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# ON $S$-NOETHERIAN RINGS

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Abstract. Let $R$ be a commutative ring and $S \subseteq R$ a given multiplicative set. Let $(M, \leq)$ be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then it is shown, under some additional conditions, that the generalized power series ring $[[R^{M,\leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and $M$ is finitely generated.

## 1. Introduction

Let $R$ be a commutative ring and $S \subseteq R$ a given multiplicative set. According to [2], an ideal $I$ of $R$ is called $S$-finite if $sI \subseteq J \subseteq I$ for some $s \in S$ and some finitely generated ideal $J$. $R$ is called $S$-Noetherian if each ideal of $R$ is $S$-finite. Clearly every Noetherian ring is $S$-Noetherian for any multiplicative set $S$.

Let $X_1, \dots, X_n$ be indeterminates. It was showed in [2], Proposition 10, that if $S \subseteq R$ is an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $R$ is $S$-Noetherian, then $R[[X_1, \dots, X_n]]$ is $S$-Noetherian. It was proved in [3], Theorem 4.3, that if $(M, \leq)$ is a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$, then the generalized power series ring $[[R^{M,\leq}]]$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated. By the technique developed in [3] we show that if $(M, \leq)$ satisfies the condition that $0 \leq m$ for every $m \in M$ and $S \subseteq R$ is an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors, then $[[R^{M,\leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and $M$ is finitely generated.

Throughout this note all rings are commutative with identity and all monoids are commutative. Any concept and notation not defined here can be found in [2], [3] and [6].

## 2. Generalized power series rings

Let $(M, \leq)$ be an ordered set. Recall that $(M, \leq)$ is artinian if every strictly decreasing sequence of elements of $M$ is finite, and that $(M, \leq)$ is narrow if every

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subset of pairwise order-incomparable elements of $M$ is finite. Let $M$ be a commutative monoid. Unless stated otherwise, the operation of $M$ shall be denoted additively, and the neutral element by 0.

Let $(M, \leq)$ be a strictly ordered monoid (that is, $(M, \leq)$ is an ordered monoid satisfying the condition that, if $m_1, m_2, m \in M$ and $m_1 < m_2$, then $m_1 + m < m_2 + m$), and $R$ a ring. Let $[[R^{M,\leq}]]$ be the set of all maps $f : M \longrightarrow R$ such that $\mathrm{supp}\,(f) = \{m \in M \mid f(m) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{M,\leq}]]$ is an abelian additive group. For every $m \in M$ and $f, g \in [[R^{M,\leq}]]$, let $X_m(f,g) = \{(u,v) \in M \times M \mid m = u + v, f(u) \neq 0, g(v) \neq 0\}$. It follows from [9], 1.16, that $X_m(f,g)$ is finite. This fact allows us to define the operation of convolution:

$$(fg)(m) = \sum_{(u,v) \in X_m(f,g)} f(u)\, g(v)\,.$$

With this operation, and pointwise addition, $[[R^{M,\leq}]]$ becomes a commutative ring, which is called the ring of generalized power series. The elements of $[[R^{M,\leq}]]$ are called generalized power series with coefficients in $R$ and exponents in $M$.

For example, if $M = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If $M$ is a commutative monoid and $\leq$ is the trivial order, then $[[R^{M,\leq}]] = R[M]$, the monoid-ring of $M$ over $R$. Further examples are given in [5] and [6]. Results for rings of generalized power series appeared in [3], [5]-[11].

Any monoid $M$ has the algebraic or natural preorder defined by $a \preceq b$ if $a + c = b$ for some $c \in M$. In general, $a \preceq b \preceq a$ does not imply $a = b$, so $\preceq$ is not always a partial order on M. The symbol $\preceq$ will always be used for the algebraic preorder of a monoid in this paper.

Recall from [3] that if $(M, \leq)$ and $(N, \leq)$ are ordered monoids, then a strict monoid homomorphism $\sigma : (M, \leq) \longrightarrow (N, \leq)$ is a monoid homomorphism $\sigma : M \longrightarrow N$ which is strictly increasing with respect to the partial orders $\leq$.

**Lemma 2.1.** *Let $(M, \leq)$, where $|M| > 1$, be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then for some commutative free monoid $F$, there exists a surjective strict monoid homomorphism $\sigma : (F, \preceq) \longrightarrow (M, \preceq)$.*

**Proof.** It follows from [3], Lemma 3.1 and Lemma 3.2. □

Note from the proof of [3], Lemma 3.2, that if $M$ is finitely generated, then the free monoid $F$ can be chosen finitely generated.

**Lemma 2.2.** *Let $\alpha : R \longrightarrow R'$ be a surjective ring homomorphism and $S \subseteq R$ a multiplicative set of $R$. If $R$ is $S$-Noetherian, then $R'$ is $\alpha(S)$-Noetherian.*

**Proof.** It follows from the definition. □

Let $m \in M$. We define a mapping $e_m \in [[R^{M,\leq}]]$ as follows:

$$e_m(m) = 1\,, \quad e_m(x) = 0\,, \quad m \neq x \in M\,.$$

Let $r \in R$. Define a mapping $c_r \in [[R^{M,\leq}]]$ as follows:

$$c_r(0) = r\,, \quad c_r(m) = 0\,, 0 \neq m \in M\,.$$

Then $R$ is isomorphic to the subring $\{c_r | r \in R\}$ of $[[R^{S,\leq}]]$. Thus if $S$ is a multiplicative set of $R$ then $C(S) = \{c_r | r \in S\}$ is a multiplicative set of $[[R^{M,\leq}]]$. In the following we will say $[[R^{M,\leq}]]$ is $S$-Noetherian if $[[R^{M,\leq}]]$ is $C(S)$-Noetherian.

It was proved in [3], Theorem 4.3, that if $(M, \leq)$ satisfies the condition that $0 \leq m$ for every $m \in M$, then $[[R^{M,\leq}]]$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated. For $S$-Noetherian rings we have the following result. Recall from [1] that a multiplicative set $S$ of a ring $R$ is said to be anti-Archimedean if $(\cap_{n\geq 1} s^n R) \cap S \neq \emptyset$ for every $s \in S$. Clearly every multiplicative set consisting of units is anti-archimedean.

**Theorem 2.3.** *Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Let $(M, \leq)$ be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then $[[R^{M,\leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and $M$ is finitely generated.*

**Proof.** We complete the proof by adapting the proof of [3], Theorem 4.3. Suppose that $[[R^{M,\leq}]]$ is $S$-Noetherian. Let $\{m_n \mid n \in \mathbb{N}\}$ be an infinite sequence in $M$. We will show that there exist $i < j$ in $\mathbb{N}$ such that $m_i \preceq m_j$. Consider the ascending chain of ideals of $[[R^{M,\leq}]]$: $[[R^{M,\leq}]]e_{m_1} \subseteq [[R^{M,\leq}]]e_{m_1} + [[R^{M,\leq}]]e_{m_2} \subseteq \cdots \subseteq [[R^{M,\leq}]]e_{m_1} + \cdots + [[R^{M,\leq}]]e_{m_i} \subseteq \ldots$. Denote that $I = \sum_{i=1}^{\infty}([[R^{M,\leq}]]e_{m_1} + \cdots + [[R^{M,\leq}]]e_{m_i})$. Then $I$ is an ideal of $[[R^{M,\leq}]]$. Since $[[R^{M,\leq}]]$ is $S$-Noetherian, there exist $s \in S$ and a finitely generated ideal $J$ of $[[R^{M,\leq}]]$ such that $c_s I \subseteq J \subseteq I$. Clearly there exists an integer $k$ such that $J \subseteq [[R^{M,\leq}]]e_{m_1} + \cdots + [[R^{M,\leq}]]e_{m_k}$. Thus $c_s e_{m_{k+1}} = f_1 e_{m_1} + f_2 e_{m_2} + \cdots + f_k e_{m_k}$ for some $f_1, f_2, \ldots, f_k \in [[R^{M,\leq}]]$. Hence $m_{k+1} \in \cup_{i=1}^{k} \operatorname{supp}(f_i e_{m_i}) \subseteq \cup_{i=1}^{k} \big(\operatorname{supp}(f_i) + m_i\big)$. This implies that $m_{k+1} = t + m_i$ for some $i < k+1$ and $t \in M$. Thus $m_i \preceq m_{k+1}$. Hence we have shown that for any infinite sequence $\{m_n \mid n \in \mathbb{N}\}$ in $M$ there exist $i < j$ in $\mathbb{N}$ such that $m_i \preceq m_j$. Thus, by ([3], Lemma 3.3), $M$ is finitely generated.

Let

$$W = \big\{f \in [[R^{M,\leq}]] \mid f(0) = 0\big\}\,.$$

For any $f \in W$ and any $g \in [[R^{M,\leq}]]$,

$$(gf)(0) = \sum_{(u,v)\in X_0(g,f)} g(u)\, f(v) = g(0)\, f(0) = 0\,,$$

which implies that $gf \in W$. Similarly $fg \in W$. Now it is easy to see that $W$ is an ideal of $[[R^{M,\leq}]]$. Define a mapping $\alpha : R \longrightarrow [[R^{M,\leq}]]/W$ via

$$\alpha(r) = c_r + W\,, \quad \forall\ r \in R\,.$$

Clearly $\alpha$ is a homomorphism of rings. For any $f \in [[R^{M,\leq}]]$, $f + W = c_{f(0)} + W = \alpha\big(f(0)\big)$, which implies that $\alpha$ is an epimorphism. Clearly $\alpha$ is a monomorphism. Thus there is an isomorphism of rings $R \cong [[R^{M,\leq}]]/W$. Now it follows from Lemma 2.2 that $R$ is $S$-Noetherian.

Now suppose that $R$ is $S$-Noetherian and $M$ is finitely generated. If $|M| = 1$, then $[[R^{M,\leq}]] \cong R$. Thus the result is clear. Now suppose that $M$ is nontrivial. From Lemma 2.1 there exists a strict monoid surjection $\sigma : \big((\mathbb{N} \cup \{0\})^n, \preceq\big) \longrightarrow (M, \preceq)$ for some $n \in \mathbb{N}$. Since $0 \leq m$ for each $m \in M$, we have $a \preceq b \Longrightarrow a \leq b$ for all $a, b \in M$. In other words, the identity map from $(M, \preceq)$ to $(M, \leq)$ is a strict monoid surjection. Composing these two maps gives a strict monoid surjection $\theta : \big((\mathbb{N} \cup \{0\})^n, \preceq\big) \longrightarrow (M, \leq)$, and so $[[R^{M,\leq}]]$ is a homomorphic image of the ring $[[R^{(\mathbb{N}\cup\{0\})^n,\preceq}]]$. From [2], Proposition 10, it follows that $[[R^{(\mathbb{N}\cup\{0\})^n,\preceq}]]$ is $S$-Noetherian. Thus, by Lemma 2.2, $[[R^{M,\leq}]]$ is $S$-Noetherian. □

**Remark 2.4.** Note that the direct implication in Theorem 2.3 holds without further assumptions on $S$. But the following example (see [2]) shows that the assumptions on $S$ is needed for the converse. Let $(V, M)$ be a rank-one nondiscrete valuation domain. Then $V$ is $S$-Noetherian where $S = V - \{0\}$, but $V[[x]]$ is not $S$-Noetherian by [2]. In fact, $V[[x]]_S$ is not Noetherian by part (3) of [4], Theorem 3.13.

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a numerical monoid. It is well-known that any numerical monoid is finitely generated (see 1.3 of [6]). Thus we have the following result.

**Corollary 2.5.** *Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Let $M$ be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. Then $[[R^{M,\leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian.*

Let $p_1, \ldots, p_n$ be prime numbers. Set

$$N(p_1, \ldots, p_n) = \left\{p_1^{m_1} p_2^{m_2} \ldots p_n^{m_n} \mid m_1, m_2, \ldots, m_n \in \mathbb{N} \cup \{0\}\right\}.$$

Then $N(p_1, \ldots, p_n)$ is a submonoid of $(\mathbb{N}, \cdot)$. Let $\leq$ be the usual natural order.

**Corollary 2.6.** *Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Then the ring $[[R^{N(p_1,\ldots,p_n),\leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian.*

**Corollary 2.7.** *Let $(M_1, \leq_1), \ldots, (M_n, \leq_n)$ be strictly ordered monoids satisfying the condition that $0 \leq_i m_i$ for every $m_i \in M_i$. Denote by (lex $\leq$) the lexicographic order on the monoid $M_1 \times \cdots \times M_n$. Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Then the following statements are equivalent.*

(1) *The ring $[[R^{M_1\times\cdots\times M_n,(\mathrm{lex}\leq)}]]$ is $S$-Noetherian.*

(2) *$R$ is $S$-Noetherian and each $M_i$ is finitely generated.*

**Proof.** It is easy to see that $(S_1 \times \cdots \times S_n, (\mathrm{lex}\ \leq))$ is a strictly ordered monoid and $(0, \ldots, 0)(\mathrm{lex}\ \leq)(m_1, \ldots, m_n)$ for each $(m_1, \ldots, m_n) \in M_1 \times \cdots \times M_n$. Thus, by Theorem 2.3, $[[R^{M_1\times\cdots\times M_n,(\mathrm{lex}\leq)}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and each $M_i$ is finitely generated.

## 3. Laurent series rings

Let $X_1, \dots, X_n$ be indeterminates. It was showed in [2], Proposition 10 that if $S \subseteq R$ is an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $R$ is $S$-Noetherian, then $R[[X_1, \dots, X_n]]$ is $S$-Noetherian. For Laurent series rings we have a same result.

**Theorem 3.1.** *Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $X$ an indeterminate. If $R$ is $S$-Noetherian, then so is $R[[X, X^{-1}]]$.*

**Proof.** Let $A$ be an ideal of $R[[X, X^{-1}]]$. We will show that $A$ is $S$-finite. For any $0 \neq f \in R[[X, X^{-1}]]$, we denote by $\pi(f)$ the smallest integer $k$ such that $f(k) \neq 0$. For every $k \in \mathbb{Z}$, set

$$I_k = \{f(k) \mid f \in A, \pi(f) = k\},$$

and $I = \cup_{k \in \mathbb{Z}} I_k$. Let $J$ be the ideal of $R$ generated by $I$. Since $R$ is $S$-Noetherian, there exist $w \in S$, $f_1, \dots, f_m \in A$ such that $wJ \subseteq \sum_{i=1}^m f_i(k_i)R$, where $k_i = \pi(f_i)$, $i = 1, \dots, m$.

Consider any $0 \neq f \in A$. Suppose that $\pi(f) = k$. Then there exist $r_{ik} \in R$ such that $wf(k) = \sum_{i=1}^m f_i(k_i) r_{ik}$. Set $g_{k+1} = wf - \sum_{i=1}^m f_i X^{k-k_i} r_{ik}$. Then $\pi(g_{k+1}) \geq k+1$. Clearly $g_{k+1} \in A$. Thus there exist $r_{i,k+1} \in R$, $i = 1, \dots, m$, such that $wg_{k+1}(k+1) = \sum_{i=1}^m f_i(k_i) r_{i,k+1}$. Set $g_{k+2} = wg_{k+1} - \sum_{i=1}^m f_i X^{k+1-k_i} r_{i,k+1}$. Then $\pi(g_{k+2}) \geq k+2$. Continuing in this manner, for any $n > 0$, we get $r_{i,k+n} \in R$ and $g_{k+n} \in A$ such that $g_{k+n+1} = wg_{k+n} - \sum_{i=1}^m f_i X^{k+n-k_i} r_{i,k+n}$ and $\pi(g_{k+n}) \geq k+n$. Thus

$$\begin{aligned} w^n f &= w^{n-1} g_{k+1} + w^{n-1} \sum_{i=1}^m f_i X^{k-k_i} r_{ik} \\ &= \cdots = g_{k+n} + \sum_{j=1}^n \sum_{i=1}^m f_i X^{k+j-1-k_i} w^{n-j} r_{i,k+j-1} \\ &= g_{k+n} + \sum_{i=1}^m f_i \Big( \sum_{j=1}^n X^{k+j-1-k_i} w^{n-j} r_{i,k+j-1} \Big). \end{aligned}$$

Since $S$ is anti-Archimedean, there exists $t \in (\cap w^j R) \cap S$. Thus $t = w^j r_j$ for some $r_j \in R$. Since $w$ is a nonzerodivisor, we have $r_n w^{n-j} = r_j$ for $j \leq n$. So $tf = r_n g_{k+n} + \sum_{i=1}^m f_i \Big( \sum_{j=1}^n X^{k+j-1-k_i} r_j r_{i,k+j-1} \Big)$. Now it is easy to see that

$$tf = \sum_{i=1}^m f_i \Big( \sum_{j=1}^\infty X^{k+j-1-k_i} r_j r_{i,k+j-1} \Big) \in \sum_{i=1}^m f_i R[[X, X^{-1}]].$$

Hence $tA \subseteq \sum_{i=1}^m f_i R[[X, X^{-1}]]$. Consequently, $R[[X, X^{-1}]]$ is $S$-Noetherian.

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