Definition 1.8 ([10]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | g \neq b)$ the counting function of those a -points of f , counted according to multiplicity, which are not the b -points of g .

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel. Let f, g, F_1, G_1 be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1} \right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1} \right)$$

and

$$H = \left(\frac{F_1''}{F_1'} - \frac{2F_1'}{F_1-1} \right) - \left(\frac{G_1''}{G_1'} - \frac{2G_1'}{G_1-1} \right).$$

Lemma 2.1. *If f, g share “(1, 1)” and $h \neq 0$. Then*

$$N(r, 1; f | \leq 1) \leq N(r, 0; h) + S(r, f) \leq N(r, \infty; h) + S(r, f) + S(r, g).$$

Proof. Since f, g share “(1, 1)” it follows that if z_0 be a common simple 1-point of f and g , then in some neighborhoods of z_0 we have $h = (z - z_0)\phi(z)$, where $\phi(z)$ is analytic at z_0 . Hence by the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned} N(r, 1; f | \leq 1) &= N^E(r, 1; f, g | \leq 1) + S(r, f) \\ &\leq N(r, 0; h) + S(r, f) \leq N(r, \infty; h) + S(r, f) + S(r, g) \quad \square \end{aligned}$$

Lemma 2.2. *If f, g share “(1, 1)” and $h \neq 0$. Then*

$$\begin{aligned} N(r, \infty; h) &\leq \overline{N}(r, 0; f | \geq 2) + \overline{N}(r, 0; g | \geq 2) \\ &\quad + \overline{N}(r, \infty; f | \geq 2) + \overline{N}(r, \infty; g | \geq 2) \\ &\quad + \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g') + S(r), \end{aligned}$$

where $\overline{N}_0(r, 0; f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r, 0; g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g , (ii) multiple poles of f and g , (iii) the common zeros of $f - 1$ and $g - 1$ whose multiplicities are different, (iii) those 1-points of f (g) which are not the 1-points of g (f), (iv) zeros of f' which are not the zeros of $f(f - 1)$, (v) zeros of g' which are not zeros of $g(g - 1)$. Since all the poles of h are simple the lemma follows from above. This proves the lemma. \square

Lemma 2.3. *If for a positive integer k , $N_k(r, 0; f' \mid f \neq 0)$ denotes the counting function of those zeros of f' which are not the zeros of f , where a zero of f' with multiplicity m is counted m times if $m \leq k$ and k times if $m > k$ then*

$$N_k(r, 0; f' \mid f \neq 0) \leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) - \sum_{p=k+1}^{\infty} \bar{N}\left(r, 0; \frac{f'}{f} \mid \geq p\right) + S(r, f).$$

Proof. By the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned} N(r, 0; f' \mid f \neq 0) &= N\left(r, 0; \frac{f'}{f}\right) \leq N\left(r, \infty; \frac{f'}{f}\right) + S(r, f) \\ &= \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + S(r, f). \end{aligned}$$

Now

$$\begin{aligned} N_k\left(r, 0; \frac{f'}{f}\right) + \sum_{p=k+1}^{\infty} \bar{N}\left(r, 0; \frac{f'}{f} \mid \geq p\right) &= N\left(r, 0; f' \mid f \neq 0\right) \\ &\leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + S(r, f). \end{aligned}$$

The lemma follows from above as $N_k(r, 0; \frac{f'}{f}) = N_k(r, 0; f' \mid f \neq 0)$. \square

Lemma 2.4. *Let f, g share “(1, 2)” and $h \neq 0$. Then*

$$\begin{aligned} T(r, f) &\leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) \\ &\quad - \sum_{p=3}^{\infty} \bar{N}\left(r, 0; \frac{g'}{g} \mid \geq p\right) + S(r, f) + S(r, g). \end{aligned}$$

Proof. Since f and g share “(1, 2)” it follows that f and g share “(1, 1)”. Also we note that $\bar{N}_L(r, 1; f) + \bar{N}_L(r, 1; g) \leq \bar{N}(r, 1; g \mid \geq 3)$. So by the second fundamental theorem Lemmas 2.1, 2.2 and 2.3 we get

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + \bar{N}(r, 1; f) - N_0(r, 0; f') + S(r, f) \\ &\leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + N(r, 1; f \mid \leq 1) + \bar{N}(r, 1; f \mid \geq 2) - N_0(r, 0; f') \\ &\leq N_2(r, 0; f) + N_2(r, \infty; f) + \bar{N}(r, 0; g \mid \geq 2) + \bar{N}(r, \infty; g \mid \geq 2) \\ &\quad + \bar{N}(r, 1; g \mid \geq 2) + \bar{N}(r, 1; g \mid \geq 3) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} &\leq N_2(r, 0; f) + N_2(r, \infty; f) + \overline{N}(r, 0; g \geq 2) + \overline{N}(r, \infty; g \geq 2) \\ &\quad + N_2(r, 0; g' \mid g \neq 0) + S(r, f) + S(r, g) \\ &\leq N_2(r, 0; f) + N_2(r, \infty; f) + N_2(r, 0; g) + N_2(r, \infty; g) \\ &\quad - \sum_{p=3}^{\infty} \overline{N}\left(r, 0; \frac{g'}{g} \geq p\right) + S(r, f) + S(r, g). \end{aligned} \quad \square$$

Lemma 2.5 ([17]). *Let f be a nonconstant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^n a_k f^k}{\sum_{j=0}^m b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

Lemma 2.6. *Let $F_1 = \frac{f^n(af^2+bf+c)f'}{f^n}$ and $G_1 = \frac{g^n(ag^2+bg+c)g'}{g^n}$, where $a \neq 0$ and $|b| + |c| \neq 0$. Then $S(r, F_1) = S(r, f)$ and $S(r, G_1) = S(r, g)$.*

Proof. Using Lemma 2.5 we see that

$$T(r, F_1) \leq (n + 2)T(r, f) + T(r, f') + S(r, f) = (n + 4) T(r, f) + S(r, f)$$

and

$$(n + 2)T(r, f) = T(r, f^n(af^2 + bf + c)) + 0(1) \leq T(r, F_1) + T(r, f') + S(r, f),$$

that is,

$$T(r, F_1) \geq n T(r, f) + S(r, f).$$

Hence $S(r, F_1) = S(r, f)$. In the same way we can prove $S(r, G_1) = S(r, g)$. □

Lemma 2.7 ([21]). *If $h \equiv 0$ and*

$$\limsup_{r \rightarrow \infty} \frac{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; g)}{T(r)} < 1, \quad r \in I$$

then $f \equiv g$ or $f \cdot g \equiv 1$.

Lemma 2.8. *Let f, g be two nonconstant meromorphic functions. Then*

$$f^n(af^2 + bf + c)f'g^n(ag^2 + bg + c)g' \not\equiv \alpha^2,$$

where $a \neq 0$ and $|b| + |c| \neq 0$ and $n (> 7)$ is an integer.

Proof. If possible, let

$$(2.1) \quad f^n(af^2 + bf + c)f'g^n(ag^2 + bg + c)g' \equiv \alpha^2.$$

We consider the following cases.

Case 1. The roots of the equation $az^2 + bz + c = 0$ are distinct and suppose they are β_1 and β_2 .

Subcase 1.1. One of β_1 and β_2 say $\beta_2 = 0$. Then (2.1) reduces to

$$\alpha^2 f^{n+1}(f - \beta_1)f'g^{n+1}(g - \beta_1)g'' \equiv \alpha^2.$$

Let z_0 be a zero of f with multiplicity $p (\geq 1)$ which is not a zero or pole of α . Clearly z_0 is a pole of g with multiplicity $q (\geq 1)$ such that

$$(2.2) \quad (n+1)p + p - 1 = (n+2)q + q + 1,$$

i.e.

$$q = (n+2)(p-q) - 2 \geq n.$$

Again from (2.2) we get

$$(n+2)p = (n+3)q + 2 = (n+2)q + q + 2 \geq (n+1)(n+2), \quad \text{i.e., } p \geq n+1.$$

Noting that α is a small function we obtain

$$N(r, 0; f) \geq (n+1)\overline{N}(r, 0; f) + S(r, f).$$

Next suppose z_1 be a zero of $f - \beta_1$ with multiplicity $p (\geq 1)$ which is not a zero or pole of α . Then z_1 be a pole of g with multiplicity $q (\geq 1)$ such that

$$2p - 1 = (n+1)q + 2q + 1 \quad \text{i.e., } p \geq \frac{n+5}{2}.$$

Let $\overline{N}_\otimes(r, 0; f')$ ($\overline{N}_\otimes(r, 0; g')$) denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f - \beta_1)$ ($g(g - \beta_1)$). Since a pole of f is either a zero of $g(g - \beta_1)$ or a zero of g' or a zero or pole of α we note that

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, \beta_1; g) + \overline{N}_\otimes(r, 0; g') + S(r) \\ &\leq \frac{1}{n+1}N(r, 0; g) + \frac{2}{n+5}N(r, \beta_1; g) + \overline{N}_\otimes(r, 0; g') + S(r) \\ &\leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right)T(r, g) + \overline{N}_\otimes(r, 0; g') + S(r). \end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, \beta_1; f) + \overline{N}(r, \infty; f) - \overline{N}_\otimes(r, 0; f') + S(r, f) \\ &\leq \frac{1}{n+1}N(r, 0; f) + \frac{2}{n+5}N(r, \beta_1; f) + \left(\frac{1}{n+1} + \frac{2}{n+5}\right)T(r, g) \\ &\quad + \overline{N}_\otimes(r, 0; g') - \overline{N}_\otimes(r, 0; f') + S(r), \end{aligned}$$

i.e.,

$$(2.3) \quad \left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right)T(r, f) \leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right)T(r, g) + \overline{N}_\otimes(r, 0; g') - \overline{N}_\otimes(r, 0; f') + S(r).$$

In a similar manner we get

$$(2.4) \quad \left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right)T(r, g) \leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right)T(r, f) + \overline{N}_\otimes(r, 0; f') - \overline{N}_\otimes(r, 0; g') + S(r).$$

Adding (2.3) and (2.4) we get

$$\left(1 - \frac{2}{n+1} - \frac{4}{n+5}\right) \{T(r, f) + T(r, g)\} \leq S(r),$$

which is a contradiction for $n > 7$. Hence this subcase does not hold.

Subcase 1.2. Both the roots β_1 and β_2 are non zero.

Let z_0 be a zero of f with multiplicity $p (\geq 1)$ which is not a zero or pole of α . Then from (2.1) we get z_0 is a pole of g with multiplicity $q (\geq 1)$ such that

$$(2.5) \quad np + p - 1 = (n + 3)q + 1$$

i.e., $q \geq \frac{n-1}{2}$. So from (2.5) we get

$$(n + 1)p \geq \frac{(n + 3)(n - 1) + 4}{2}, \quad \text{i.e., } p \geq \frac{n + 1}{2}.$$

So from above we have

$$N(r, 0; f) \geq \frac{n + 1}{2} \overline{N}(r, 0; f) + S(r, f), \quad \text{and so } \Theta(0; f) \geq 1 - \frac{2}{n + 1}.$$

Next suppose z_1 be a zero of $f - \beta_1$ with multiplicity $p (\geq 1)$ and it is not a zero or pole of α . Then z_1 be a pole of g with multiplicity $q (\geq 1)$ such that

$$2p - 1 = (n + 3)q + 1, \quad \text{i.e., } p = \frac{(n + 3)q + 2}{2} \geq \frac{n + 5}{2}.$$

$$N(r, \beta_1; f) \geq \frac{n + 5}{2} \overline{N}(r, 0; f) + S(r, f), \quad \text{and so } \Theta(\beta_1; f) \geq 1 - \frac{2}{n + 5}.$$

Similarly we can deduce that

$$\Theta(\beta_2; f) \geq 1 - \frac{2}{n + 5}.$$

Since $\Theta(0; f) + \Theta(\beta_1; f) + \Theta(\beta_2; f) \leq 2$, it follows that

$$3 - \frac{4}{n + 5} - \frac{2}{n + 1} \leq 2, \quad \text{or } \frac{4}{n + 5} + \frac{2}{n + 1} \geq 1$$

which is a contradiction for $n > 7$. Hence this subcase also does not hold.

Case 2. The roots of the equation $az^2 + bz + c = 0$ are equal say $\beta_1 = \beta_2 = \beta$.

Let z_0 be a zero of f with multiplicity $p (\geq 1)$ which is not a zero or pole of α . Then z_0 is a pole of g with multiplicity $q (\geq 1)$ such that $np + p - 1 = (n + 3)q + 1$, i.e.

$$q \geq \frac{n - 1}{2} \quad \text{and so } p \geq \frac{n + 1}{2}.$$

Hence

$$N(r, 0; f) \geq \frac{n + 1}{2} \overline{N}(r, 0; f) + S(r, f).$$

Next suppose z_1 be a zero of $f - \beta$ with multiplicity $p (\geq 1)$ which is not a zero or pole of α . Then z_1 be a pole of g with multiplicity $q (\geq 1)$ such that

$$3p - 1 = (n + 3)q + 1 \geq n + 4, \quad \text{i.e., } p \geq \frac{n + 5}{3}.$$

Let $\bar{N}_{\oplus}(r, 0; f')$ ($\bar{N}_{\oplus}(r, 0; g')$) denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f - \beta)$ ($g(g - \beta)$). Now proceeding in the same way as done in Subcase 1.1 we note that

$$\bar{N}(r, \infty; f) \leq \left(\frac{2}{n+1} + \frac{3}{n+5} \right) T(r, g) + \bar{N}_{\oplus}(r, 0; g') + S(r).$$

By the second fundamental theorem we get

$$\begin{aligned} T(r, f) &\leq \bar{N}(r, 0; f) + \bar{N}(r, \beta; f) + \bar{N}(r, \infty; f) - \bar{N}_{\oplus}(r, 0; f') + S(r, f) \\ &\leq \frac{2}{n+1} N(r, 0; f) + \frac{3}{n+5} N(r, \beta; f) + \left(\frac{2}{n+1} + \frac{3}{n+5} \right) T(r, g) \\ &\quad + \bar{N}_{\oplus}(r, 0; g') - \bar{N}_{\oplus}(r, 0; f') + S(r), \end{aligned}$$

i.e.,

$$(2.6) \quad \left(1 - \frac{2}{n+1} - \frac{3}{n+5} \right) T(r, f) \leq \left(\frac{2}{n+1} + \frac{3}{n+5} \right) T(r, g) + \bar{N}_{\oplus}(r, 0; g') - \bar{N}_{\oplus}(r, 0; f') + S(r).$$

In a similar manner we get

$$(2.7) \quad \left(1 - \frac{2}{n+1} - \frac{3}{n+5} \right) T(r, g) \leq \left(\frac{2}{n+1} + \frac{3}{n+5} \right) T(r, f) + \bar{N}_{\oplus}(r, 0; f') - \bar{N}_{\oplus}(r, 0; g') + S(r).$$

Adding (2.6) and (2.7) we get

$$\left(1 - \frac{4}{n+1} - \frac{6}{n+5} \right) \{T(r, f) + T(r, g)\} \leq S(r),$$

which is a contradiction for $n > 7$. This proves the lemma. \square

Lemma 2.9. *Let $F = f^{n+1} \left[\frac{af^2}{n+3} + \frac{bf}{n+2} + \frac{c}{n+1} \right]$ and $G = g^{n+1} \left[\frac{ag^2}{n+3} + \frac{bg}{n+2} + \frac{c}{n+1} \right]$, where $n (\geq 5)$ is an integer $a \neq 0$, $|b| + |c| \neq 0$. Then $F' \equiv G'$ implies $F \equiv G$.*

Proof. Let $F' \equiv G'$, then $F = G + d$ where d is a constant. If possible let $d \neq 0$. Then by the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} (n+3)T(r, f) &\leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F) + \bar{N}(r, d; F) + S(r, F) \\ &\leq \bar{N}(r, \infty; f) + \bar{N}(r, 0; f) + \bar{N}(r, \beta_1; f) + \bar{N}(r, \beta_2; f) \\ &\quad + \bar{N}(r, 0; g) + \bar{N}(r, \beta_1; g) + \bar{N}(r, \beta_2; g) + S(r, f) \\ (2.8) \quad &\leq 4T(r, f) + 3T(r, g) + S(r, f), \end{aligned}$$

where β_1 and β_2 are the roots of the equation $az^2 + bz + c = 0$. In a similar manner we get

$$(2.9) \quad (n+3)T(r, g) \leq 3T(r, f) + 4T(r, g) + S(r, g).$$

Adding (2.8) and (2.9) we get

$$(n-4)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

which is a contradiction for $n \geq 5$. So $d = 0$ and the lemma follows. \square

Lemma 2.10 ([4]). *Let*

$$Q(\omega) = (n - 1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n - 2) (\omega^{n-1} - 1)^2,$$

then

$$Q(\omega) = (\omega - 1)^4(\omega - \beta_1) (\omega - \beta_2) \dots (\omega - \beta_{2n-6}),$$

where $\beta_j \in C \setminus \{0, 1\}$ ($j = 1, 2, \dots, 2n - 6$), which are distinct respectively.

Lemma 2.11. *Let F and G be given as in Lemma 2.9 and $n (\geq 3)$ be an integer. Suppose $F \equiv G$. Then the following holds.*

- (i) *If $b \neq 0, c = 0$ and $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ then $f \equiv g$.*
- (ii) *If $b \neq 0, c \neq 0$, and the roots of the equation $az^2 + bz + c = 0$ are distinct and one of f and g is non entire meromorphic functions having only multiple poles then $f \equiv g$.*
- (iii) *If $b \neq 0, c \neq 0$, and the roots of the equation $az^2 + bz + c = 0$ coincides then $f \equiv g$.*
- (iv) *If $b = 0, c \neq 0$ then either $f \equiv g$ or $f \equiv -g$.
If n is an even integer then the possibility $f \equiv -g$ does not arise.*

Proof. We consider the following cases.

Case 1. Suppose $c = 0$ and $b \neq 0$. Then $F \equiv G$ implies

$$(2.10) \quad f^{n+2} \left(\frac{a}{n+3} f + \frac{b}{n+2} \right) \equiv g^{n+2} \left(\frac{a}{n+3} g + \frac{b}{n+2} \right).$$

Let us assume $f \not\equiv g$. We consider two cases:

Subcase 1.1. Let $y = \frac{g}{f}$ be a constant. Since $y \neq 1$, from (2.10) it follows that $y^{n+2} \neq 1, y^{n+3} \neq 1$ and $f \equiv -\frac{b(n+3)(1-y^{n+2})}{a(n+2)(1-y^{n+3})}$, a constant, which is impossible.

Subcase 1.2. Let $y = \frac{g}{f}$ be nonconstant. Noting that $f \not\equiv g$ clearly the poles of f comes from the zeros of $y - u_k$ where $u_k = \exp(\frac{2k\pi i}{n+3}), k = 1, 2, \dots, n + 2$. So we have

$$\sum_{k=1}^{n+2} \bar{N}(r, u_k; y) \leq \bar{N}(r, \infty; f).$$

By the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} n T(r, y) &\leq \sum_{k=1}^{n+2} \bar{N}(r, u_k; y) + S(r, y) \leq \bar{N}(r, \infty; f) + S(r, y) \\ &\leq (1 - \Theta(\infty; f) + \varepsilon) T(r, f) + S(r, y) \\ &= (n + 2) (1 - \Theta(\infty; f) + \varepsilon) T(r, y) + S(r, y), \end{aligned}$$

i.e.,

$$(2.11) \quad \left[\frac{n}{n+2} - 1 + \Theta(\infty; f) - \varepsilon \right] T(r, y) \leq S(r, y),$$

where $\varepsilon > 0$ be arbitrary. In a similar manner we can obtain

$$(2.12) \quad \left[\frac{n}{n+2} - 1 + \Theta(\infty; g) - \varepsilon \right] T(r, y) \leq S(r, y).$$

Adding (2.11) and (2.12) we get

$$(2.13) \quad \left(\Theta(\infty; f) + \Theta(\infty; g) - \frac{4}{n+2} - 2\varepsilon \right) T(r, y) \leq S(r, y).$$

Since $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ we can choose a $\delta > 0$ such that

$$\Theta(\infty; f) + \Theta(\infty; g) = \frac{4}{n+2} + \delta.$$

So for $0 < \varepsilon < \frac{\delta}{2}$ from (2.13) we can deduce a contradiction. Hence $f \equiv g$.

Case 2. Suppose $b \neq 0$ and $c \neq 0$. Then $F \equiv G$ implies

$$(2.14) \quad Af^{n+3} + Bf^{n+2} + Cf^{n+1} \equiv Ag^{n+3} + Bg^{n+2} + Cg^{n+1},$$

where $A = \frac{a}{n+3}$, $B = \frac{b}{n+2}$ and $C = \frac{c}{n+1}$.

Let us assume $f \not\equiv g$.

Subcase 2.1. Suppose the roots of the equation $az^2 + bz + c = 0$ are distinct. Since (2.14) implies f, g share (∞, ∞) without loss of generality we may assume that g has some multiple poles. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$Ag^2(\eta^{n+3} - 1) + Bg(\eta^{n+2} - 1) + C(\eta^{n+1} - 1) \equiv 0,$$

i.e.,

$$(2.15) \quad Ag^2 \equiv -Bg \frac{\eta^{n+2} - 1}{\eta^{n+3} - 1} - C \frac{\eta^{n+1} - 1}{\eta^{n+3} - 1}.$$

Let z_0 be a pole of g which is not a root of $\eta - u_k = 0$, where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \dots, n+2$ with multiplicity p . Then from (2.15) we have

$$2p = p \quad \text{i.e.,} \quad p = 0,$$

which is impossible. The other poles of the right hand side of (2.15) are the roots of $\eta - u_k = 0$ where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \dots, n+2$. Suppose z_1 is a zero of $\eta - u_k$ of multiplicity r . From (2.15) we see that z_1 is a pole of g with multiplicity s (say) such that

$$2s = r + s \quad \text{i.e.,} \quad r = s.$$

Since g has no simple pole it follows that $\eta - u_k$ has no simple zero for $k = 1, 2, \dots, n+2$. Hence

$$\Theta(u_k; \eta) \geq \frac{1}{2}$$

for $k = 1, 2, \dots, n+2$. Since $\sum_{k=1}^{n+2} \Theta(u_k; \eta) \leq 2$ and $n \geq 3$ we arrive at a contradiction.

Subcase 2.2. Suppose the roots of the equation $az^2 + bz + c = 0$ coincides and so we obtain $b^2 = 4ac$. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$(2.16) \quad \begin{aligned} & a(n+2)(n+1)g^2(\eta^{n+3} - 1) + b(n+3)(n+1)g(\eta^{n+2} - 1) \\ & + c(n+3)(n+2)(\eta^{n+1} - 1) \equiv 0. \end{aligned}$$

Since η is not constant using Lemma 2.10 we get from (2.16) that

$$\begin{aligned} & \left[(n+2)(n+1)g(\eta^{n+3} - 1) + \frac{b}{2a}(n+3)(n+1)(\eta^{n+2} - 1) \right]^2 \\ & = - (n+3)(n+1) \left[\frac{c}{a}(n+2)^2(\eta^{n+3} - 1)(\eta^{n+1} - 1) \right. \\ & \quad \left. - \frac{b^2}{4a^2}(n+3)(n+1)(\eta^{n+2} - 1)^2 \right] = -\frac{c}{a}(n+3)(n+1)Q(\eta), \end{aligned}$$

where $Q(\eta) = (\eta - 1)^4(\eta - \beta_1)(\eta - \beta_2) \dots (\eta - \beta_{2n})$ and $\beta_j \in C \setminus \{0, 1\}$ ($j = 1, 2, \dots, 2n$) which are distinct. This implies that every zero of $\eta - \beta_j$ ($j = 1, 2, \dots, 2n$) has a multiplicity of at least 2, i.e., $\Theta(\beta_j; \eta) \geq \frac{1}{2}$ for ($j = 1, 2, \dots, 2n$).

But $\sum_{j=1}^{2n} \Theta(\beta_j; \eta) \leq 2$ which implies $n \leq 2$. This is a contradiction. So η is constant and from (2.15) we have $(\eta^{n+1} - 1) = 0$ and $(\eta^{n+2} - 1) = 0$ which implies $\eta = 1$ and so $f \equiv g$.

Case 3. Suppose $b = 0$ and $c \neq 0$. Then (2.14) reduces to

$$\left[\frac{a}{n+3}f^2 + \frac{c}{n+1} \right] f^{n+1} \equiv \left[\frac{a}{n+3}g^2 + \frac{c}{n+1} \right] g^{n+1}.$$

Now proceeding in the line of Lemma 2.4 in [12] we can prove $f \equiv g$ and $f \equiv -g$ and if n is an even integer then the possibility of $f \equiv -g$ does not arise. \square

Lemma 2.12 ([19]). *Let f be a nonconstant meromorphic function. Then*

$$N(r, 0; f^{(k)}) \leq k\bar{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

Lemma 2.13. *Let F and G be given as in Lemma 2.9 and F_1, G_1 be given by Lemma 2.6. If γ_1, γ_2 are the roots of $\frac{a}{n+3}z^2 + \frac{b}{n+2}z + \frac{c}{n+1} = 0$ and β_1, β_2 are the roots of $az^2 + bz + c = 0$. Then*

$$\begin{aligned} T(r, F) & \leq T(r, F_1) + N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) \\ & \quad - N(r, \beta_1; f) - N(r, \beta_2; f) - N(r, 0; f') + S(r). \end{aligned}$$

Proof. Clearly $F' = \alpha F_1$ and $G' = \alpha G_1$. By the first fundamental theorem and Lemmas 2.5, 2.6 we obtain

$$\begin{aligned}
T(r, F) &= T(r, \frac{1}{F}) + O(1) = N(r, 0; F) + m(r, \frac{1}{F}) + O(1) \\
&\leq N(r, 0; F) + m(r, \frac{F'}{F}) + m(r, 0; F') + O(1) \\
&= T(r, F') + N(r, 0; F) - N(r, 0; F') + S(r, F) \\
&\leq T(r, F_1) + (n+1)N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) - nN(r, 0; f) \\
&\quad - N(r, \beta_1; f) - N(r, \beta_2; f) - N(r, 0; f') + S(r) \\
&= T(r, F_1) + N(r, 0; f) + N(r, \gamma_1; f) + N(r, \gamma_2; f) - N(r, \beta_1; f) \\
&\quad - N(r, \beta_2; f) - N(r, 0; f') + S(r). \quad \square
\end{aligned}$$

3. PROOF OF THE THEOREM

Proof of Theorem 1.1. Let F, G be defined as in Lemma 2.9 and F_1 and G_1 be defined as in Lemma 2.6. Then it follows that F' and G' share “ $(\alpha; 2)$ ” and hence F_1 and G_1 share “ $(1, 2)$ ”. Suppose $H \neq 0$. Then by Lemmas 2.4, 2.6 and (2.6) we get

$$\begin{aligned}
(3.1) \quad T(r, F_1) &\leq N_2(r, 0; F_1) + N_2(r, \infty; F_1) + N_2(r, 0; G_1) \\
&\quad + N_2(r, \infty; G_1) + S(r, f) + S(r, g) \\
&\leq 2\bar{N}(r, 0; f) + N(r, \beta_1; f) + N(r, \beta_2; f) + 2\bar{N}(r, 0; g) \\
&\quad + N(r, \beta_1; g) + N(r, \beta_2; g) + 2\bar{N}(r, \infty; f) + 2\bar{N}(r, \infty; g) \\
&\quad + N(r, 0; f') + \bar{N}(r, 0; g') + S(r).
\end{aligned}$$

Now from Lemmas 2.5, 2.12 and 2.13 we can obtain from (3.1) for $\varepsilon (> 0)$

$$\begin{aligned}
(3.2) \quad (n+3)T(r, f) &\leq 2\bar{N}(r, 0; f) + 2\bar{N}(r, \infty; f) + 3T(r, f) + 2\bar{N}(r, 0; g) \\
&\quad + 2\bar{N}(r, \infty; g) + 2T(r, g) + N(r, 0; g') + S(r) \\
&\leq 5T(r, f) + 5T(r, g) + 2\bar{N}(r, \infty; f) + 3\bar{N}(r, \infty; g) + S(r) \\
&\leq (15 - 2\Theta(\infty; f) - 3\Theta(\infty; g) + 2\varepsilon) T(r) + S(r).
\end{aligned}$$

In a similar manner we can obtain

$$(3.3) \quad (n+3)T(r, g) \leq (15 - 3\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon) T(r) + S(r).$$

From (3.2) and (3.3) we get

$$(3.4) \quad [n - 12 + 2\Theta(\infty; f) + 2\Theta(\infty; g) + \min\{\Theta(\infty; f); \Theta(\infty; g)\} - 2\varepsilon] T(r) \leq S(r).$$

Since $\varepsilon (> 0)$ is arbitrary, (3.4) implies a contradiction. Hence $H \equiv 0$.

Since for $\varepsilon > 0$ we have

$$\begin{aligned} \overline{N}(r, 0; f') &\leq T(r, f') - m\left(r, \frac{1}{f'}\right) \\ &\leq m(r, f) + N(r, \infty; f) + \overline{N}(r, \infty; f) - m\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq (2 - \Theta(\infty; f) + \varepsilon)T(r, f) - m\left(r, \frac{1}{f'}\right) + S(r, f). \end{aligned}$$

We note that

$$\begin{aligned} &\overline{N}(r, 0; F_1) + \overline{N}(r, \infty; F_1) + \overline{N}(r, 0; G_1) + \overline{N}(r, \infty; G_1) \\ &\leq \overline{N}(r, 0; f) + \overline{N}(r, \beta_1; f) + \overline{N}(r, \beta_2; f) + \overline{N}(r, \infty; f) + \overline{N}(r, 0; f') \\ &\quad + \overline{N}(r, 0; g) + \overline{N}(r, \beta_1; g) + \overline{N}(r, \beta_2; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 0; g') \\ &\leq (12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) + 2\varepsilon)T(r) \\ (3.5) \quad &- m(r, 0; f') - m(r, 0; g') + S(r). \end{aligned}$$

Also using Lemma 2.5 we get

$$\begin{aligned} T(r, F') + m\left(r, \frac{1}{f'}\right) &= m(r, f^n(af^2 + bf + c)f') + m\left(r, \frac{1}{f'}\right) \\ &\quad + N(r, \infty; f^n(af^2 + bf + c)f') \geq m(r, f^n(af^2 + bf + c)) \\ &\quad + N(r, \infty; f^n(af^2 + bf + c)) = T(r, f^n(af^2 + bf + c)) \\ (3.6) \quad &= (n + 2)T(r, f) + O(1). \end{aligned}$$

Similarly

$$(3.7) \quad T(r, G') + m\left(r, \frac{1}{g'}\right) \geq (n + 2)T(r, g) + O(1).$$

From (3.6) and (3.7) we get

$$(3.8) \quad \max \{T(r, F_1), T(r, G_1)\} \geq (n + 2)T(r) - m\left(r, \frac{1}{f'}\right) - m\left(r, \frac{1}{g'}\right) + O(1).$$

By (3.5) and (3.8) applying Lemma 2.7 we get either $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$.

Now from Lemma 2.8 it follows that $F_1G_1 \not\equiv 1$. Again $F_1 \equiv G_1$ implies $F' \equiv G'$. So from Lemmas 2.9 and 2.11 the theorem follows. \square

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DEPARTMENT OF MATHEMATICS, KALYANI GOVERNMENT ENGINEERING COLLEGE
WEST BENGAL 741235, INDIA
E-mail: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com